Bull. Korean Math. Soc. **46** (2009), No. 3, pp. 535–543
DOI 10.4134/BKMS.2009.46.3.535

# SOME REMARKS ON THE HELTON CLASS OF AN OPERATOR

Insook Kim, Yoenha Kim, Eungil Ko, and Ji Eun Lee

Abstract. In this paper we study some properties of the Helton class of an operator. In particular, we show that the Helton class preserves the quasinilpotent property and Dunford's boundedness condition $(B)$. As corollaries, we get that the Helton class of some quadratically hyponormal operators or decomposable subnormal operators satisfies Dunford's boundedness condition $(B)$.

## 1. Introduction

Let $H$ be a complex (separable) Hilbert space and let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on $H$. If $T \in \mathcal{L}(H)$, we write $\sigma(T)$ for the spectrum of $T$. An operator $T \in \mathcal{L}(H)$ is said to be *normal* if $T$ and $T^*$ commute, where $T^*$ is the adjoint of $T$. An operator $T \in \mathcal{L}(H)$ is said to be *subnormal* if there is a Hilbert space $K$ containing $H$ and a normal operator $N$ on $K$ such that $N$ leaves $H$ invariant and $T = N\mid_H$. An operator $T \in \mathcal{L}(H)$ is said to be *hyponormal* if $T^*T \geq TT^*$. An operator $T$ is *quadratically hyponormal* if and only if $aT^2+bT+cI$ is hyponormal for all $a, b, c \in \mathbb{C}$. Indeed, every subnormal operator is quadratically hyponormal. But the converse is not true from [10]. An operator $T$ is said to be *normaloid* if $\| T \| = r(T)$, where $r(T)$ is the spectral radius of $T$. An operator $T$ is said to be *spectraloid* if $w(T) = r(T)$, where $w(T)$ denotes the numerical radius of $T$.

An operator $T \in \mathcal{L}(H)$ is said to have the *single valued extension property* if for any analytic function $f : D \to H,\ D \subset \mathbb{C}$ open, with $(\lambda - T)f(\lambda) \equiv 0$, it results $f(\lambda) \equiv 0$. For an operator $T \in \mathcal{L}(H)$ having the single valued extension property and for $x \in H$ we can consider the set $\rho_T(x)$ of elements $\lambda_0 \in \mathbb{C}$ such that there exists an analytic function $f(\lambda)$ defined in a neighborhood of $\lambda_0$, with values in $H$, which verifies $(\lambda - T)f(\lambda) \equiv x$. Throughout this paper, we denote $\sigma_T(x) = \mathbb{C}\backslash\rho_T(x)$.

---

Received July 4, 2008.
2000 *Mathematics Subject Classification.* Primary 47B20, Secondary 47A10.
*Key words and phrases.* Helton class, quasinilpotent property, Dunford's boundedness condition $(B)$.
This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD)(KRF-2007-314-C00011).

Assume that $T$ has the single valued extension property. If there exists a constant $k$ such that for every $x, y \in H$ with $\sigma_T(x) \cap \sigma_T(y) = \emptyset$ we have

$$\| x \| \leq k \| x + y \|,$$

where $k$ is independent of $x$ and $y$, we say that an operator $T$ satisfies *Dunford's boundedness condition* $(B)$.

Let $R$ and $S$ be in $\mathcal{L}(H)$ and let $C(R, S) : \mathcal{L}(H) \to \mathcal{L}(H)$ be defined by $C(R, S)(A) = RA - AS$. Then

$$C(R, S)^k(I) = \sum_{j=0}^{k} (-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} R^j S^{k-j}. \tag{1}$$

Let $R \in \mathcal{L}(H)$. If there is an integer $k \geq 1$ such that an operator $S$ satisfies $C(R, S)^k(I) = 0$, we say that *$S$ belongs to Helton class of $R$ with order $k$*. We denote this by $S \in \text{Helton}_k(R)$.

In this paper we study some properties of the Helton class of an operator. In particular, we show that the Helton class preserves the quasinilpotent property and Dunford's boundedness condition $(B)$. As corollaries, we get that the Helton class of some quadratically hyponormal operators or decomposable subnormal operators satisfies Dunford's boundedness condition $(B)$.

## 2. Some properties

In this section, we study some properties of the Helton class of operators. First, we start with the following proposition.

**Proposition 2.1.** *Let $R$ and $S$ be in $\mathcal{L}(H)$ and $S \in \text{Helton}_k(R)$ for some integer $k \geq 1$. Then the following statements hold*:

(1) *$X^{-1}SX \in \text{Helton}_k(X^{-1}RX)$, where $X$ is an invertible operator,*
(2) *$S_2 S_1 \notin \text{Helton}_k(I)$ if $S_1 S_2 \in \text{Helton}_k(I)$ and $0 \in \sigma(S_1)$,*
(3) *$S|_{\mathcal{M}} \in \text{Helton}_k(R|_{\mathcal{M}})$ if $\mathcal{M} \in Lat(R) \cap Lat(S)$, where $Lat(R)$ consists of all invariant subspaces (including $(0)$ and $H$) for $R$, and*
(4) *$S_1 \oplus S_2 \in \text{Helton}_k(R_1 \oplus R_2)$ if $S_i \in \text{Helton}_k(R_i)$ for $i = 1, 2$.*

Next we consider some relations of invertible operators through the Helton class.

**Proposition 2.2.** *Let $S$ and $R$ be invertible in $\mathcal{L}(H)$.*

(1) *If $R^j S^{k-j} = S^{k-j} R^j$ for $j = 0, 1, \ldots, k$, then $S \in \text{Helton}_k(R)$ if and only if $R^{-1} \in \text{Helton}_k(S^{-1})$.*
(2) *$S \in \text{Helton}_k(R)$ if and only if $S^{-1} \in \text{Helton}_k(R^{-1})$.*

*Proof.* (1) Assume that $R^j S^{k-j} = S^{k-j} R^j$ for $j = 0, 1, \ldots, k$. If $S \in \text{Helton}_k(R)$, then the following equation holds:

$$0 = S^{-k} \left[ \sum_{j=0}^{k} (-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} R^j S^{k-j} \right] R^{-k} = \sum_{j=0}^{k} (-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} (S^{-1})^j (R^{-1})^{k-j}.$$

Thus $R^{-1} \in \mathrm{Helton}_k(S^{-1})$. The converse implication is similar.

(2) If $S^{-1} \in \mathrm{Helton}_k(R^{-1})$, then the following equation holds:

$$0 = R^k \left[\sum_{j=0}^{k}(-1)^{k-j}\begin{pmatrix} k \\ j \end{pmatrix}(R^{-1})^j(S^{-1})^{k-j}\right]S^k = \sum_{j=0}^{k}(-1)^{k-j}\begin{pmatrix} k \\ j \end{pmatrix}R^{k-j}S^j.$$

If $k$ is even, then $(-1)^{k-j} = (-1)^j$. Since $\binom{k}{j} = \binom{k}{k-j}$,

$$\begin{aligned} 0 = \sum_{j=0}^{k}(-1)^{k-j}\begin{pmatrix} k \\ j \end{pmatrix}R^{k-j}S^j &= \sum_{j=0}^{k}(-1)^{j}\begin{pmatrix} k \\ k-j \end{pmatrix}R^{k-j}S^j \\ &= \sum_{i=0}^{k}(-1)^{k-i}\begin{pmatrix} k \\ i \end{pmatrix}R^{i}S^{k-i}. \end{aligned}$$

Hence we have $S \in \mathrm{Helton}_k(R)$. If $k$ is odd, then $(-1)^{k+1-j} = (-1)^j$. So by the similar method we can show that $S \in \mathrm{Helton}_k(R)$. The converse implication is similar. □

Next we study the Helton class of upper triangular operator matrices.

**Lemma 2.3.** *Let*

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} & \cdots & R_{1n} \\ 0 & R_{22} & R_{23} & \cdots & R_{2n} \\ 0 & 0 & R_{33} & \cdots & R_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & R_{nn} \end{pmatrix} \quad \textit{and} \quad S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & \cdots & S_{1n} \\ 0 & S_{22} & S_{23} & \cdots & S_{2n} \\ 0 & 0 & S_{33} & \cdots & S_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & S_{nn} \end{pmatrix}$$

*be operator matrices, where $R_{ij}, S_{ij} \in \mathcal{L}(H)$ for all $i = 1, 2, \ldots, n$ and $j = i, i+1, \ldots, n$. Then $S \in \mathrm{Helton}_2(R)$ if and only if $S_{ii} \in \mathrm{Helton}_2(R_{ii})$ for all $i = 1, 2, \ldots, n$ and $\sum_{m=j}^{k} R_{jm}(R_{mk} - S_{mk}) = \sum_{m=j}^{k}(R_{jm} - S_{jm})S_{mk}$ for all $1 \le j < k \le n$.*

*Proof.* If $S \in \mathrm{Helton}_2(R)$, then $\sum_{j=0}^{2}(-1)^{2-j}\binom{2}{j}R^jS^{2-j} = 0$. In particular, if $n = 2$, then $S_{ii}^2 - 2R_{ii}S_{ii} + R_{ii}^2 = 0$ for all $i = 1, 2$, and

$$S_{11}S_{12} + S_{12}S_{22} - 2(R_{11}S_{12} + R_{12}S_{22}) + R_{11}R_{12} + R_{12}R_{22} = 0.$$

Thus $R_{11}(R_{12} - S_{12}) + R_{12}(R_{22} - S_{22}) = (R_{11} - S_{11})S_{12} + (R_{12} - S_{12})S_{22}$. If $n \ge 3$, then we get that $S_{ii}^2 - 2R_{ii}S_{ii} + R_{ii}^2 = 0$ for all $i = 1, 2, \ldots, n$, and

$$\begin{aligned} &(S_{jj}S_{j,k} + S_{j,j+1}S_{j+1,k} + \cdots + S_{j,k-1}S_{k-1,k} + S_{j,k}S_{kk}) \\ &- 2\,(R_{jj}S_{j,k} + R_{j,j+1}S_{j+1,k} + \cdots + R_{j,k-1}S_{k-1,k} + R_{j,k}S_{kk}) \\ &+ (R_{jj}R_{j,k} + R_{j,j+1}R_{j+1,k} + \cdots + R_{j,k-1}R_{k-1,k} + R_{j,k}R_{kk}) = 0 \end{aligned}$$

for all $1 \le j < k \le n$. So we obtain

$$\sum_{m=j}^{k} R_{jm}(R_{mk} - S_{mk}) = \sum_{m=j}^{k}(R_{jm} - S_{jm})S_{mk}$$

for all $1 \le j < k \le n$. The other implication is obvious. □

**Theorem 2.4.** *Let $R$ and $S$ be operator matrices as in Lemma 2.3 and $S \in \text{Helton}_2(R)$. If $R_{ii}$ has the single valued extension property for $i = 1, 2, \ldots, n$, then $S_{ii}$ has the single valued extension property for $i = 1, 2, \ldots, n$.*

*Proof.* Since $S_{ii} \in \text{Helton}_2(R_{ii})$ by Lemma 2.3, the proof follows from [7]. □

Next we study the Helton class of operators on 2-dimensional Hilbert space.

**Lemma 2.5.** *Let $A$ be a $2 \times 2$ matrix on $\mathbb{C}^2$, $B \in \text{Helton}_2(A)$, and $A$ be similar to $R = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$. Then we get the following cases:*
(i) *if $a \neq c$, then $B$ is similar to $R = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$,*
(ii) *if $a = c$ and $b = 0$, then $B$ is similar to $S = \left(\begin{smallmatrix} a & \beta \\ 0 & a \end{smallmatrix}\right)$ for any $\beta$ or $S = \left(\begin{smallmatrix} a & 0 \\ \gamma & a \end{smallmatrix}\right)$ for any $\gamma$, and*
(iii) *if $a = c$ and $b \neq 0$, then $B$ is similar to $S = \left(\begin{smallmatrix} a & \beta \\ 0 & a \end{smallmatrix}\right)$ for any $\beta$.*

*Proof.* (i) Since $A$ is similar to $R = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$, where $a \neq c$, there exists an invertible matrix $X$ such that $X^{-1}AX = R$. Since $B \in \text{Helton}_2(A)$, by Proposition 2.1 $X^{-1}BX \in \text{Helton}_k(R)$. Set $S = X^{-1}BX$ with $S = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. Since $S^2 - 2RS + R^2 = 0$,

$$\begin{aligned} 0 &= S^2 - 2RS + R^2 \\ &= \begin{pmatrix} a^2 - 2(a\alpha + b\gamma) + \alpha^2 + \beta\gamma & ab + bc - 2(a\beta + b\delta) + \alpha\beta + \beta\delta \\ -2c\gamma + \gamma\alpha + \delta\gamma & c^2 - 2c\delta + \gamma\beta + \delta^2 \end{pmatrix}. \end{aligned}$$

Hence we get the following equations:

$$a^2 - 2(a\alpha + b\gamma) + \alpha^2 + \beta\gamma = 0, \tag{2}$$

$$ab + bc - 2(a\beta + b\delta) + \alpha\beta + \beta\delta = 0, \tag{3}$$

$$-2c\gamma + \gamma\alpha + \delta\gamma = 0, \tag{4}$$

$$c^2 - 2c\delta + \gamma\beta + \delta^2 = 0. \tag{5}$$

From (4), we get $\gamma = 0$ or $\alpha + \delta = 2c$.

If $\gamma = 0$, then we get the following equations:

$$a^2 - 2a\alpha + \alpha^2 = 0, \tag{6}$$

$$ab + bc - 2(a\beta + b\delta) + \alpha\beta + \beta\delta = 0, \tag{7}$$

$$c^2 - 2c\delta + \delta^2 = 0. \tag{8}$$

From (6) and (8), we obtain $\alpha = a$ and $\delta = c$. Hence from (7) we have $\beta = b$ since $a \neq c$ by hypothesis. Therefore

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = R.$$

Assume that $\alpha + \delta = 2c$. We consider two cases: $\beta = 0$ and $\beta \neq 0$. If $\beta = 0$, then from (5) we have $c = \delta$ and from (3) we get $b(a - c) = 0$, which implies $b = 0$ since $a \neq c$. So from (2) we have $a = \alpha$ and it implies that

$a+\delta = a+c = 2c$. Thus $a = c$, which contradicts the hypothesis. If $\beta \neq 0$, then from (5) and $\delta = 2c-\alpha$

$$(c-\delta)^2 = (\alpha - c)^2 = -\gamma\beta. \tag{9}$$

Since $\delta = 2c - \alpha$, from (3) we get that

$$\begin{aligned} 0 &= (a+c-2\delta)b + (\alpha+\delta-2a)\beta \\ &= (a+2\alpha-3c)b + 2(c-a)\beta. \end{aligned}$$

So we have

$$\beta = \frac{(a+2\alpha-3c)b}{2(a-c)}, \tag{10}$$

since $a \neq c$. From (2) and (9) we obtain that

$$(a-c)(a-2\alpha+c) = 2b\gamma. \tag{11}$$

Hence from (9) and (10) we get

$$\gamma = -\frac{(\alpha-c)^2}{\beta} = -\frac{2(a-c)}{(a+2\alpha-3c)b}(\alpha-c)^2$$

and finally from (11) we have $a = c$, which contradicts the hypothesis.

(ii) If $a = c$ and $A$ is similar to $R = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$, then $A = R$. Since any $2 \times 2$ matrix $B$ is similar to an upper triangular matrix or a lower triangular matrix, first we assume that $B$ is similar to $S = \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right)$. Then there exists an invertible matrix $Y$ such that $Y^{-1}BY = S$. Since $B \in \mathrm{Helton}_2(A)$, by Proposition 2.1 $S \in \mathrm{Helton}_2(R)$. Then by the similar calculation as in the case of (i) we obtain $S = \left(\begin{smallmatrix} a & \beta \\ 0 & a \end{smallmatrix}\right)$ for any $\beta$. For the case of $S = \left(\begin{smallmatrix} \alpha & 0 \\ \gamma & \delta \end{smallmatrix}\right)$, we get that $B$ is similar to $S = \left(\begin{smallmatrix} a & 0 \\ \gamma & a \end{smallmatrix}\right)$ for any $\gamma$.

(iii) If $a = c$ and $A$ is similar to $R = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$, where $b \neq 0$, then by the similar method as (i) we obtain that $B$ is similar to $S = \left(\begin{smallmatrix} a & \beta \\ 0 & a \end{smallmatrix}\right)$ for any $\beta$. □

From Lemma 2.5 we can get the following theorem.

**Theorem 2.6.** *Let $A$ be a $2 \times 2$ matrix on $\mathbb{C}^2$. If $B \in \mathrm{Helton}_2(A)$, then $\sigma(A) = \sigma(B)$ and $B$ is similar to $A$ if $A$ has two distinct eigenvalues.*

*Proof.* The first statement is clear from Lemma 2.5. Since the matrix $A$ has two distinct eigenvalues we have the case of (i) in Lemma 2.5. So $A$ and $B$ are similar to $R$, which means that $B$ is similar to $A$. □

Recall that an operator $S \in \mathcal{L}(H)$ is *quasinilpotent* if $\sigma(S) = \{0\}$, or equivalently $\lim_{n\to\infty} \|S^n\|^{\frac{1}{n}} = 0$. Next we show that the Helton class preserves the quasinilpotent property.

**Theorem 2.7.** *Let $S \in \mathrm{Helton}_k(R)$. Then $R$ is quasinilpotent if and only if $S$ is quasinilpotent.*

*Proof.* Suppose that $R$ is quasinilpotent. Then $\lim_{n\to\infty}\|R^n\|^{\frac{1}{n}}=0$ and this means $\lim_{n\to\infty}\|C(0,R)^n(I)\|^{\frac{1}{n}}=0$. Since $S\in \text{Helton}_k(R)$, $C(R,S)^k(I)=0$ and hence $\lim_{n\to\infty}\|C(R,S)^n(I)\|^{\frac{1}{n}}=0$. So for a given $\epsilon>0$, there exists a constant $M\ge 1$ such that $||R^n||=||C(0,R)^n(I)||<M(\frac{\epsilon}{2})^n$ and $||C(R,S)^n(I)||<M(\frac{\epsilon}{2})^n$ for all $n\ge 0$. Then

$$\begin{aligned} ||S^n|| &= ||C(0,S)^n(I)|| = \left|\left|\sum_{j=0}^{\infty}\binom{n}{j}C(0,R)^j(I)\ C(R,S)^{n-j}(I)\right|\right| \\ &\le \sum_{j=0}^{\infty}\binom{n}{j}||C(0,R)^j(I)||\ ||C(R,S)^{n-j}(I)|| \\ &< \sum_{j=0}^{\infty}\binom{n}{j}M^2\left(\frac{\epsilon}{2}\right)^j\left(\frac{\epsilon}{2}\right)^{n-j} \\ &= M^2\epsilon^n \end{aligned}$$

for all $n\ge 0$. So $\limsup_{n\to\infty}||S^n||^{\frac{1}{n}}\le\limsup_{n\to\infty}(M^{\frac{2}{n}}\,\epsilon)=\epsilon$. Since $\epsilon>0$ is arbitrary, $\limsup_{n\to\infty}||S^n||^{\frac{1}{n}}=0$. Hence $S$ is quasinilpotent.

Conversely, suppose that $S$ is quasinilpotent. Then $S^*$ is quasinilpotent. Since $S\in\text{Helton}_k(R)$, $R^*\in\text{Helton}_k(S^*)$. Thus $R^*$ is quasinilpotent by the same argument as above. So $R$ is quasinilpotent. □

A bounded linear operator $S$ on $H$ is called a *generalized scalar operator* if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^\infty(\mathbb{C})\to\mathcal{L}(H)$$

such that $\Phi(z)=S$, where as usual $z$ stands for the identity function on $\mathbb{C}$ and $C_0^\infty(\mathbb{C})$ stands for the space of compactly supported functions on $\mathbb{C}$ that are continuously infinitely differentiable.

**Corollary 2.8.** *Let $R\in\mathcal{L}(H)$ be quasinilpotent and $S\in\text{Helton}_k(R)$. Then*
(1) *if $S$ is a generalized scalar operator, then $S$ is nilpotent, and*
(2) *if $S$ is hyponormal, then $S$ is a zero operator.*

*Proof.* The first statement follows from Theorem 2.7 and [8]. (2) Since $S$ is hyponormal and quasinilpotent by Theorem 2.7, $S$ is a zero operator. □

Next we show that the Helton class preserves Dunford's boundedness condition $(B)$. First, we give some direct proof for the following lemma which is known in [8].

**Lemma 2.9** ([8])**.** *If $R$ has the single valued extension property and $S\in\text{Helton}_k(R)$, then $\sigma_R(x)\subset\sigma_S(x)$ for any $x\in H$.*

*Proof.* If $S\in\text{Helton}_k(R)$ and $R$ has the single valued extension property, then $S$ has the single valued extension property from [7]. Next, let $f(\lambda)$ be an analytic function which verifies $(\lambda - S)f(\lambda) = x$. Since $(\lambda - S)f^{(n)}(\lambda) = -nf^{(n-1)}(\lambda)$ for every positive integer $n$, it is an easy calculation to show that

$$(\lambda - S)^k f^{(k-1)}(\lambda) = (-1)^{k-1}(k-1)!x.$$

Set

$$g(\lambda) = (-1)^{k-2}\frac{1}{(k-1)!}\sum_{j=1}^{k}\binom{k}{j}(R-\lambda)^{j-1}(\lambda - S)^{k-j}f^{(k-1)}(\lambda).$$

Then $g(\lambda)$ is analytic and

$$\begin{aligned}(R-\lambda)g(\lambda) &= (-1)^{k-2}\frac{1}{(k-1)!}\sum_{j=1}^{k}\binom{k}{j}(R-\lambda)^{j}(\lambda - S)^{k-j}f^{(k-1)}(\lambda)\\ &= (\lambda - S)^k(-1)^{k-1}\frac{1}{(k-1)!}f^{(k-1)}(\lambda) = x.\end{aligned}$$

Thus $\rho_S(x) \subset \rho_R(x)$ for any $x \in H$. Hence $\sigma_R(x) \subset \sigma_S(x)$ for any $x \in H$. □

**Theorem 2.10.** *Let* $R \in \mathcal{L}(H)$ *and let* $S \in \mathrm{Helton}_k(R)$. *Then*

(1) $r(R) = r(S)$,

(2) *if* $R$ *is normaloid, then* $\| R \| \leq \| S \|$,

(3) *if* $R$ *is spectraloid, then* $w(R) \leq w(S)$, *where* $w(R)$ *and* $w(S)$ *are the numerical radii of* $R$ *and* $S$, *respectively, and*

(4) *if* $R$ *satisfies the single valued extension property and Dunford's boundedness condition* $(B)$, *then so does* $S$.

*Proof.* (1) Since $r(S) = \sup\{|\lambda| : \lambda \in \sigma(S)\}$ and $\sigma(S)$ is compact, there exists $\lambda_0 \in \sigma(S)$ such that $r(S) = |\lambda_0|$. In fact, the maximum is attained on the boundary of $\sigma(S)$, i.e., $\lambda_0 \in \partial\sigma(S)$. Since $S \in \mathrm{Helton}_k(R)$, $\sigma_{ap}(S) \subset \sigma_{ap}(R)$ by [9]. So $\lambda_0 \in \partial\sigma(S) \subset \sigma_{ap}(S) \subset \sigma_{ap}(R)$ and hence $|\lambda_0| \leq r(R)$. Thus we have $r(S) \leq r(R)$. On the other hand, we obtain $r(R^*) \leq r(S^*)$, because $R^* \in \mathrm{Helton}_k(S^*)$. Since $r(R^*) = r(R)$ and $r(S^*) = r(S)$, we have $r(R) \leq r(S)$. So $r(R) = r(S)$.

(2) If $R$ is normaloid, then $r(R) = \|R\|$. Since $r(S) \leq \|S\|$, we have $\|R\| \leq \|S\|$.

(3) If $R$ is spectraloid, then $w(R) = r(R)$. Since $r(S) \leq w(S)$ by [5], we get that $w(R) \leq w(S)$.

(4) By [7], $S$ has the single valued extension property. Assume that $x$ and $y$ are any vectors in $H$ such that $\sigma_S(x) \cap \sigma_S(y) = \emptyset$. Since $\sigma_R(x) \subset \sigma_S(x)$ and $\sigma_R(y) \subset \sigma_S(y)$ from Lemma 2.9, $\sigma_R(x) \cap \sigma_R(y) = \emptyset$. Since $R$ satisfies Dunford's boundedness condition $(B)$, there exists a constant $k$ such that $||x|| \leq k||x+y||$, where $k$ is independent of $x$ and $y$. Hence $S$ satisfies Dunford's boundedness condition $(B)$. □

Let $S_1(S) := \{\delta \subset \mathbb{C}$ : the vectors of the form $x + y$ with $\sigma_S(x) \subset \delta$ and $\sigma_S(y) \subset \mathbb{C} \setminus \delta$ are dense in $H\}$.

**Corollary 2.11.** *Let $R$ have the single valued extension property and let $S \in \text{Helton}_k(R)$. If $R$ satisfies Dunford's boundedness condition $(B)$, then for any $\delta \in S_1(S)$ there exists one and only one bounded idempotent $E(\delta)$ on $H$ with the properties that $E(\delta)x = x$ if $\sigma_S(x) \subset \delta$ and $E(\delta)x = 0$ if $\sigma_S(x) \subset \mathbb{C} \setminus \delta$.*

*Proof.* Since $S$ has the single valued extension property from [7] and satisfies Dunford's boundedness condition $(B)$ from Theorem 2.10, the proof follows from [4, p. 242]. □

Recall that the residual spectrum of $R$, $\sigma_r(R)$, is the set of $\lambda \in \sigma(R)$ for which $R - \lambda$ is one to one and $\text{ran}(R - \lambda)$ is not dense in $H$.

**Corollary 2.12.** *Suppose that $R$ is quadratically hyponormal and $S \in \text{Helton}_k(R)$. If $\sigma_r(S) = \emptyset$, then for any $x, y \in H$ with $\sigma_S(x) \cap \sigma_S(y) = \emptyset$, we have $\langle x, y \rangle = 0$. Moreover, in this case, $S$ satisfies Dunford's boundedness condition $(B)$.*

*Proof.* If $R$ is a quadratically hyponormal operator, it is well-known that $R$ has the single valued extension property. So $S$ has the single valued extension property from [7]. Assume that $\sigma_S(x) \cap \sigma_S(y) = \emptyset$. Since $\sigma_R(x) \subset \sigma_S(x)$ and $\sigma_R(y) \subset \sigma_S(y)$ from Lemma 2.9, $\sigma_R(x) \cap \sigma_R(y) = \emptyset$. To obtain $\langle x, y \rangle = 0$ we need to show that $\sigma_r(R) \subset \sigma_r(S)$. If $\lambda \in \sigma_r(R)$, then $R - \lambda$ is one to one and $\text{ran}(R - \lambda)$ is not dense in $H$. So $\lambda \notin \sigma_p(R)$. Since $\sigma_p(S) \subset \sigma_p(R)$ by [9], $\lambda \notin \sigma_p(S)$ and hence $S - \lambda$ is one to one. We also have $\overline{\text{ran}(S - \lambda)} \neq H$, since $\Gamma(R) \subset \Gamma(S)$, where $\Gamma(R) = \{\lambda \in \mathbb{C} : \overline{\text{ran}(R - \lambda)} \neq H\}$ by [9]. Thus $\sigma_r(R) \subset \sigma_r(S)$ and hence $\sigma_r(R) = \emptyset$. So $\langle x, y \rangle = 0$ follows from [10]. Moreover, since $\langle x, y \rangle = 0$, $\|x\| \leq \|x + y\|$ holds. Hence $S$ satisfies Dunford's boundedness condition $(B)$. □

Recall that a closed linear subspace $\mathcal{Y}$ of $H$ is called a *spectral maximal space* of $T$ if $\mathcal{Y}$ is invariant to $T$ and if $\mathcal{Z}$ is another closed linear subspace of $H$, invariant to $T$, such that $\sigma(T \mid_{\mathcal{Z}}) \subset \sigma(T \mid_{\mathcal{Y}})$, then $\mathcal{Z} \subset \mathcal{Y}$. An operator $T \in \mathcal{L}(H)$ is called *decomposable* if for every finite open covering $\{G_i\}_{i=1}^n$ of $\sigma(T)$ there exists a system $\{\mathcal{M}_i\}_{i=1}^n$ of spectral maximal spaces of $T$ such that $\sigma(T|_{\mathcal{M}_i}) \subset G_i$ for every $i = 1, 2, \ldots, n$ and $H = \sum_{i=1}^n \mathcal{M}_i$.

**Corollary 2.13.** *Let $R$ be a decomposable subnormal operator and $S \in \text{Helton}_k(R)$. Then for any $x, y \in H$ with $\sigma_S(x) \cap \sigma_S(y) = \emptyset$, we have $\langle x, y \rangle = 0$. Moreover, in this case, $S$ satisfies Dunford's boundedness condition $(B)$.*

*Proof.* Since $R$ has the single valued extension property, $S$ has the single valued extension property from [7]. If $\sigma_S(x) \cap \sigma_S(y) = \emptyset$, then $\sigma_R(x) \cap \sigma_R(y) = \emptyset$ from Lemma 2.9. Hence $\langle x, y \rangle = 0$ from [11]. Moreover, since $\langle x, y \rangle = 0$, $\|x\| \leq \|x + y\|$ holds. Hence $S$ satisfies Dunford's boundedness condition $(B)$. □

**Corollary 2.14.** *Let $S \in \text{Helton}_k(R)$. Suppose that $R$ is a quadratically hyponormal operator and $\sigma_r(S) = \emptyset$ or $R$ is a decomposable subnormal operator. If there exist nonzero vectors $x$ and $y$ such that $\sigma_S(x) \cap \sigma_S(y) = \emptyset$, then $S$ has a nontrivial invariant subspace.*

*Proof.* Let $\lambda_0 \in \rho_S(x)$. Then there exists an analytic function $f(\lambda)$ defined in a neighborhood of $\lambda_0$, with values in $H$, which verifies $(\lambda - S)f(\lambda) \equiv x$. Since $(\lambda - S)S^n f(\lambda) \equiv S^n x$ for any nonnegative integer $n$, $\lambda_0 \in \rho_S(S^n x)$. Thus $\sigma_S(S^n x) \subset \sigma_S(x)$. Hence $\sigma_S(S^n x) \cap \sigma_S(y) = \emptyset$. By Corollaries 2.12 and 2.13, we get that $\langle S^n x, y\rangle = 0$ for any nonnegative integer $n$. Hence $\vee_{n\geq 0}\{S^n x\}$ is a nontrivial invariant subspace for $S$. $\square$

## References

[1] I. Colojoară and C. Foiaş, *Theory of Generalized Spectral Operators*, Mathematics and its Applications, Vol. 9. Gordon and Breach, Science Publishers, New York-London-Paris, 1968.

[2] J. B. Conway, *A Course in Functional Analysis*, Springer-Verlag, 1985.

[3] S. Djordjević, I. H. Jeon, and E. Ko, *Weyl's theorem through local spectral theory*, Glasg. Math. J. **44** (2002), no. 2, 323–327.

[4] N. Dunford, *Spectral operators*, Pacific J. Math. **4** (1954), 321–354.

[5] P. R. Halmos, *A Hilbert Space Problem Book*, Springer-Verlag, 1980.

[6] R. A. Horn and C. A. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[7] Y. Kim, E. Ko, and J. Lee, *Operators with the single valued extension property*, Bull. Korean Math. Soc. **43** (2006), no. 3, 509–517.

[8] K. Laursen and M. Neumann, *An Introduction to Local Spectral Theory*, London Mathematical Society Monographs. New Series, 20. The Clarendon Press, Oxford University Press, New York, 2000.

[9] J. Lee, *The Helton class of operators and rank one perturbations of the unilateral shift*, Ph. D. thesis, Ewha Womans University, 2008.

[10] J. G. Stampfli, *Analytic extensions and spectral localization*, J. Math. Mech. **16** (1966), 287–296.

[11] B. L. Wadhwa, *Spectral, M-hyponormal and decomposable operators*, Ph. D. thesis, Indiana University, 1971.

Insook Kim
Department of Mathematics
Ewha Women's University
Seoul 120-750, Korea
*E-mail address*: ikim99@hanmail.net

Yoenha Kim
Department of Mathematics
Ewha Women's University
Seoul 120-750, Korea
*E-mail address*: yoenha@ewhain.ne

Eungil Ko
Department of Mathematics
Ewha Women's University
Seoul 120-750, Korea
*E-mail address*: eiko@ewha.ac.kr

Ji Eun Lee
Department of Mathematics
Ewha Women's University
Seoul 120-750, Korea
*E-mail address*: jieun7@ewhain.net