Bull. Korean Math. Soc. **46** (2009), No. 3, pp. 577–589
DOI 10.4134/BKMS.2009.46.3.577

η-PARALLEL CONTACT 3-MANIFOLDS

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Abstract. In this paper, we give a classification of contact 3-manifolds whose Ricci tensors are η-parallel.

1. Introduction

It is well-known that for a 3-dimensional Riemannian manifold its curvature tensor R is expressed only in terms of the Ricci tensor S, the metric tensor g and the scalar curvature r. Hence, in studying the 3-dimensional Riemannian geometry, we see at once that the condition of local symmetry ($\nabla R = 0$) is equivalent to the Ricci-parallel condition ($\nabla S = 0$).

Recently Boeckx and the present author [7] proved that a locally symmetric contact Riemannian manifold is either Sasakian and of constant curvature 1 or locally isometric to the unit tangent sphere bundle (with its standard contact metric structure) of a Euclidean space. (In the 3-dimensional case, we may also refer to [5].) This result says that the local symmetry is rather a strong condition in contact Riemannian geometry and hence, it is natural to consider a weaker condition, that is *η-parallel*. Let $M = (M; \eta, g, \varphi, \xi)$ be a contact Riemannian manifold. Then the *contact form* η determines the *contact distribution* D which is given by the kernel of η. We say that the Ricci tensor S is η-parallel if S satisfies $g((\nabla_X S)Y, Z) = 0$ for any $X, Y, Z \in D$. In this paper, we shall study 3-dimensional contact Riemannian manifolds whose Ricci-tensors are η-parallel.

On the other hand, given a contact manifold $M = (M; \eta)$ one may raise a following natural question (cf. [16]): Which metric is most proper among Riemannian metrics associated with η? One method of finding the nice Riemannian metrics is to study the criticality in the variational sense. In particular, in [2] and [16] the authors showed that M satisfies $\nabla_\xi h = 2h\varphi$ if and only if it has the critical metric of the Dirichlet energy functional $E(g) = \int_M \|L_\xi g\|^2 dM$ defined on the set of all Riemannian metrics associated with the given contact form η, where $h = \frac{1}{2} L_\xi \varphi$ and L_ξ is the Lie derivative with respect to ξ. (Chern-Hamilton [10] first studied critical metrics of the Dirichlet energy functional in

Received July 17, 2008; Revised September 12, 2008.
2000 *Mathematics Subject Classification.* 53C15, 53C35, 53D10.
Key words and phrases. contact 3-manifolds, η-parallel Ricci tensors, critical metrics.

dimension three.) And, in dimension three, Perrone [13] showed that $\nabla_\xi h = 0$ is the critical condition of the functional $I(g) = \int_M \|r\|^2 dM$.

The main purpose of this paper is to determine the 3-dimensional contact Riemannian manifolds whose Ricci tensors are η-parallel under the condition that $\nabla_\xi h = 2ah\varphi$, where $a \in \mathbb{R}$. More precisely, we prove:

Main Theorem. *Let M be a 3-dimensional contact Riemannian manifold which satisfies $\nabla_\xi h = 2ah\varphi$, $a \in \mathbb{R}$. Then the Ricci tensor S is η-parallel if and only if M is locally isometric to a Sasakian φ-symmetric space or a unimodular Lie group with a left invariant contact metric structure which is not Sasakian.*

In [6], the authors gave a classification of a 3-dimensional Sasakian φ-symmetric space (complete and simply connected Sasakian locally φ-symmetric space), namely, the standard unit sphere S^3; $SU(2)$, $\widetilde{SL(2,\mathbb{R})}$ (the universal covering space of $SL(2,\mathbb{R})$) or the Heisenberg group H with a left invariant Sasakian metric, respectively. In the process of the proof of the Main Theorem, we have also:

Corollary A. *Let M be a 3-dimensional complete and simply connected contact Riemannian manifold which satisfies $\nabla_\xi h = 0$ $(a = 0)$. Then the Ricci tensor S is η-parallel if and only if M is isometric to one of the followings*:

(1) *the standard unit sphere S^3; $SU(2)$, $\widetilde{SL(2,\mathbb{R})}$ or the Heisenberg group H with a left invariant Sasakian metric, respectively*;
(2) *$SU(2)$, $\widetilde{SL(2,\mathbb{R})}$ with a special left-invariant contact metric which is not Sasakian, respectively*;
(3) *flat manifold.*

Remark 1. In [11] (Theorem 5.1) they also studied 3-dimensional contact Riemannian manifolds satisfying $\nabla_\xi h = 0$ (or equivalently, $\nabla_\xi(h\varphi) = 0$)), which they have η-parallel Ricci tensors. But, we found that their proof is not complete. In fact, they made a gap obtaining the equation (5.3) in p. 162.

Corollary B. *Let M be a 3-dimensional complete and simply connected contact Riemannian manifold which satisfies $\nabla_\xi h = 2h\varphi$ $(a = 1)$. Then the Ricci tensor S is η-parallel if and only if M is isometric to one of the followings*:

(1) *the standard unit sphere S^3; $SU(2)$, $\widetilde{SL(2,\mathbb{R})}$ or the Heisenberg group H with a left invariant Sasakian metric, respectively*;
(2) *$\widetilde{SL(2,\mathbb{R})}$ with a special left-invariant contact metric which is not Sasakian.*

Finally, we remark that non-Sasakian, non-unimodular, 3-dimensional Lie groups with left-invariant contact metric structures have the property $\nabla_\xi h = 2ah\varphi$, but their Ricci tensors are not η-parallel. We also find a non-homogeneous example which satisfies $\nabla_\xi h = 2ah\varphi$, but its Ricci tensor is not η-parallel (see the end of Section 3).

2. Preliminaries

All manifolds in the present paper are assumed to be connected and of class C^∞. A $(2n+1)$-dimensional manifold M^{2n+1} is said to be a contact manifold if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, we have a unique vector field ξ, which is called the characteristic vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well-known that there exists a Riemannian metric g and a $(1,1)$-tensor field φ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \varphi Y), \quad \varphi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

where X and Y are vector fields on M. From (2.1) it follows that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

A Riemannian manifold M equipped with structure tensors (η, g, φ, ξ) satisfying (2.1) is said to be a contact Riemannian manifold and is denoted by $M = (M; \eta, g, \varphi, \xi)$. Given a contact Riemannian manifold M, we define a $(1,1)$-tensor field h by $h = \frac{1}{2} L_\xi \varphi$, where L denotes Lie differentiation. Then we may observe that h is symmetric and satisfies

$$h\xi = 0 \quad \text{and} \quad h\varphi = -\varphi h, \tag{2.3}$$

$$\nabla_X \xi = -\varphi X - \varphi h X, \tag{2.4}$$

where ∇ is the Levi-Civita connection. From (2.3) and (2.4) we see that each trajectory of ξ is a geodesic. We denote by R the Riemannian curvature tensor defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z$$

for all vector fields X, Y, Z. Along a trajectory of ξ, the Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ is a symmetric $(1,1)$-tensor field. We have

$$(\text{trace } R_\xi) = g(S\xi, \xi) = 2n - (\text{trace } h^2), \tag{2.5}$$

$$\nabla_\xi h = \varphi - \varphi R_\xi - \varphi h^2. \tag{2.6}$$

A contact Riemannian manifold for which ξ is Killing is called a K-contact manifold. It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$. For a contact Riemannian manifold M one may define naturally an almost complex structure J on $M \times \mathbb{R}$;

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, M is said to be normal or Sasakian. It is known that M is normal if and only if M satisfies

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. A Sasakian manifold is characterized by a condition

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{2.7}$$

for all vector fields X and Y on the manifold. It is also well-known that M is Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.8}$$

for all vector fields X and Y. For more details about contact Riemannian manifolds we refer to [1].

For a contact Riemannian manifold M, the tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = D_p \oplus \{\xi\}_p$ (direct sum), where we denote $D_p = \{v \in T_pM \mid \eta(v) = 0\}$. Then $D : p \to D_p$ defines a distribution orthogonal to ξ. The $2n$-dimensional distribution D is called the *contact distribution.* Now, we define a contact Riemannian manifold whose Ricci operator S is η-parallel.

Definition 2.1. A contact Riemannian manifold $M = (M; \eta, g, \varphi, \xi)$ is said to have an η-parallel Ricci tensor if $g((\nabla_U S)V, W) = 0$ for all vector fields $U, V, W \in D$.

Finally, in this section we recall the definition of Sasakian locally φ-symmetric spaces ([15]).

Definition 2.2. A Sasakian manifold $M = (M; \eta, g, \varphi, \xi)$ is said be locally φ-symmetric if $(*)$ $\varphi^2(\nabla_U R)(V, W)X = 0$ for all vector fields $U, V, W, X \in D$.

We may extend the above Definition 2.2 to contact Riemannian manifolds. Namely, a contact Riemannian manifold is said to be locally φ-symmetric if it satisfies $(*)$ ([4]).

3. Contact 3-manifolds

In this section we prove the Main Theorem. For a 3-dimensional contact Riemannian manifold M, it is known that their associated CR-structures are integrable. Tanno ([16]) showed that M always satisfies

$$(\nabla_X \varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX). \tag{3.1}$$

From (2.7) and (3.1) we immediately get:

Lemma 3.1. *A 3-dimensional contact Riemannian manifold is Sasakian if and only if $h = 0$.*

It is well-known that the curvature tensor R of a 3-dimensional Riemannian manifold is expressed by

$$\begin{aligned} R(Y, X)Z = {} & \rho(X, Z)Y - \rho(Y, Z)X + g(X, Z)SY - g(Y, Z)SX \\ & - \frac{1}{2}r\{g(X, Z)Y - g(Y, Z)X\} \end{aligned} \tag{3.2}$$

for all vector fields X, Y, Z, where $\rho(Y, X) = g(SY, X)$ and r is the scalar curvature of the manifold. If $h = 0$ on M, then from Lemma 3.1 we see that M is Sasakian. Moreover, since S is η-parallel, from (3.2) we see that M is locally φ-symmetric. So, we consider on M the maximal open subset U_1 on which $h \neq 0$ and the maximal open subset U_2 on which h is identically zero. (U_2 is the union of all points p in M such that $h = 0$ in a neighborhood of p). $U_1 \cup U_2$ is open and dense in M. Suppose that M is non-Sasakian. Then U_1 is non-empty and there is a local orthonormal frame field $\{e_1 = e, e_2 = \varphi e, e_3 = \xi\}$ on U_1 such that $h(e_1) = \lambda e_1$, $h(e_2) = -\lambda e_2$ for some positive function λ. In U_1 we have (cf. [9])

Lemma 3.2. *Let $M = (M^3; \eta, g, \varphi, \xi)$ be a 3-dimensional contact Riemannian manifold which satisfies*

$$\nabla_\xi h = 2ah\varphi, \tag{3.3}$$

$a \in \mathbb{R}$. Then we have

$$\begin{aligned}
&\nabla_\xi e_1 = -ae_2, && \nabla_\xi e_2 = ae_1,\\
&\nabla_{e_1}\xi = -(\lambda+1)e_2, && \nabla_{e_2}\xi = -(\lambda-1)e_1,\\
&\nabla_{e_1}e_1 = \frac{1}{2\lambda}\{e_2(\lambda)+A\}e_2, && \nabla_{e_2}e_2 = \frac{1}{2\lambda}\{e_1(\lambda)+B\}e_1,\\
&\nabla_{e_1}e_2 = -\frac{1}{2\lambda}\{e_2(\lambda)+A\}e_1 + (\lambda+1)\xi, &&\\
&\nabla_{e_2}e_1 = -\frac{1}{2\lambda}\{e_1(\lambda)+B\}e_2 + (\lambda-1)\xi. &&
\end{aligned} \tag{3.4}$$

Furthermore, we have the Ricci operator S as follows:

$$\begin{aligned}
Se_1 &= \left(\frac{r}{2} - 1 + \lambda^2 + 2a\lambda\right) e_1 + \xi(\lambda)e_2 + A\xi,\\
Se_2 &= \xi(\lambda)e_1 + \left(\frac{r}{2} - 1 + \lambda^2 - 2a\lambda\right) e_2 + B\xi,\\
S\xi &= Ae_1 + Be_2 + 2(1-\lambda^2)\xi,
\end{aligned} \tag{3.5}$$

where $A = \rho(\xi, e_1)$ and $B = \rho(\xi, e_2)$.

We use the following notational conventions: $\Gamma_{ijk} = g(\nabla_{e_i}e_j, e_k)$, $\rho_{ij} = \rho(e_i, e_j)$, $\nabla_i\rho_{jk} = (\nabla_{e_i}\rho)(e_j, e_k)$ and $\nabla_m R_{ijkl} = g((\nabla_m R)(e_i, e_j)e_k, e_l)$ for $m, i, j, k, l = 1, 2, 3$.

Proof of the Main Theorem. Let $M = (M^3; \eta, g, \varphi, \xi)$ be a 3-dimensional contact Riemannian manifold which satisfies (3.3) and its Ricci tensor S is η-parallel. Then, since $(\nabla_\xi h)e_1 = (\xi\lambda)e_1 + (\lambda - h)\nabla_\xi e_1$, from (3.3) we have

$$\xi\lambda = 0. \tag{3.6}$$

From (3.5) and (3.6) we get

$$\rho_{12} = 0. \tag{3.7}$$

If we apply the second Bianchi identity in (3.2), then we have

$$2\nabla_2\rho_{12} + 2\nabla_3\rho_{13} + \nabla_1\rho_{11} - \nabla_1\rho_{22} - \nabla_1\rho_{33} = 0, \tag{3.8}$$

$$2\nabla_1\rho_{21} + 2\nabla_3\rho_{23} - \nabla_2\rho_{11} + \nabla_2\rho_{22} - \nabla_2\rho_{33} = 0. \tag{3.9}$$

Since M has η-parallel Ricci tensor, we have

$$\nabla_\alpha\rho_{\beta\gamma} = 0 \tag{3.10}$$

for $\alpha, \beta, \gamma = 1, 2$.

Here, we divide our arguments into two cases: (I) $a = 0$, (II) $a \neq 0$.

(I) $a = 0$; From (3.5) we get at once $\rho_{11} = \rho_{22}$. Further, from (3.4), (3.5), (3.6) and (3.10) we see that $e_1(\rho_{11}) = 0$, and hence we obtain

$$\begin{aligned}\nabla_1\rho_{22} &= e_1(\rho_{22}) - 2\sum_{k=1}^{3}\Gamma_{12k}\rho(e_k, e_2)\\ &= -2(\lambda+1)\rho_{23} \quad (\because\ (3.4) \text{ and } (3.7)).\end{aligned}$$

Similarly, we get $e_2(\rho_{22}) = 0$, and hence we obtain

$$\nabla_2\rho_{11} = -2(\lambda-1)\rho_{13}.$$

By (3.10) we have

$$(1+\lambda)\rho_{23} = 0 \text{ and } (1-\lambda)\rho_{13} = 0. \tag{3.11}$$

On the other hand, differentiating (3.7) covariantly in the direction e_1, e_2, respectively, then by using (3.4) we get

$$\begin{aligned}\nabla_1\rho_{12} &= -(1+\lambda)\rho_{13},\\ \nabla_2\rho_{12} &= (1-\lambda)\rho_{23},\end{aligned} \tag{3.12}$$

respectively. Together with (3.10) we have

$$(1+\lambda)\rho_{13} = 0 \text{ and } (1-\lambda)\rho_{23} = 0. \tag{3.13}$$

The equations (3.11) and (3.13) yield

$$\rho_{13} = \rho_{31} = 0,\ \rho_{23} = \rho_{32} = 0. \tag{3.14}$$

From (3.4) and (3.14) we have

$$\nabla_3\rho_{13} = \nabla_3\rho_{31} = 0,\ \nabla_3\rho_{23} = \nabla_3\rho_{32} = 0. \tag{3.15}$$

Thus, from (3.8), (3.9), (3.10) and (3.15) we have

$$\nabla_1\rho_{33} = \nabla_2\rho_{33} = 0.$$

Since $\rho_{33} = 2 - 2\lambda^2$ (the equation (2.5)), together with (3.4), (3.6) and (3.14) we see that λ is constant on M, where we have used the continuity of λ.

Thus, together with (3.4) and (3.14), we have

$$[e_1, e_2] = 2e_3,\ [e_2, e_3] = (1-\lambda)e_1,\ [e_3, e_1] = (1+\lambda)e_2.$$

By virtue of the Milnor's classification for 3-dimensional unimodular Lie group ([12]), we see that M is locally isometric to one of the followings:

(i) $SU(2)$ (or $SO(3)$) with a left invariant metric when $0 < \lambda < 1$;
(ii) $SL(2, \mathbb{R})$ (or $O(1, 2)$) with a left invariant metric when $\lambda > 1$;
(iii) flat when $\lambda = 1$. (This gives the proof of Corollary A.)

(II) $a \neq 0$; Differentiating (3.7) covariantly in the direction e_1, e_2, respectively, and by using (3.4) and (3.10), then we have

$$\begin{aligned} \Gamma_{112}(\rho_{11} - \rho_{22}) - (\lambda + 1)\rho_{13} = 0, \\ \Gamma_{221}(\rho_{11} - \rho_{22}) + (\lambda - 1)\rho_{23} = 0, \end{aligned} \tag{3.16}$$

respectively. From (3.5) and (3.16), then we have

$$\begin{aligned} 4\lambda a \Gamma_{112} &= (\lambda + 1)\rho_{13}, \\ 4\lambda a \Gamma_{221} &= (-\lambda + 1)\rho_{23}. \end{aligned} \tag{3.17}$$

Also, if we differentiate $\rho_{11} - \rho_{22} = 4\lambda a$ covariantly in the direction e_1, e_2, respectively, then together with (3.10), we have

$$2(e_1\lambda)a = -(\lambda + 1)\rho_{23}, \quad 2(e_2\lambda)a = (\lambda - 1)\rho_{13}, \tag{3.18}$$

respectively. If we put $X = e_2$, $Y = e_1$, $Z = e_3$ in (3.2), then we get

$$R(e_1, e_2)e_3 = \rho_{23}e_1 - \rho_{13}e_2. \tag{3.19}$$

On the other hand, together with (3.4) we calculate

$$\begin{aligned} R(e_1, e_2)e_3 &= \nabla_{e_1}(\nabla_{e_2}e_3) - \nabla_{e_2}(\nabla_{e_1}e_3) - \nabla_{[e_1,e_2]}e_3 \\ &= \{-(e_1\lambda) + 2\lambda\Gamma_{221}\}e_1 + \{(e_2\lambda) - 2\lambda\Gamma_{112}\}e_2 \end{aligned} \tag{3.20}$$

and comparing this with (3.19) we have

$$\begin{aligned} -(e_2\lambda) + 2\lambda\Gamma_{112} &= \rho_{13}, \\ -(e_1\lambda) + 2\lambda\Gamma_{221} &= \rho_{23}. \end{aligned} \tag{3.21}$$

From (3.18) and (3.21) we have

$$4\lambda a\Gamma_{112} = (\lambda + 2a - 1)\rho_{13}, \quad 4\lambda a\Gamma_{221} = -(\lambda - 2a + 1)\rho_{23}. \tag{3.22}$$

By using (3.17) and (3.22) we can see that $\rho_{13} = \rho_{23} = 0$ if $a \neq 1$. At first we consider the case $a \neq 1$. Then from (3.18) we see that λ is constant on M. Therefore, together with (3.2), (3.4), (3.6) and (3.18), we have

$$[e_1, e_2] = 2e_3, \ [e_2, e_3] = (1 - \lambda - a)e_1, \ [e_3, e_1] = (1 + \lambda - a)e_2.$$

With the help of Milnor's result ([12]), we see that M is locally isometric to a unimodular Lie group with a left invariant (non-Sasakian) contact Riemannian metric.

Next, we consider the case $a = 1$. Then further from (3.2) we have

$$R(e_2, e_3)e_1 = \rho_{13}e_2, \quad R(e_3, e_1)e_2 = -\rho_{23}e_1. \tag{3.23}$$

Also, together with (3.4) we calculate

$$(3.24)\qquad \begin{aligned} R(e_2,e_3)e_1 &= \nabla_{e_2}(\nabla_{e_3}e_1) - \nabla_{e_3}(\nabla_{e_2}e_1) - \nabla_{[e_2,e_3]}e_1 \\ &= (e_3\Gamma_{221} + \lambda\Gamma_{112})e_2, \\ R(e_3,e_1)e_2 &= \nabla_{e_3}(\nabla_{e_1}e_2) - \nabla_{e_1}(\nabla_{e_3}e_2) - \nabla_{[e_3,e_1]}e_2 \\ &= (-e_3\Gamma_{112} - \lambda\Gamma_{221})e_2. \end{aligned}$$

If we consider (3.24) with (3.23) we have

$$(3.25)\qquad \begin{aligned} e_3\Gamma_{221} + \lambda\Gamma_{112} &= \rho_{13}, \\ e_3\Gamma_{112} + \lambda\Gamma_{221} &= \rho_{23}. \end{aligned}$$

Since the Ricci tensor is η-parallel, from (3.8) and (3.9) we get

$$(3.26)\qquad 2\nabla_3\rho_{13} = \nabla_1\rho_{33}, \quad 2\nabla_3\rho_{23} = \nabla_2\rho_{33}.$$

But, from (2.5) using (3.4) we have

$$\begin{aligned} \nabla_1\rho_{33} &= -4\lambda(e_1\lambda) + 2(\lambda+1)\rho_{23}, \\ \nabla_2\rho_{33} &= -4\lambda(e_2\lambda) + 2(\lambda-1)\rho_{13}, \end{aligned}$$

hence from (3.26) we have also

$$(3.27)\qquad \begin{aligned} 2\nabla_3\rho_{13} &= -4\lambda(e_1\lambda) + 2(\lambda+1)\rho_{23}, \\ 2\nabla_3\rho_{23} &= -4\lambda(e_2\lambda) + 2(\lambda-1)\rho_{13}. \end{aligned}$$

Now, if we differentiate two equations in (3.22)($a = 1$) covariantly in the characteristic direction $\xi = e_3$, respectively, then together with (3.4) and (3.6) we get

$$\begin{aligned} 4\lambda(e_3\Gamma_{221}) &= (1-\lambda)(\nabla_3\rho_{13} + \rho_{13}), \\ 4\lambda(e_3\Gamma_{112}) &= (\lambda+1)(\nabla_3\rho_{13} - \rho_{23}), \end{aligned}$$

respectively, and so, with (3.22), (3.25) and (3.27), we have

$$(3.28)\qquad \rho_{13} = (\lambda-1)(e_2\lambda), \quad \rho_{23} = -(\lambda+1)(e_1\lambda).$$

Thus from (3.18) and (3.28) we have

$$(\lambda^2 - 2\lambda - 1)(e_2\lambda) = 0, \quad (\lambda^2 + 2\lambda - 1)(e_1\lambda) = 0.$$

From these and (3.6) we find that λ is constant on M, and again in (3.28) we see that $\rho_{13} = \rho_{23} = 0$. Therefore, together with (3.4), we have

$$(3.29)\qquad [e_1,e_2] = 2e_3,\ [e_2,e_3] = -\lambda e_1,\ [e_3,e_1] = \lambda e_2.$$

Due to the same reason as the case (I) $a = 1$, we see that M is locally isometric to $SL(2,\mathbb{R})$ (or $O(1,2)$) with a left invariant (non-Sasakian) contact metric structure. (Hence, we have Corollary B.)

Conversely, we see that a Sasakian locally φ-symmetric space satisfies (3.3) ($h = 0$) and at the same time S is η-parallel. In fact, we have

$$(\nabla_U\rho)(V,W) = g((\nabla_U R)(e_1,V)W,e_1) + g((\nabla_U R)(\varphi e_1,V)W,\varphi e_1) + g((\nabla_U R)(\xi,V)W,\xi)$$

and from (2.4) and (2.8) we obtain $g((\nabla_U R)(\xi, V)W, \xi) = 0$ for $U, V, W \in D$. (We note that a Sasakian locally φ-symmetric space implies that the Ricci tensor is η-parallel in general.) Now, we consider a 3-dimensional unimodular Lie group with a left-invariant metric structure. Its Lie algebra structure is given by (cf. [12])

$$[e_1, e_2] = c_1 e_3, \ [e_2, e_3] = c_2 e_1, \ [e_3, e_1] = c_3 e_2 \tag{3.30}$$

for some constants $c_1(\neq 0), c_2, c_3$. Let $\{\omega_i\}$ be the dual 1-forms to the vector fields $\{e_i\}$. By using (3.30) we get $d\omega_3(e_1, e_2) = -d\omega_3(e_2, e_1) = -\frac{c_1}{2}$ and $d\omega_3(e_i, e_j) = 0$ otherwise. Further we easily find that $(\omega_3 \wedge d\omega_3)(e_1, e_2, e_3) = -\frac{c_1}{6}(\neq 0)$, and hence ω_3 is a contact form and e_3 is the characteristic vector field. Define a Riemannian metric g and a $(1, 1)$-tensor field φ by

$$g(e_i, e_j) = \delta_{ij}, \ d\omega_3(e_i, e_j) = g(e_i, \varphi e_j)$$

for $i, j = 1, 2, 3$. In order $(\omega_3, g, \varphi, e_3)$ to be a contact Riemannian structure, it must satisfy that $g(\varphi e_i, \varphi e_j) = g(e_i, e_j) - \omega(e_i)\omega(e_j)$ for $i, j = 1, 2, 3$, so that $c_1 = 2$.

We recall the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) =& Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &+ g(Y, [Z, X]) + g(Z, [X, Y]) - g(X, [Y, Z]) \end{aligned}$$

for X, Y, Z are smooth vector fields on the manifold. Then the Koszul formula and (3.30) give

$$\begin{aligned} \Gamma_{123} &= \frac{1}{2}(c_3 - c_2 + 2), \\ \Gamma_{213} &= \frac{1}{2}(c_3 - c_2 - 2), \\ \Gamma_{312} &= \frac{1}{2}(c_3 + c_2 - 2), \\ &\text{all others are zero.} \end{aligned} \tag{3.31}$$

Then a straightforward computation together with (3.31) yields

$$\begin{aligned} R(e_1, e_2)e_2 &= \left(\frac{1}{4}(c_3 - c_2)^2 - 3 + c_3 + c_2\right) e_1, \\ R(e_1, e_3)e_3 &= \left(-\frac{1}{4}(c_3 - c_2)^2 - \frac{1}{2}({c_3}^2 - {c_2}^2) + 1 - c_2 + c_3\right) e_1, \\ R(e_2, e_1)e_1 &= \left(\frac{1}{4}(c_3 - c_2)^2 - 3 + c_3 + c_2\right) e_2, \\ R(e_2, e_3)e_3 &= \left(\frac{1}{4}(c_3 + c_2)^2 - {c_2}^2 + 1 + c_2 - c_3\right) e_2, \\ R(e_3, e_1)e_1 &= \left(-\frac{1}{4}(c_3 - c_2)^2 - \frac{1}{2}({c_3}^2 - {c_2}^2) + 1 - c_2 + c_3\right) e_3, \end{aligned} \tag{3.32}$$

$$R(e_3, e_2)e_2 = \left(\frac{1}{4}(c_3 + c_2)^2 - {c_2}^3 + 1 + c_2 - c_3\right) e_3.$$

From the definition of the Ricci tensor and (3.32) we get

$$\begin{aligned} Se_1 &= \left(-\frac{1}{2}(c_3^2 - c_2^2) - 2 + 2c_3\right) e_1, \\ Se_2 &= \left(\frac{1}{2}(c_3^2 - c_2^2) - 2 + 2c_2\right) e_2, \\ Se_3 &= \left(-\frac{1}{2}(c_3 - c_2)^2 + 2\right) e_3. \end{aligned} \tag{3.33}$$

From (2.4) and (3.31) we obtain

$$he_1 = \frac{c_3 - c_2}{2} e_1, \ he_2 = -\frac{c_3 - c_2}{2} e_2. \tag{3.34}$$

From (3.31) and (3.34), we can see that all the unimodular Lie groups with the contact left invariant Riemannian metrics satisfy (3.3) with $a = \frac{1}{2}(2 - c_2 - c_3)$ or $c_2 = c_3, (h = 0)$. And we can check that S is η-parallel for all the unimodular Lie groups. Therefore summing up all the arguments so far then we complete the proof of Main Theorem. □

We close this section, by showing two examples of contact Riemannian 3-manifolds which have the property $\nabla_\xi h = 2ah\varphi$, but the Ricci tensors are not η-parallel. One is homogeneous space and another is a non-homogeneous case.

(1) Non-unimodular Lie groups with left invariant (non-Sasakian) contact metric structures also have the property $\nabla_\xi h = 2ah\varphi$, but the Ricci tensors are not η-parallel. Indeed, let M be a 3-dimensional non-unimodular Lie group with left invariant contact metric structure. Then we know that (cf. [8]) there exists an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\} \in \mathfrak{m}$ such that

$$[e_1, e_2] = \alpha e_2 + 2e_3, \ [e_2, e_3] = 0, \ [e_3, e_1] = \gamma e_2, \tag{3.35}$$

where $\alpha \neq 0$. Moreover, M is Sasakian if and only if $\gamma = 0$. From (3.35), by using the Koszul formula we have

$$\begin{aligned} &\Gamma_{123} = \frac{\gamma + 2}{2}, \\ &\Gamma_{212} = -\alpha, \\ &\Gamma_{213} = \frac{\gamma - 2}{2}, \\ &\Gamma_{312} = \frac{\gamma - 2}{2}, \\ &\text{all others are zero.} \end{aligned} \tag{3.36}$$

Then, by the definition of the curvature tensor, we have

$$R(e_1, e_2)e_2 = \left(\frac{\gamma^2 + 4\gamma - 12}{4} - \alpha^2\right) e_1,$$

$$
\begin{aligned}
R(e_1,e_3)e_3 &= \left(\frac{-3\gamma^2+8\gamma+4}{4}\right)e_1,\\
R(e_2,e_1)e_1 &= \left(\frac{\gamma^2+4\gamma-12}{4}-\alpha^2\right)e_2+\alpha\gamma e_3,\\
R(e_2,e_3)e_1 &= \frac{(\gamma-2)^2}{4}\,e_2,\\
R(e_3,e_1)e_1 &= \alpha\gamma e_2+\left(\frac{-3\gamma^2+4\gamma+4}{4}\right)e_3,\\
R(e_3,e_2)e_2 &= \frac{(\gamma-2)^2}{4}\,e_3.
\end{aligned}\tag{3.37}
$$

From these, we have the Ricci tensor

$$
\begin{aligned}
Se_1 &= \left(-\alpha^2-2+2\gamma-\frac{\gamma^2}{2}\right)e_1,\\
Se_2 &= \left(-\alpha^2-2+\frac{\gamma^2}{2}\right)e_2+\alpha\gamma e_3,\\
Se_3 &= \alpha\gamma e_2+\left(2-\frac{\gamma^2}{2}\right)e_3.
\end{aligned}\tag{3.38}
$$

On the other hand, from (2.4) and (3.36) we have

$$he_1=\gamma/2e_1,\quad he_2=-\gamma/2e_2.\tag{3.39}$$

From (3.36) and (3.39), we see that $\nabla_\xi h=2ah\varphi$, where $2a=2-\gamma$. But, from (3.36) and (3.38), we obtain

$$
\begin{aligned}
&(\nabla_{e_1}\rho)(e_2,e_2)=-\alpha\gamma\cdot(\gamma+2)/2,\quad (\nabla_{e_2}\rho)(e_1,e_2)=\alpha\gamma(\gamma-2),\\
&(\nabla_{e_1}\rho)(e_1,e_1)=(\nabla_{e_1}\rho)(e_2,e_2)=(\nabla_{e_2}\rho)(e_1,e_1)=(\nabla_{e_2}\rho)(e_2,e_2)=0.
\end{aligned}\tag{3.40}
$$

So, we see that for a non-Sasakian M ($\gamma\neq 0$) its Ricci tensor is not η-parallel.

(2) (This example appeared in [14].) Let M be the open submanifold $\{(x,y,z)\in\mathbb{R}^3 \mid x\neq 0\}$ of Cartesian 3-space together with a contact form $\eta=xydx+dz$. The characteristic vector field of this contact 3-manifold is $\xi=\partial/\partial z$. Take a global frame field

$$e_1=-\frac{2}{x}\frac{\partial}{\partial y},\ e_2=\frac{\partial}{\partial x}-\frac{4z}{x}\frac{\partial}{\partial y}-xy\frac{\partial}{\partial z},\ e_3=\xi$$

and define a Riemannian metric g with respect to $\{e_1,e_2,e_3\}$ to be an orthonormal frame. Moreover, define an endomorphism field φ by $\varphi e_1=e_2$, $\varphi e_2=-e_1$ and $\varphi\xi=0$. Then (φ,ξ,g) is an associated almost contact metric structure for η. The endomorphism field h satisfies $he_1=e_1$, $he_2=-e_2$. Hence, M is not Sasakian. Perrone showed that this contact Riemannian 3-manifold is non-homogeneous. We calculate that

$$(\nabla_\xi h)e_i=-4e_i,\quad i=1,2.$$

And from the straightforward computations we have

$$g((\nabla_{e_1} S)e_1, e_2) = \frac{8}{x} \neq 0.$$

So, this is a non-homogeneous example which satisfies $\nabla_\xi h = -4h\varphi$ ($a = -2$), but its Ricci tensor is not η-parallel.

Remark 2. A 3-dimensional contact Riemannian manifold which has η-parallel Ricci tensor is locally φ-symmetric. But, the converse does not hold in general. Actually the above examples (1) with $\gamma = 2$ and (2) are locally φ-symmetric spaces (cf. [8], [14]). In this context, we do not know so far an example of a locally φ-symmetric contact Riemannian manifold which does not satisfy $\nabla_\xi h = 2ah\varphi$. So, we may raise the following question: *Does a locally φ-symmetric contact Riemannian manifold always satisfy the property $\nabla_\xi h = 2ah\varphi$?* A locally symmetric contact Riemannian manifold satisfies $\nabla_\xi h = 0$ (cf. [3], [7]).

Acknowledgement. The authors thank to the referee for careful reading of the manuscript and giving the valuable comments and suggestions for the revised version.

References

[1] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, 203. Birkhauser Boston, Inc., Boston, 2002.

[2] ———, *Critical associated metrics on contact manifolds*, J. Austral. Math. Soc. Ser. A **37** (1984), no. 1, 82–88.

[3] ———, *When is the tangent sphere bundle locally symmetric?*, Geometry and topology, 15–30, World Sci. Publishing, Singapore, 1989.

[4] D. E. Blair, T. Koufogiorgos, and R. Sharma, *A classification of* 3*-dimensional contact metric manifolds with* $Q\phi = \phi Q$, Kodai Math. J. **13** (1990), no. 3, 391–401.

[5] D. E. Blair and R. Sharma, *Three-dimensional locally symmetric contact metric manifolds*, Boll. Un. Mat. Ital. A (7) **4** (1990), no. 3, 385–390.

[6] D. E. Blair and L. Vanhecke, *Symmetries and* ϕ*-symmetric spaces*, Tohoku Math. J. (2) **39** (1987), no. 3, 373–383.

[7] E. Boeckx and J. T. Cho, *Locally symmetric contact metric manifolds*, Monatsh. Math. **148** (2006), no. 4, 269–281.

[8] E. Boeckx, P. Bueken, and L. Vanhecke, ϕ*-symmetric contact metric spaces*, Glasg. Math. J. **41** (1999), no. 3, 409–416.

[9] G. Calvaruso, D. Perrone, and L. Vanhecke, *Homogeneity on three-dimensional contact metric manifolds*, Israel J. Math. **114** (1999), 301–321.

[10] S. S. Chern and R. S. Hamilton, *On Riemannian metrics adapted to three-dimensional contact manifolds*, With an appendix by Alan Weinstein. Lecture Notes in Math., 1111, Workshop Bonn 1984 (Bonn, 1984), 279–308, Springer, Berlin, 1985.

[11] F. Gouli-Andreou and P. J. Xenos, *On* 3*-dimensional contact metric manifolds with* $\nabla_\xi \tau = 0$, J. Geom. **62** (1998), no. 1-2, 154–165.

[12] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. **21** (1976), no. 3, 293–329.

[13] D. Perrone, *Torsion and critical metrics on contact three-manifolds*, Kodai Math. J. **13** (1990), no. 1, 88–100.

[14] ______, *Weakly ϕ-symmetric contact metric spaces*, Balkan J. Geom. Appl. **7** (2002), no. 2, 67–77.

[15] T. Takahashi, *Sasakian ϕ-symmetric spaces*, Tohoku Math. J. (2) **29** (1977), no. 1, 91–113.

[16] S. Tanno, *Variational problems on contact Riemannian manifolds*, Trans. Amer. Math. Soc. **314** (1989), no. 1, 349–379.

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