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HELICOIDAL SURFACES AND THEIR GAUSS MAP IN MINKOWSKI 3-SPACE II

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Abstract. We classify and characterize the rational helicoidal surfaces in a three-dimensional Minkowski space satisfying pointwise 1-type like problem on the Gauss map.

1. Introduction

Nash's imbedding theorem enables us to view every Riemannian manifold as a submanifold of a Euclidean space. In that sense, one way to study a Riemannian manifold is to apply the theory of submanifolds in a Euclidean space. Since B.-Y. Chen ([3]) introduced the notion of finite type immersion of submanifolds in a Euclidean space late 1970's, many works have been carried out in this area. Further, the notion of finite type can be extended to any smooth functions on a submanifold of a Euclidean space or a pseudo-Euclidean space. In dealing with submanifolds of a Euclidean or a pseudo-Euclidean space, the Gauss map is a useful tool to examine the character of submanifolds in a Euclidean space. For the last few years, two of the present authors and D. W. Yoon introduced and studied the notion of pointwise 1-type Gauss map in a Euclidean or a pseudo-Euclidean space ([4], [5], [7], [8]), namely the Gauss map G on a submanifold M of a Euclidean space or a pseudo-Euclidean space is said to be of *pointwise 1-type* if

$$\Delta G = F(G + C) \tag{1.1}$$

for a non-zero smooth function F on M and a constant vector C, where Δ denotes the Laplace operator defined on M.

On the other hand, a helicoidal surface is well known as a kind of generalization of some ruled surfaces and surfaces of revolution in a Euclidean space or a Minkowski space ([1], [2], [6]). Recently, two of the authors, H. Liu and D. W. Yoon have classified the helicoidal surfaces with pointwise 1-type Gauss

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map in a Minkowski 3-space $\mathbb{L}^3$ ([5]). Then, we may have a natural question as follows:

What helicoidal surfaces have the *harmonic Gauss map*, that is, $\Delta G = 0$? Or, what helicoidal surfaces satisfy equation (1.1) whether the function F is non-zero or zero?

In this paper, we mainly focus on the study of the helicoidal surfaces with harmonic Gauss map in a Minkowski 3-space and find the all solution spaces of the so-called rational helicoidal surfaces satisfying (1.1). As a consequence, we have the following characterizations:

Theorem A. *Let M be a helicoidal surface with space-like or time-like axis in a Minkowski 3-space $\mathbb{L}^3$. Then, a plane is the only rational helicoidal surface with harmonic Gauss map.*

Theorem B. *There exists no rational helicoidal surface with harmonic Gauss map which has null axis in Minkowski 3-space $\mathbb{L}^3$.*

Theorem C. *Let M be a rational helicoidal surface with time-like axis in a Minkowski 3-space $\mathbb{L}^3$. Then, the Gauss map G of M satisfies the condition $\Delta G = F(G + C)$ for some smooth function F and constant vector C if and only if M is an open part of a plane, a circular cylinder, a right cone, a right helicoid of type II or a helicoidal surface of elliptic type in $\mathbb{L}^3$.*

2. Preliminaries

Let $\mathbb{L}^3$ be a Minkowski 3-space with the Lorentz metric

$$\langle\cdot,\cdot\rangle = -{dx_0}^2 + {dx_1}^2 + {dx_2}^2,$$

where (x_0, x_1, x_2) is a system of the canonical coordinates in $\mathbb{R}^3$. Let M be a connected 2-dimensional surface in $\mathbb{L}^3$ and $x : M \to \mathbb{L}^3$ a smooth non-degenerate isometric immersion. A surface M is said to be *space-like* (resp. *time-like*) if the induced metric on M is positive definite (resp. indefinite). Assuming that M is orientable, we can always choose a unit normal vector field G globally defined on M. In such a case, the unit normal vector field G can be regarded as a map $G : M \to \mathbb{H}^2_+$ if M is space-like and as a map $G : M \to \mathbb{S}^2_1$ if M is time-like, where $\mathbb{H}^2_+ = \{x \in \mathbb{L}^3 \mid \langle x, x\rangle = -1, x_2 > 0\}$ is the *hyperbolic space* and $\mathbb{S}^2_1 = \{x \in \mathbb{L}^3 \mid \langle x, x\rangle = 1\}$ is the *de Sitter space*. The map G is also called the *Gauss map* of the surface M. For the matrix $\tilde{g} = (\tilde{g}_{ij})$ consisting of the components of the induced metric on M, we denote by $\tilde{g}^{-1} = (\tilde{g}^{ij})$ (resp. $\mathcal{G}$) the inverse matrix (resp. the determinant) of the matrix $(\tilde{g}_{ij})$. The Laplacian Δ on M is, in turn, given by

$$\Delta = -\frac{1}{\sqrt{|\,\mathcal{G}\,|}} \sum_{i,j} \frac{\partial}{\partial x^i}\Big(\sqrt{|\,\mathcal{G}\,|}\; \tilde{g}^{ij} \frac{\partial}{\partial x^j}\Big).$$

Let e be a non-zero vector in $\mathbb{L}^3$ and $\mathbf{S}(e)$ the set of screw motions fixing e in $\mathbb{L}^3$. In particular, if e is non-null, the screw motions fixing e belong to

$\mathbf{O}(e)$, the set of orthogonal transformations with positive determinant. Then, a *helicoidal motion* around the axis in the e-direction is defined by

$$g_t(x) = A(t)x^T + (ht)e,\ x = (x_0,\ x_1,\ x_2) \in \mathbb{L}^3,\ t \in \mathbb{R},\ A \in \mathbf{S}(e),$$

where h is a constant and x^T is the transpose of the vector x.

Let $\gamma : I = (a, b) \subset \mathbb{R} \to \Pi$ be a plane curve in $\mathbb{L}^3$ and l a straight line in Π which does not intersect the curve γ. A *helicoidal surface* M with the axis l and pitch h in $\mathbb{L}^3$ is a non-degenerate surface which is invariant under the action of the helicoidal motion g_t. Depending on the axis being space-like, time-like or null, there are three types of screw motions. If the axis l is space-like (resp. time-like), then l is transformed to the x_1-axis or x_2-axis (resp. x_0-axis) by the Lorentz transformation. Therefore, we may consider x_2-axis (resp. x_0-axis) as the axis if l is space-like (resp. time-like). If the axis l is null, then we may assume that the axis is the line spanned by the vector $(1, 1, 0)$.

We now consider the helicoidal surfaces in $\mathbb{L}^3$ with space-like, time-like or null axis respectively.

Case 1. The axis l is space-like.

Without loss of generality we may assume that the profile curve γ lies in the x_1x_2-plane or x_0x_2-plane. Hence, the curve γ can be represented by

$$\gamma(u) = (0,\ f(u),\ g(u)) \text{ or } \gamma(u) = (f(u),\ 0,\ g(u))$$

for smooth functions f and g on an open interval $I = (a, b)$. Therefore, the surface M may be parameterized by

$$x(u, v) = (f(u)\sinh v,\ f(u)\cosh v,\ g(u) + hv),\ f(u) > 0,\ h \in \mathbb{R} \tag{2.1}$$

or

$$x(u, v) = (f(u)\cosh v,\ f(u)\sinh v,\ g(u) + hv),\ f(u) > 0,\ h \in \mathbb{R}. \tag{2.2}$$

Case 2. The axis l is time-like.

In this case, we may assume that the profile curve γ lies in the x_0x_1-plane. So the curve γ is given by $\gamma(u) = (g(u), f(u), 0)$ for a positive function $f = f(u)$ on an open interval $I = (a, b)$. Hence, the surface M can be expressed by

$$x(u, v) = (g(u) + hv,\ f(u)\cos v,\ f(u)\sin v),\ f(u) > 0,\ h \in \mathbb{R}. \tag{2.3}$$

Case 3. The axis l is null.

In this case, we may assume that the profile curve γ lies in the x_0x_1-plane of the form $\gamma(u) = (f(u),\ g(u),\ 0)$, where $f = f(u)$ is a positive function and $g = g(u)$ is a function satisfying $p(u) = f(u) - g(u) \neq 0$ for all $u \in I$. Under the cubic screw motion, its parametrization has the form

$$x(u, v) = \left(f(u) + \frac{v^2}{2}p(u) + hv,\ g(u) + \frac{v^2}{2}p(u) + hv,\ p(u)v\right),\ h \in \mathbb{R}. \tag{2.4}$$

3. Helicoidal surfaces with time-like axis in Minkowski 3-space

In this section, we study the helicoidal surfaces with harmonic Gauss map which has time-like axis in Minkowski 3-space $\mathbb{L}^3$.

Suppose that M is a helicoidal surface in $\mathbb{L}^3$ with time-like axis parameterized by (2.3) for some smooth functions f and g.

First, if f is constant, the parametrization of M can be written as

$$x(u,v) = (g(u) + hv,\ a\cos v,\ a\sin v),\ h \in \mathbb{R}$$

for a non-zero constant a. By a straightforward computation, we see that the Laplacian ΔG of the Gauss map G satisfies $\Delta G = \frac{1}{a^2}G$. Hence, M does not have the harmonic Gauss map. In fact, it has non-proper pointwise 1-type Gauss map of the first kind ([5]). Therefore, we may assume that f is not constant. Then, we may put $f(u) = u$ and thus M is parameterized by

$$x(u,v) = (g(u) + hv,\ u\cos v,\ u\sin v),\ u > 0,\ h \in \mathbb{R}. \tag{3.1}$$

If M is space-like, that is, $u^2 - u^2{g'}^2 - h^2 > 0$, then the Gauss map G and its Laplacian ΔG are obtained as follows:

$$G = \frac{1}{\sqrt{u^2-u^2{g'}^2-h^2}}(-u,\ -ug'\cos v + h\sin v,\ -ug'\sin v - h\cos v)$$

and

$$\Delta G = -\frac{1}{(u^2-u^2{g'}^2-h^2)^{\frac{7}{2}}}(D(u),\ A(u)\sin v + B(u)\cos v,\ -A(u)\cos v + B(u)\sin v),$$

where we have put

$$\begin{aligned} A(u) = h\{&2h^4 - 4h^4{g'}^2 + (-7h^4g'g'')u + (-2h^2 + 2h^2{g'}^2 - h^4{g''}^2 - h^4g'g''')u^2 \\ &+ (8h^2g'g'' + h^2{g'}^3g'')u^3 + (3h^2{g'}^2{g''}^2 - h^2{g'}^3g''' + 2h^2{g''}^2 + 2h^2g'g''')u^4 \\ &+ (-g'g'' + {g'}^3g'')u^5 + (-{g''}^2 - 3{g'}^2{g''}^2 - g'g''' + {g'}^3g''')u^6\}, \end{aligned}$$

$$\begin{aligned} B(u) = &-3h^6g'' + (-6h^4g' + 8h^4{g'}^3 - h^6g''')u + (7h^4g'' + 7h^4{g'}^2g'')u^2 \\ &+ (7h^2g' - 12h^2{g'}^3 + 5h^2{g'}^5 + 4h^4g'{g''}^2 + 3h^4g''' - h^4{g'}^2g''')u^3 \\ &+ (-5h^2g'' - 6h^2{g'}^2g'' + 2h^2{g'}^4g'')u^4 + \{-g'(1-{g'}^2)^3 - 8h^2g'{g''}^2 \\ &- 3h^2g''' + 2h^2{g'}^2g'''\}u^5 + (g'' - {g'}^2g'')u^6 + (-{g'}^2g''' + 4g'{g''}^2 + g''')u^7 \end{aligned}$$

and

$$\begin{aligned} D(u) = u\{&-2h^4 + 4h^4{g'}^2 + (7h^4g'g'')u + (2h^2 - 2h^2{g'}^2 + h^4{g''}^2 + h^4g'g''')u^2 \\ &+ (-8h^2g'g'' - h^2{g'}^3g'')u^3 + (-3h^2{g'}^2{g''}^2 + h^2{g'}^3g''' - 2h^2{g''}^2 \\ &- 2h^2g'g''')u^4 + (g'g'' - {g'}^3g'')u^5 + ({g''}^2 + 3{g'}^2{g''}^2 + g'g''' - {g'}^3g''')u^6\}. \end{aligned}$$

Suppose that M has harmonic Gauss map, that is, its Gauss map G satisfies $\Delta G = 0$. Then, we obtain that the functions $A(u)$, $B(u)$ and $D(u)$ are all vanishing.

First, we consider the case that M is a helicoidal surface of polynomial kind with harmonic Gauss map, that is, g is a polynomial in u. Then we may put

$$g(u) = a_n u^n + a_{n-1}u^{n-1} + \cdots + a_1 u + a_0,$$

where n is nonnegative integer and a_n is non-zero constant.

Considering the constant terms of $B(u)$, it is easy to see that $h = 0$, that is, M is a surface of revolution. Therefore, $A(u) = 0$. Also, $B(u)$ and $D(u)$ are reduced to respectively:

$$\begin{aligned} B(u) &= -g'(1-g'^2)^3u^5 + (g'' - g'^2g'')u^6 + (-g'^2g''' + 4g'g''^2 + g''')u^7, \\ D(u) &= (g'g'' - g'^3g'')u^5 + (g''^2 + 3g'^2g''^2 + g'g''' - g'^3g''')u^6. \end{aligned}$$

Assume that $\deg g(u) \geq 2$, where $\deg g(u)$ means the degree of the polynomial $g(u)$. Then, the term $-g'(1-g'^2)^3u^5$ in $B(u)$ includes the highest degree in u and its leading coefficient must be zero, that is, $n^7a_n^7 = 0$. Thus, $a_n = 0$, a contradiction.

Assuming $\deg g(u) = 1$, $B(u) = -a_1(1-a_1^2)^3u^5$. Hence, $a_1^2 = 1$, which is a contradiction since M is non-degenerate.

If g is constant, then $B(u) = 0$ and $D(u) = 0$. Hence, the Gauss map is harmonic. In this case, the parametrization of M in (3.1) is reduced to

$$x(u, v) = (a,\ u\cos v,\ u\sin v),\ u > 0$$

for some constant a. This means that M is part of a plane.

Conversely, it is obvious that the Gauss map of a plane is harmonic. By a similar process as above, the same conclusion can be made in case of time-like surface. Consequently, we have:

Theorem 3.1. *Let M be a helicoidal surface of polynomial kind with time-like axis in a Minkowski 3-space $\mathbb{L}^3$. Then, M has the harmonic Gauss map if and only if M is part of a plane.*

Next, consider M is of rational kind, that is, $g(u)$ is a rational function. Suppose that M is a genuine helicoidal surface of rational kind with harmonic Gauss map, i.e., $h \neq 0$. Then we may put

$$g(u) = p(u) + \frac{r(u)}{q(u)}\ , \tag{3.2}$$

where $p(u)$ is a polynomial in u and the polynomials $r(u)$ and $q(u)$ are relatively prime with $\deg r(u) < \deg q(u)$ and $\deg q(u) \geq 1$. Let $\deg p(u) = l,\ \deg r(u) = n$ and $\deg q(u) = m$ with $n < m$ and $m \geq 1$ where $l,\ m$ and n are some nonnegative integers. Then, we may put

$$\begin{aligned} p(u) &= a_l u^l + a_{l-1}u^{l-1} + \cdots + a_1 u + a_0, \\ q(u) &= b_m u^m + b_{m-1}u^{m-1} + \cdots + b_1 u + b_0, \\ r(u) &= c_n u^n + c_{n-1}u^{n-1} + \cdots + c_1 u + c_0. \end{aligned} \tag{3.3}$$

Putting (3.2) in the equation $B(u)$ and multiplying $q^{14}(u)$ with thus obtained equation, we get a polynomial $q^{14}(u)B(u)$ in u.

Assume that $\deg p(u) \geq 2$. By an algebraic computation, we see that the degree of the polynomial is $7l + 14m - 2$ and so its coefficient $l^7 a_l^7 b_m^{14}$ must be zero. But, this is a contradiction.

Assuming $\deg p(u) = 1$, the leading coefficient of the polynomial is $-a_1(1 - a_1^2)^3 b_m^{14}$. It must be zero and so $a_1^2 = 1$. In this case, we can consider two cases according to the value of $m - n$.

If $m - n > 1$, then the polynomial includes the term of the degree $14m + 1$ with the coefficient $2h^4 a_1 b_m^{14}$. Hence it must be zero, a contradiction.

Suppose $m - n = 1$. Since the Gauss map of M is harmonic, the polynomials $q^{10}(u)A(u)$ and $q^{14}(u)B(u)$ are vanishing. With the help of (3.2) and (3.3), we have $b_0 = 0$. So we may put

$$q(u) = b_m u^m + \cdots + b_2 u^2 + b_1 u, \ b_m \neq 0.$$

Then, an algebraic computation shows that the polynomial $q^{10}(u)A(u)$ has the lowest degree 4 with the coefficient $4h^2 b_1^6 c_0^4$. Similarly, the polynomial $q^{14}(u)B(u)$ has the lowest degree 5 with the coefficient $-b_1^7 c_0^7$. Therefore, $b_1 c_0 = 0$.

If we assume $c_0 \neq 0$, then $b_1 = 0$ and we have

$$q(u) = b_m u^m + \cdots + b_2 u^2, \ b_m \neq 0.$$

By considering the coefficients of the terms with the lowest degree in $q^{10}(u)A(u)$ and $q^{14}(u)B(u)$, we get $b_2 c_0 = 0$. Hence, $b_2 = 0$. Inductively, $b_3, \ldots, b_{m-1}$ are zero. So we put

$$q(u) = b_m u^m, \ b_m \neq 0.$$

Then, the polynomial $q^{14}(u)B(u)$ has the lowest degree $7m - 2$ with the coefficient $(-m b_m c_0)^7$. It must be zero, a contradiction. Thus, we conclude that $c_0 = 0$. Hence, $g(u)$ can be written as

$$g(u) = \pm u + a_0 + \frac{r(u)}{q(u)} \ ,$$

where $r(u) = c_n u^{n-1} + \cdots + c_1$ and $q(u) = b_m u^{m-1} + \cdots + b_1$ with $c_n \neq 0$ and $b_m \neq 0$. By a similar process as above, we obtain $b_1, \ldots, b_{m-1} = 0$ and $c_1, \ldots, c_{n-1} = 0$. Consequently, we get

$$g(u) = \pm u + a_0 + \frac{c}{u} \ , \ c \neq 0.$$

Hence, $q^{14}(u)B(u)$ has the coefficient $-c^7$ of the lowest degree which is 5 and it must be zero. Thus, $c = 0$, that is, g is a polynomial in u.

Finally, if p is constant, then the degree of $q^{14}(u)B(u)$ is $13m + n + 4$ and its leading coefficient is $-(m - n)^2(m - n + 2) b_m^{13} c_n$. This must be zero, a contradiction.

By a similar argument as above, we lead to a contradiction in case of surfaces of revolution. In case of time-like surface, we have the same result. Consequently, we have:

Theorem 3.2. *Let M be a helicoidal surface with time-like axis in a Minkowski 3-space $\mathbb{L}^3$. Then, there exists no helicoidal surface of rational kind with harmonic Gauss map except polynomial kind.*

Combining the above theorems we have the following:

Theorem 3.3 (Characterization)**.** *Let M be a rational helicoidal surface with time-like axis in a Minkowski 3-space $\mathbb{L}^3$. Then, M has the harmonic Gauss map if and only if it is part of a plane.*

Combining the results above and [5], we have the following characterization.

Theorem 3.4 (Characterization)**.** *Let M be a rational helicoidal surface with time-like axis in a Minkowski 3-space $\mathbb{L}^3$. Then, the Gauss map G of M satisfies the condition $\Delta G = F(G+C)$ for some smooth function F and constant vector C if and only if M is an open part of a plane, a circular cylinder, a right cone, a right helicoid of type II or a helicoidal surface of elliptic type in $\mathbb{L}^3$.*

4. Helicoidal surfaces with null axis in Minkowski 3-space

In this section, we investigate the helicoidal surfaces with harmonic Gauss map which has null axis in $\mathbb{L}^3$.

Suppose that M is a helicoidal surface with null axis parameterized by

$$x(u,v) = \left(f(u) + \frac{v^2}{2}p(u) + hv,\ g(u) + \frac{v^2}{2}p(u) + hv,\ p(u)v \right),\ h \in \mathbb{R},$$

where $p(u) = f(u) - g(u) \neq 0$. Since the induced metric on M is non-degenerate, $(f(u)-g(u))^2(f'^2(u)-g'^2(u))+h^2(f'(u)-g'(u))^2$ never vanishes and so $f'(u)-g'(u) \neq 0$ everywhere. Thus, we may change the variable in such a way that $p(u) = f(u) - g(u) = -2u$.

Let $k(u) = f(u) + u$. Then, the functions f and g in the profile curve γ look like

$$f(u) = k(u) - u \text{ and } g(u) = k(u) + u.$$

Thus, the parametrization of M becomes

$$x(u,v) = (k(u) - u - uv^2 + hv,\ k(u) + u - uv^2 + hv,\ -2uv).$$

We now suppose that M is space-like, that is, $4u^2k'(u) - h^2 > 0$. By a direct computation, the Gauss map G and its Laplacian ΔG are obtained as follows:

$$G = \frac{1}{\sqrt{4u^2k'(u) - h^2}}(uk'(u) + u + uv^2 - vh,\ uk'(u) - u + uv^2 - vh,\ 2uv - h)$$

and

$$\Delta G = -\frac{1}{(4u^2k'(u) - h^2)^{\frac{7}{2}}}(2uX + Y\ ,\ -2uX + Y,\ 2(2uv - h)X)\ ,$$

where we have put
(4.1)
$$X = X(u) = h^4 + 4h^2k'u^2 + 9h^2k''u^3 + h^2k'''u^4 - 4k'k''u^5 + 8k''^2u^6 - 4k'k'''u^6$$
and
(4.2)
$$\begin{aligned} Y &= Y(u,v) \\ &= 10h^4k'u + 7h^4k''u^2 - 32h^2k'^2u^3 + h^4k'''u^3 - 14h^2k'k''u^4 + 32k'^3u^5 \\ &\quad + 6h^2k''^2u^5 - 6h^2k'k'''u^5 + 8k'^2k''u^6 - 8k'k''^2u^7 + 8k'^2k'''u^7 - 2h^5v \\ &\quad - 8h^3k'u^2v - 18h^3k''u^3v - 2h^3k'''u^4v + 8hk'k''u^5v - 16hk''^2u^6v \\ &\quad + 8hk'k'''u^6v + 2h^4uv^2 + 8h^2k'u^3v^2 + 18h^2k''u^4v^2 + 2h^2k'''u^5v^2 \\ &\quad - 8k'k''u^6v^2 + 16k''^2u^7v^2 - 8k'k'''u^7v^2. \end{aligned}$$

Suppose that M has harmonic Gauss map, that is, its Gauss map G satisfies $\Delta G = 0$. Then the above equations $X(u)$ and $Y(u,v)$ are vanishing. Hence, the equation $Y(u,v)$ in (4.2) can be rewritten as
$$Y(u,v) = Y_1(u) + Y_2(u)v + Y_3(u)v^2,$$
where we put
$$\begin{aligned} Y_1(u) =& 10h^4k'u + 7h^4k''u^2 - 32h^2k'^2u^3 + h^4k'''u^3 - 14h^2k'k''u^4 + 32k'^3u^5 \\ & + 6h^2k''^2u^5 - 6h^2k'k'''u^5 + 8k'^2k''u^6 - 8k'k''^2u^7 + 8k'^2k'''u^7, \end{aligned}$$
$$Y_2(u) = -2h(h^4 + 4h^2k'u^2 + 9h^2k''u^3 + h^2k'''u^4 - 4k'k''u^5 + 8k''^2u^6 - 4k'k'''u^6),$$
$$Y_3(u) = 2u(h^4 + 4h^2k'u^2 + 9h^2k''u^3 + h^2k'''u^4 - 4k'k''u^5 + 8k''^2u^6 - 4k'k'''u^6).$$
Since $X(u)$ and $Y(u,v)$ are vanishing, we have $Y_1(u) = 0$. Moreover, $Y_1(u)$ can be written as $Y_1(u) = -2k'uX(u) + uZ(u)$ and we also get $Z(u) = 0$, where
(4.3)
$$\begin{aligned} Z(u) = &\ 12h^4k' + 7h^4k''u - 24h^2k'^2u^2 + h^4k'''u^2 + 4h^2k'k''u^3 + 32k'^3u^4 \\ & + 6h^2k''^2u^4 - 4h^2k'k'''u^4 + 8k'k''^2u^6. \end{aligned}$$

Let M be a helicoidal surface of polynomial kind with harmonic Gauss map, that is, k is a polynomial in u. Then we may put
$$k(u) = a_nu^n + a_{n-1}u^{n-1} + \cdots + a_1u + a_0,$$
where n is nonnegative integer and a_n is non-zero constant.

Considering the constant terms in $X(u)$, it is easy to see that $h = 0$. Therefore, the equations $X(u)$ and $Z(u)$ can be written as
$$X(u) = -4k'k''u^5 + 8k''^2u^6 - 4k'k'''u^6 \text{ and } Z(u) = 32k'^3u^4 + 8k'k''^2u^6.$$

Assume that $\deg k(u) \geq 2$. Considering the equation $X(u)$, we can easily lead to a contradiction.

If $\deg k(u) = 1$, then $X(u) = 0$ and $Z(u) = 32a_1^3 u^4$. Hence, $Z(u)$ cannot be zero and so we have a contradiction.

If k is constant, then $X(u) = 0$ and $Z(u) = 0$. But, in this case, it contradicts that M is non-degenerate, i.e., $4u^2k'(u) \neq 0$. Hence, M does not have harmonic Gauss map.

By a similar argument as above, we have the same results in case of time-like helicoidal surface of polynomial kind with null axis. Thus, we have:

Theorem 4.1. *Suppose that M is a helicoidal surface of polynomial kind with null axis in a Minkowski 3-space $\mathbb{L}^3$. Then M does not have harmonic Gauss map.*

We now consider a helicoidal surface of rational kind with harmonic Gauss map, that is, k is a rational function in u. Then we may put

$$k(u) = p(u) + \frac{r(u)}{q(u)},$$

where $p(u)$ is a polynomial in u, $r(u)$ and $q(u)$ are relatively prime polynomials with $\deg r(u) < \deg q(u)$ and $\deg q(u) \geq 1$.

Suppose that M is a genuine helicoidal surface of rational kind, that is, $h \neq 0$. With the help of (4.1) and (4.3), we get

$$u^2Z(u) - h^2X(u) = (4u^2k' - h^2)(h^4 - 4h^2k'u^2 + 2h^2k''u^3 + 8{k'}^2u^4 + 2{k''}^2u^6).$$

Since $X(u)$ and $Z(u)$ vanishes identically,

$$(4u^2k' - h^2)(h^4 - 4h^2k'u^2 + 2h^2k''u^3 + 8{k'}^2u^4 + 2{k''}^2u^6) = 0.$$

Because M is a nondegenerate surface, i.e., $4u^2k' - h^2 \neq 0$,

$$h^4 - 4h^2k'u^2 + 2h^2k''u^3 + 8{k'}^2u^4 + 2{k''}^2u^6 = 0. \tag{4.4}$$

From the equation (4.4), we get

$$(2k''u^3 + h^2)^2 + (4u^2k' - h^2)^2 = 0.$$

It is easily seen that this is a contradiction because of $4u^2k' - h^2 \neq 0$. Thus, $h = 0$.

If $h = 0$, the equation $Z(u)$ in (4.3) can be reduced as

$$Z(u) = 8u^2k'({k''}^2u^4 + 4u^2{k'}^2).$$

Since M is nondegenerate, ${k''}^2u^4 + 4u^2{k'}^2 = 0$, which implies k is constant, a contradiction.

Similarly, we prove that a time-like helicoidal surface of rational kind does not have harmonic Gauss map. Consequently, we have:

Theorem 4.2. *Let M be a helicoidal surface with null axis in a Minkowski 3-space $\mathbb{L}^3$. Then, there exists no rational helicoidal surface with harmonic Gauss map.*

Combining the results we obtained above and those in [5], we have the following:

Theorem 4.3 (Characterization). *Let M be a helicoidal surface of rational kind with null axis in a Minkowski 3-space $\mathbb{L}^3$. Then, the Gauss map G of M satisfies $\Delta G = F(G+C)$ for some smooth function F and constant vector C if and only if it is part of an Enneper's surface of second kind, a de Sitter space, a hyperbolic space, a helicoidal surface of Enneper type, a helicoidal surface of hyperbolic type or a helicoidal surface of de Sitter type in $\mathbb{L}^3$.*

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