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NEW COMPLEXITY ANALYSIS OF PRIMAL-DUAL IMPS FOR P_* LAPS BASED ON LARGE UPDATES

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Abstract. In this paper we present new large-update primal-dual interior point algorithms for P_* linear complementarity problems(LAPS) based on a class of kernel functions, $\psi(t) = \frac{t^{p+1}-1}{p+1} + \frac{1}{\sigma}(e^{\sigma(1-t)} - 1)$, $p \in [0,1]$, $\sigma \geq 1$. It is the first to use this class of kernel functions in the complexity analysis of interior point method(IPM) for P_* LAPS. We showed that if a strictly feasible starting point is available, then new large-update primal-dual interior point algorithms for P_* LAPS have $O((1+2\kappa)n^{\frac{1}{p+1}} \log n \log \frac{n}{\varepsilon})$ complexity bound. When $p = 1$, we have $O((1+2\kappa)\sqrt{n} \log n \log \frac{n}{\varepsilon})$ complexity which is so far the best known complexity for large-update methods.

1. Introduction

In this paper we consider the following linear complementarity problem (LCP):

$$s = Mx + q,\ xs = 0,\ x \geq 0,\ \ s \geq 0, \tag{1}$$

where $M \in \mathbb{R}^{n\times n}$ is a $P_*(\kappa)$ matrix and $x, s, q \in \mathbb{R}^n$, and xs denotes the componentwise product of vectors x and s.

LAPS have many applications, e.g., linear and quadratic programming, finding a Nash-equilibrium in bimatrix games, economies with institutional restrictions upon prices, contact problems with friction, optimal stopping in Markov chains, circuit simulation, free boundary problems, and calculating the interval hull of linear systems of interval equations ([14]).

Kojima et al. ([6, 10]) first introduced the primal-dual IPM for linear optimization(LO) problem. Since then many other algorithms have been developed based on the primal-dual strategy. Subsequently, Kojima, Mizuno, and

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Yoshise ([8]) generalized the algorithm in [6] to monotone linear complementarity problems, i.e., $P_*(0)$ LAPS, where the target is reduced with a small-update. They also proposed an $O(\sqrt{n}L)$ potential reduction algorithm ([9]). Since then an interior point algorithm's quality is measured by the fact whether it can be generalized to $P_*(\kappa)$ LAPS or not ([5]). Several variants of the Mizuno-Todd-Ye type predictor-corrector interior point algorithm are proposed. First, Miao ([11]) extended the Mizuno-Todd-Ye predictor-corrector method to $P_*(\kappa)$ LAPS. His algorithm uses the l_2 neighborhood of the central path and has $O((1+\kappa)\sqrt{n}L)$ iteration complexity. Later, Illés and Nagy ([5]) give a version of Mizuno-Todd-Ye predictor-corrector interior point algorithm for the $P_*(\kappa)$ LCP and show $O((1+\kappa)^{\frac{3}{2}}\sqrt{n}L)$ iteration complexity.

Most of the classical primal-dual interior point method (IPM) for LO are based on the use of the logarithmic barrier function, e.g. see [13]. Peng et al. ([12]) introduced self-regular barrier functions for primal-dual IMPS for LO and obtained the best complexity result for large-update primal-dual IMPS for LO with some specific self-regular barrier functions. Recently, Bai et al. ([2]) proposed a new class of barrier functions which are called eligible and not logarithmic barrier nor self-regular and they presented a unified computational framework for the complexity analysis of the algorithm. They greatly simplified the analysis of IMPS.

In this paper we generalized the large-update primal-dual interior point algorithm ([1]) for LO to P_* LCP. The analysis in this paper uses the analysis scheme presented in [2]. Since we define a neighborhood and a search direction based on kernel functions which are neither logarithmic nor self-regular and don't use the condition (iii) of eligible function (Definition 2.5), the analysis is different from the ones in [5], [7], [8], [9], and [11]. When $p = 1$, we have $O((1+2\kappa)\sqrt{n}\log n \log\frac{n}{\varepsilon})$ complexity which is similar to the one in [1] which is so far the best known complexity for large-update IPM. In [1], they analyzed the complexity bound of the algorithm for LO based on all eligible conditions. But we used three eligible conditions.

This paper is organized as follows. In Section 2 we recall some basic definitions and introduce the algorithm. In Section 3 we give the properties of the kernel function. In Section 4 we compute the feasible step size and the amount of decrease of the proximity function during an inner iteration. Finally, in Section 5 we obtain the complexity result of the algorithm.

We use the following notations throughout the paper : $\mathbb{R}^n_+$ denotes the set of n dimensional nonnegative vectors and $\mathbb{R}^n_{++}$, the set of n dimensional positive vectors. For $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, $x_{\min} = \min\{x_1, x_2, \ldots, x_n\}$, i.e., the minimal component of x, $\|x\|$ is the 2-norm of x, and X is the diagonal matrix from vector x, i.e., $X = \mathrm{diag}(x)$. xs denotes the componentwise product (Hadamard product) of vectors x and s. $x^T s$ is the scalar product of the vectors x and s. e is the n-dimensional vector of ones and I is the n-dimensional identity matrix. J is the index set, i.e., $J = \{1, 2, \ldots, n\}$. We write $f(x) = O(g(x))$ if

$|f(x)| \leq k|g(x)|$ for some positive constant k and $f(x) = \Theta(g(x))$ if $k_1|g(x)| \leq |f(x)| \leq k_2|g(x)|$ for some positive constants k_1 and k_2.

2. Preliminaries

In this section we give some basic definitions and introduce the new algorithm. $P_*(\kappa)$ matrix is first introduced by Kojima et al.([7]) which is the generalization of positive semi-definite matrices, i.e., $P_*(0)$ matrix.

Definition 2.1. Let κ be a nonnegative number. A matrix $M \in \mathbb{R}^{n\times n}$ is called a $P_*(\kappa)$ matrix if

$$(1+4\kappa)\sum_{i\in J_+(x)} x_i(Mx)_i + \sum_{i\in J_-(x)} x_i(Mx)_i \geq 0$$

for all $x \in \mathbb{R}^n$, where

$$J_+(x) = \{i \in J : x_i(Mx)_i \geq 0\} \text{ and } J_-(x) = \{i \in J : x_i(Mx)_i < 0\}.$$

Definition 2.2. A matrix $M \in \mathbb{R}^{n\times n}$ is called a P_* matrix if it is a $P_*(\kappa)$ matrix for some $\kappa \geq 0$, i.e., $P_* = \bigcup_{\kappa\geq 0} P_*(\kappa)$.

Note that the class P_* contains the class PSD of positive semi-definite matrices, i.e., matrices M satisfying $x^T Mx \geq 0$ for all $x \in \mathbb{R}^n$, and the class P of matrices with all the principal minors positive. We denote the strictly feasible set of LCP (1) by $\mathcal{F}^o$, i.e.,

$$\mathcal{F}^o := \{(x,s) \in \mathbb{R}^{2n}_{++} : s = Mx + q\}.$$

Definition 2.3. A $(x,s) \in \mathcal{F}^o$ is an ε-approximate solution if and only if $x^T s \leq \varepsilon$ for $\varepsilon > 0$.

Definition 2.4. A function $f : D(\subset \mathbb{R}) \to \mathbb{R}$ is exponentially convex if and only if $f(\sqrt{x_1x_2}) \leq \frac{1}{2}(f(x_1) + f(x_2))$ for all $x_1, x_2 \in D$.

Definition 2.5. A function $\psi(\in \mathcal{C}^3) : (0,\infty) \to \mathbb{R}$ is eligible if it satisfies the following conditions:

(i) $t\psi''(t) + \psi'(t) > 0,\ t > 0,$
(ii) $\psi'''(t) < 0,\ t > 0,$
(iii) $2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0,\ 0 < t \leq 1,$
(iv) $\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0,\ t > 1,\ \beta > 1.$

In the following we cite some well-known results. For proofs and details see the book of Kojima et al. ([7]).

Proposition 2.6 (Lemma 4.1 in [7])**.** *Let $\kappa \geq 0$. If $M \in \mathbb{R}^{n\times n}$ is a $P_*(\kappa)$ matrix, then*

$$M' = \begin{pmatrix} -M & I \\ S & X \end{pmatrix}$$

is a nonsingular matrix for any positive diagonal matrices $X,\ S\ \in \mathbb{R}^{n\times n}$.

Corollary 2.7. *Let $M \in \mathbb{R}^{n\times n}$ be a $P_*(\kappa)$ matrix and $x,\ s \in \mathbb{R}^n_{++}$. Then for all $a \in \mathbb{R}^n$ the system*

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ S\Delta x + X\Delta s = a \end{cases}$$

has a unique solution $(\Delta x, \Delta s)$.

We use this corollary to prove that the modified Newton-system (7) has a unique solution.

In IMPS to find an ε-approximate solution for (1) we relax the complementarity condition, i.e., the second equation in (1), and we get the following parameterized system:

$$s = Mx + q,\ xs = \mu e,\ x > 0,\ s > 0, \tag{2}$$

where $\mu > 0$. Without loss of generality, we assume that (1) is strictly feasible, i.e., there exists (x^0, s^0) such that $s^0 = Mx^0 + q$, $x^0 > 0$, $s^0 > 0$, and moreover, we have an initial strictly feasible point with $\Psi(x^0, s^0, \mu^0) \le \tau$ for some $\mu^0 > 0$, where Ψ is the proximity function which will be defined in (6). Indeed, we may not have an available strictly feasible point (x^0, s^0). In order to solve this difficulty, we embed (1) to an artificial LCP which has a strictly feasible point([7]). For given strictly feasible point (x^0, s^0) we can always find a $\mu^0 > 0$ such that $\Psi(x^0, s^0, \mu^0) \le \tau$. Since M is a $P_*(\kappa)$ matrix and (1) is strictly feasible, (2) has a unique solution for any $\mu > 0$. We denote the solution of (2) as $(x(\mu), s(\mu))$ for given $\mu > 0$. We also call it *μ-center* for given μ and the solution set $\{(x(\mu), s(\mu)) \mid \ \mu > 0\}$ the *central path* for system (1). Note that the sequence $(x(\mu), s(\mu))$ approaches to the solution (x, s) of the system (1) as $\mu \to 0$ ([7]). For notational convenience we define the following :

$$d = \sqrt{\frac{x}{s}},\ v = \sqrt{\frac{xs}{\mu}},\ d_x = \frac{v\Delta x}{x},\ d_s = \frac{v\Delta s}{s}. \tag{3}$$

Then we have the scaled Newton-system as follows:

$$\begin{cases} -\bar{M}d_x + d_s &= 0, \\ d_x + d_s &= v^{-1} - v, \end{cases} \tag{4}$$

where $\bar{M} = DMD$ and $D = \mathrm{diag}(d)$.

Note that $v^{-1} - v$ in (4) is exactly the negative gradient of the logarithmic barrier function $\Psi_l(v) = \sum_{i=1}^n \psi_l(v_i)$, $\psi_l(t) = (t^2 - 1)/2 - \log\ t$. In this paper we replace the scaled centering equation, the second equation in (4), with

$$d_x + d_s = -\nabla\Psi(v), \tag{5}$$

where

(6)

$$\Psi(v) = \sum_{i=1}^n \psi(v_i),\ \psi(t) = \frac{t^{p+1} - 1}{p+1} + \frac{1}{\sigma}(e^{\sigma(1-t)} - 1),\ t > 0,\ p \in [0, 1],\ \sigma \ge 1.$$

Since $\lim_{t\to 0^+}\psi(t) = \frac{1}{\sigma}(e^{\sigma}-1) - \frac{1}{p+1} < \infty$, $\Psi(v)$ is a finite barrier function which is different from other barrier functions in [2]. We call $\psi(t)$ the kernel function of $\Psi(v)$. We use $\Psi(v)$ as the proximity function between the μ-center and the current iterate for the given $\mu > 0$. Then we have the following modified Newton system:

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ S\Delta x + X\Delta s = -\mu v \nabla\Psi(v). \end{cases} \tag{7}$$

Since M is a $P_*(\kappa)$ matrix and (1) is strictly feasible, the system uniquely defines a search direction $(\Delta x, \Delta s)$ by Corollary 2.7, Throughout the paper, we assume that a proximity parameter τ and a barrier update parameter θ are given and $\tau = O(n)$ and $0 < \theta < 1$, fixed. The algorithm works as follows: We assume that a strictly feasible point (x, s) is given which is in a τ-neighborhood of the given μ-center. Then after decreasing μ to $\mu_+ = (1-\theta)\mu$ for some fixed $\theta \in (0, 1)$, we solve the modified Newton system (7) to obtain the unique search direction. The positivity condition of a new iterate is ensured with the right choice of the step size α. This procedure is repeated until we find a new iterate (x_+, s_+) which is in a τ-neighborhood of the μ_+-center and then we let $\mu := \mu_+$ and $(x, s) := (x_+, s_+)$. Then μ is again reduced by the factor $1-\theta$ and we solve the modified Newton system targeting at the new μ_+-center, and so on. This process is repeated until μ is small enough, e.g. $n\mu \le \varepsilon$.

Algorithm

Input:
A threshold parameter $\tau > 1$;
an accuracy parameter $\varepsilon > 0$;
a fixed barrier update parameter θ, $0 < \theta < 1$;
starting point (x^0, s^0) and $\mu^0 > 0$ such that $\Psi(x^0, s^0, \mu^0) \le \tau$;
begin
$x := x^0$; $s := s^0$; $\mu := \mu^0$;
while $n\mu > \varepsilon$ do
begin
$\mu := (1-\theta)\mu$;
while $\Psi(v) > \tau$ do
begin
solve the modified Newton-system (7) for Δx and Δs;
determine a step size $\tilde{\alpha}$ from (20);
$x := x + \tilde{\alpha}\Delta x$;
$s := s + \tilde{\alpha}\Delta s$;
end
end

end

Remark 2.8. One distinguishes IMPS as large-update methods when $\tau = O(n)$, $\theta = \Theta(1)$ and small-update methods when $\tau = O(1)$, $\theta = \Theta(\frac{1}{\sqrt{n}})$. The small-update methods have better iteration complexity than large-update, but in practice large-update methods are more efficient than small-update.

3. Properties of the kernel function

For $\psi(t)$ in (6) we have

$$\psi^{'}(t) = t^p - e^{\sigma(1-t)},\ \psi^{''}(t) = pt^{p-1} + \sigma e^{\sigma(1-t)}, \tag{8}$$
$$\psi^{'''}(t) = p(p-1)t^{p-2} - \sigma^2 e^{\sigma(1-t)}.$$

Since $\psi^{''}(t) > 0$, $\psi(t)$ is strictly convex, $t > 0$. Note that for $p \in [0, 1]$ and $\sigma \geq 1$, $\psi(1) = \psi^{'}(1) = 0$. Hence $\psi(t)$ is determined by the second derivative, i.e., $\psi(t) = \int_1^t \int_1^\xi \psi^{''}(\varsigma) d\varsigma d\xi$. We also define the norm-based proximity measure $\delta := \delta(v)$ as follows:

$$\delta := \delta(v) = \frac{1}{2} \parallel \nabla\Psi(v) \parallel = \frac{1}{2} \parallel d_x + d_s \parallel . \tag{9}$$

Note that since $\Psi(v)$ is strictly convex and minimal at $v = e$, we have

$$\Psi(v) = 0 \ \Leftrightarrow\ \delta(v) = 0 \ \Leftrightarrow\ v = e.$$

In the following lemma we give key properties which are important in the complexity analysis.

Lemma 3.1 (Lemma 2.1 and Lemma 2.4 in [4]). *Kernel function $\psi(t)$ satisfies the following properties.*

(i) $t\psi^{''}(t) + \psi^{'}(t) \geq 0,\ t \geq \frac{1}{\sigma}$.
(ii) $\psi^{'''}(t) < 0,\ t > 0$,
(iii) $\psi^{''}(t)\psi^{'}(\beta t) - \beta\psi^{'}(t)\psi^{''}(\beta t) > 0,\ t > 1,\ \beta > 1$.
(iv) $t\psi^{'}(t) - \psi(t) \geq 0,\ t \geq 1$,
(v) $\psi(t) \leq \frac{t^{p+1}}{p+1},\ t \geq 1$.

By Lemma 3.1 (i) and Lemma 1 in [12], $\psi(t)$ is exponentially convex for $t \geq \frac{1}{\sigma}$. Let $\varrho : [0, \infty) \to [1, \infty)$ be the inverse function of $\psi(t)$ for $t \geq 1$. Then we have the following lemma.

Lemma 3.2 (Section 5.1 in [4]). *For $\sigma \geq 2$, $\varrho(s) \leq 1 + \sqrt{2(s^2 + s)},\ s \geq 0$.*

Note that at the start of outer iteration of the algorithm, just before the update of μ with the factor $1 - \theta$, we have $\Psi(v) \leq \tau$. Due to the update of μ the vector v is divided by the factor $\sqrt{1-\theta}$ with $0 < \theta < 1$ which in general leads to an increase in the value of $\Psi(v)$. Then, during the subsequent inner iterations, $\Psi(v)$ decreases until it passes the threshold τ again. Hence, during

the process of the algorithm the largest values of $\Psi(v)$ occur just after the updates of μ.

In the following lemma we give an estimate for the effect of a μ-update on the value of $\Psi(v)$.

Lemma 3.3. *Assume that $\sigma \geq 2$. If $\Psi(v) \leq \tau$, then we have for $0 < \theta < 1$*

$$\Psi(\frac{v}{\sqrt{1-\theta}}) \leq \frac{n}{(p+1)(1-\theta)^{\frac{p+1}{2}}} \left(1 + \frac{1}{n}\sqrt{2(\tau^2+n\tau)}\right)^{p+1}.$$

Proof. By the definition of ϱ and $\frac{1}{\sqrt{1-\theta}} \geq 1$, we have $\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(v)}{n}\right) \geq 1$. By Theorem 3.2 in [2], Lemma 3.2, and Lemma 3.1 (v), we have

$$\begin{aligned}\Psi(\frac{v}{\sqrt{1-\theta}}) &\leq n\psi\left(\frac{\varrho\left(\frac{\Psi(v)}{n}\right)}{\sqrt{1-\theta}}\right) \leq n\psi\left(\frac{1+\frac{1}{n}\sqrt{2(\tau^2+n\tau)}}{\sqrt{1-\theta}}\right)\\ &\leq \frac{n}{(p+1)(1-\theta)^{\frac{p+1}{2}}}\left(1+\frac{1}{n}\sqrt{2(\tau^2+n\tau)}\right)^{p+1}.\end{aligned}$$

□

Define

$$L := \frac{n}{(p+1)(1-\theta)^{\frac{p+1}{2}}}\left(1+\frac{1}{n}\sqrt{2(\tau^2+n\tau)}\right)^{p+1}. \tag{10}$$

Then L is an upper bound of $\Psi(v)$ during the process of the algorithm. By taking $\tau = O(n)$ and $\theta = \Theta(1)$, $L = O(n)$. Using Lemma 3.1 (iv), we have the following lemma.

Lemma 3.4 (Lemma 3.1 in [4])**.** *Let δ be the norm-based proximity measure as defined in* (9)*. If $\Psi(v) \geq 1$, then we have*

$$\delta \geq \frac{1}{6}\Psi(v)^{\frac{p}{p+1}}.$$

Lemma 3.5 (Lemma 2.7 in [4])**.** *Suppose that $L \geq 9$ and $\Psi(v) \leq L$. If $\sigma \geq 1 + 2\log(L+1)$, then $v_i > \frac{3}{2\sigma}$ for all $i = 1, \ldots, n$.*

In this paper we assume that $L \geq 9$. Then

$$\sigma = 1 + 2\,\log(L+1) \geq 1 + 2\log 10 \geq 5. \tag{11}$$

Using Lemma 3.5, we have $v_{\min} > \frac{3}{2\sigma}$.

4. Computation of the step size and the decrease

In this section we compute the feasible step size α such that the proximity function is decreasing and the bound for the decrease. At the start of the inner iterations we have $\tau < \Psi(v) \leq L$, where L is defined in (10).

Since M is a $P_*(\kappa)$ matrix and $M\Delta x = \Delta s$ from (7) for $\Delta x \in \mathbb{R}^n$ we have

$$(1+4\kappa)\sum_{i\in J_+}\Delta x_i\Delta s_i + \sum_{i\in J_-}\Delta x_i\Delta s_i \geq 0,$$

where Δx_i and Δs_i denote the i-th components of the vectors Δx and Δs, respectively and $J_+ = \{i \in J : \Delta x_i \Delta s_i \geq 0\}$, $J_- = J - J_+$. Since $d_x d_s = \frac{v^2\Delta x\Delta s}{xs} = \frac{\Delta x\Delta s}{\mu}$ from (3),

$$(1+4\kappa)\sum_{i\in J_+}[d_x]_i[d_s]_i + \sum_{i\in J_-}[d_x]_i[d_s]_i \geq 0, \tag{12}$$

where $[d_x]_i$ and $[d_s]_i$ denote the i-th components of the vectors d_x and d_s, respectively.

In the following lemma we compute the bound for $\|d_x\|$ and $\|d_s\|$. For the proof and details, the reader can refer to Lemma 4.2 in [3].

Lemma 4.1 (Lemma 4.2 in [3])**.** *Let δ be as defined in* (9)*. Then we have* $\sum_{i=1}^n([d_x]_i^2+[d_s]_i^2) \leq 4(1+2\kappa)\delta^2$, $\|d_x\| \leq 2\sqrt{1+2\kappa}\,\delta$*, and* $\|d_s\| \leq 2\sqrt{1+2\kappa}\,\delta$*.*

Lemma 4.2. *Suppose that $L \geq 9$, $\Psi(v) \leq L$ and $\sigma \geq 1 + 2\log(L+1)$. Let δ be the value in* (9) *and σ in* (6)*. Then we have*

$$\|(x^{-1}\Delta x, s^{-1}\Delta s)\| \leq \frac{4}{3}\sqrt{1+2\kappa}\;\sigma\delta. \tag{13}$$

Proof. Using Lemma 4.1 and Lemma 3.5, we have

$$\begin{aligned}\|(x^{-1}\Delta x, s^{-1}\Delta s)\| = \|(v^{-1}d_x, v^{-1}d_s)\| &\leq \frac{1}{v_{\min}}\sqrt{\|d_x\|^2+\|d_s\|^2} \\ &\leq \frac{1}{v_{\min}}2\sqrt{1+2\kappa}\;\delta = \frac{4}{3}\sqrt{1+2\kappa}\;\sigma\delta.\end{aligned}$$

□

Define

$$\hat{\alpha} := \frac{3}{4\sqrt{1+2\kappa}\;\sigma\delta}. \tag{14}$$

Then we have $(x_+, s_+) > 0$ for any $\alpha \in [0, \hat{\alpha}]$, where $x_+ = x + \alpha\Delta x$ and $s_+ = s + \alpha\Delta s$. Indeed, if $\Delta x > 0$, it is clear. Otherwise, there exists an index set $\bar{J}$ such that $\bar{J} = \{i \in J : \Delta x_i < 0\}$. From (13),

$$\max_{i\in\bar{J}}(-x^{-1}\Delta x)_i \leq \| - x^{-1}\Delta x\| \leq \frac{4}{3}\sqrt{1+2\kappa}\;\sigma\delta = \hat{\alpha}^{-1}.$$

Thus

$$\min_{i\in\bar{J}}(-x(\Delta x)^{-1})_i \geq \hat{\alpha} \geq \alpha$$

and hence $x_i + \alpha\Delta x_i > 0$, for any $i \in \bar{J}$ and $\alpha \in [0, \hat{\alpha}]$. Hence $x + \alpha\Delta x > 0$ for any $\alpha \in [0, \hat{\alpha}]$. By the same way, we can get the case $s_+ = s + \alpha\Delta s > 0$ for any $\alpha \in [0, \hat{\alpha}]$.

From (3), we have

$$x_+ = x\left(e + \alpha\frac{\Delta x}{x}\right) = x\left(e + \alpha\frac{d_x}{v}\right) = \frac{x}{v}(v + \alpha d_x),$$
$$s_+ = s\left(e + \alpha\frac{\Delta s}{s}\right) = s\left(e + \alpha\frac{d_s}{v}\right) = \frac{s}{v}(v + \alpha d_s).$$

Then we have

$$v_+^2 = \frac{x_+ s_+}{\mu} = (v + \alpha d_x)(v + \alpha d_s).$$

By Lemma 3.1 (i) and Lemma 1 in [12], $\Psi(v)$ is exponentially convex for $v \geq \frac{1}{\sigma}e$. Now we assume that $v + \alpha d_x \geq \frac{1}{\sigma}e$ and $v + \alpha d_s \geq \frac{1}{\sigma}e$ for some α which will be computed in (20). Then we have

$$\Psi(v_+) = \Psi(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}) \ \leq \ \frac{1}{2}\ (\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)).$$

For given $\mu > 0$ by letting $f(\alpha)$ be the difference of the new and old proximity measures, i.e.,

$$f(\alpha) = \Psi(v_+) - \Psi(v).$$

Then we have $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) := \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)) - \Psi(v).$$

Note that $f(0) = f_1(0) = 0$. By taking the derivative of $f_1(\alpha)$ with respect to α, we have

$$f_1^{'}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi^{'}(v_i + \alpha[d_x]_i)[d_x]_i + \psi^{'}(v_i + \alpha[d_s]_i)[d_s]_i).$$

Using (5) and the definition of δ, we have

$$f_1^{'}(0) = \frac{1}{2}\nabla\Psi(v)^T(d_x + d_s) = -\frac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v) = -2\delta^2. \tag{15}$$

By taking the derivative of $f_1^{'}(\alpha)$ with respect to α, we have

$$f_1^{''}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi^{''}(v_i + \alpha[d_x]_i)[d_x]_i^2 + \psi^{''}(v_i + \alpha[d_s]_i)[d_s]_i^2). \tag{16}$$

In the following lemma we obtain the upper bound for the difference of the new and old proximity measures.

Lemma 4.3 (Modification of Lemma 4.3 in [3]). *For $\alpha \in [0, \hat{\alpha}]$ we have*

$$f_1^{''}(\alpha) \leq 2(1 + 2\kappa)\ \delta^2\psi^{''}(v_{\min} - 2\alpha\sqrt{1 + 2\kappa}\ \delta).$$

Lemma 4.4 (Modification of Lemma 4.4 in [3]). *For $\alpha \in [0, \hat{\alpha}]$ we have $f_1^{'}(\alpha) \leq 0$ if α is satisfying*

$$-\psi'(v_{\min} - 2\alpha\delta\sqrt{1 + 2\kappa}) + \psi^{'}(v_{\min}) \leq \frac{2\delta}{\sqrt{1 + 2\kappa}}. \tag{17}$$

Let $\rho : [0, -\frac{1}{2}\psi^{'}(0)) \to (0, 1]$ be the inverse function of $-\frac{1}{2}\psi^{'}(t)$ for $t \in (0, 1]$. Letting $s = -\frac{1}{2}\psi^{'}(t)$, we have $t = \rho(s)$ and $2s = -\psi^{'}(t) = e^{\sigma(1-t)} - t^p$. Since $t \leq 1$, we have

$$e^{\sigma(1-t)} = 2s + t^p \leq 2s + 1. \tag{18}$$

In the following lemma, we compute the feasible step size α such that the proximity measure is decreasing when we take a new iterate for fixed μ.

Lemma 4.5 (Modification of Lemma 4.5 in [3])**.** *Let ρ be the inverse function of $-\frac{1}{2}\psi^{'}(t)$, $t \in (0, 1]$. Then the largest step size α that satisfies* (17) *is given by*

$$\bar{\alpha} := \frac{1}{2\delta\sqrt{1+2\kappa}}\left(\rho(\delta) - \rho\left(\left(1 + \frac{1}{\sqrt{1+2\kappa}}\right)\delta\right)\right). \tag{19}$$

In the following lemma we compute the lower bound for $\bar{\alpha}$ in (19).

Lemma 4.6. *Let ρ and $\bar{\alpha}$ be the values as defined in Lemma 4.5. Then we have*

$$\bar{\alpha} \geq \frac{1}{1+2\kappa}\ \frac{1}{\psi^{''}(\rho((1 + \frac{1}{\sqrt{1+2\kappa}})\delta))} \geq \frac{1}{16(1+2\kappa)\sigma\delta}.$$

Proof. By the definition of ρ, $-\frac{1}{2}\psi^{'}(\rho(\delta)) = \delta$, i.e., $-\psi^{'}(\rho(\delta)) = 2\delta$. By taking the derivative of $-\psi^{'}$ with respect to δ, we get $-\psi^{''}(\rho(\delta))\rho^{'}(\delta) = 2$. So we have $\rho^{'}(\delta) = -\frac{2}{\psi^{''}(\rho(\delta))} < 0$ since $\psi^{''} > 0$. Hence ρ is monotonically decreasing. Using (19) and the fundamental theorem of calculus, we have $\bar{\alpha} = \frac{1}{2\delta\sqrt{1+2\kappa}}(\rho(\delta) - \rho((1 + \frac{1}{\sqrt{1+2\kappa}})\delta)) = \frac{1}{2\delta\sqrt{1+2\kappa}}\int_{(1+\frac{1}{\sqrt{1+2\kappa}})\delta}^{\delta}\rho^{'}(\xi)d\xi = \frac{1}{\delta\sqrt{1+2\kappa}}\int_{\delta}^{(1+\frac{1}{\sqrt{1+2\kappa}})\delta}\frac{d\xi}{\psi^{''}(\rho(\xi))}$. Since $\delta \leq \xi \leq (1 + \frac{1}{\sqrt{1+2\kappa}})\delta$ and ρ is monotonically decreasing, $\rho(\xi) \geq \rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta)$. Since $\psi^{''}$ is monotonically decreasing, $\psi^{''}(\rho(\xi)) \leq \psi^{''}(\rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))$. Hence $\frac{1}{\psi^{''}(\rho(\xi))} \geq \frac{1}{\psi^{''}(\rho((1+\frac{1}{\sqrt{1+2\kappa}})\delta))}$. Therefore we have

$$\begin{aligned}\bar{\alpha} &\geq \frac{1}{\delta\sqrt{1+2\kappa}}\frac{1}{\psi^{''}(\rho((1 + \frac{1}{\sqrt{1+2\kappa}})\delta))}\int_{\delta}^{(1+\frac{1}{\sqrt{1+2\kappa}})\delta} d\xi \\ &= \frac{1}{1+2\kappa}\ \frac{1}{\psi^{''}(\rho((1 + \frac{1}{\sqrt{1+2\kappa}})\delta))}.\end{aligned}$$

For the notational convenience let $a := 1 + \frac{1}{\sqrt{1+2\kappa}}$. Then $a \leq 2$. Letting $t = \rho(a\delta)$, we have $-\frac{1}{2}\psi^{'}(t) = a\delta$, i.e., $2a\delta = -\psi^{'}(t)$. Using the definition of ρ and (18), we have

$$e^{\sigma(1-t)} \leq 2a\delta + 1$$

and $\frac{1}{\sigma} \le t \le 1$. Using Lemma 3.4, we have for $p \in [0,1]$ and $\frac{1}{\sigma} \le t \le 1$,

$$\begin{aligned}\bar{\alpha} &\ge \frac{1}{(1+2\kappa)}\frac{1}{\psi''(\rho(a\delta))} = \frac{1}{(1+2\kappa)}\frac{1}{\psi''(t)} = \frac{1}{(1+2\kappa)}\frac{1}{pt^{p-1}+\sigma e^{\sigma(1-t)}} \\ &\ge \frac{1}{(1+2\kappa)}\frac{1}{p\sigma^{1-p}+\sigma e^{\sigma(1-t)}} \ge \frac{1}{(1+2\kappa)}\frac{1}{\sigma(1+e^{\sigma(1-t)})} \\ &\ge \frac{1}{(1+2\kappa)}\frac{1}{\sigma(2a\delta+2)} = \frac{1}{2(1+2\kappa)}\frac{1}{\sigma(a\delta+1)} \\ &\ge \frac{1}{2(1+2\kappa)}\frac{1}{\sigma(2\delta+6\delta)} \ge \frac{1}{16(1+2\kappa)\sigma\delta}.\end{aligned}$$

□

Define

$$\tilde{\alpha} := \frac{1}{16(1+2\kappa)\sigma\delta}. \tag{20}$$

Then by Lemma 4.6 and (14), we have $\tilde{\alpha} \le \min\{\bar{\alpha}, \hat{\alpha}\}$ and hence $\tilde{\alpha}$ is a strictly feasible step size such that the proximity function decreases. Using Lemma 4.1 and (20), we have for all $i = 1, \ldots, n$

$$v_i + \tilde{\alpha}[d_x]_i \ge v_{\min} - 2\sqrt{1+2\kappa}\tilde{\alpha}\delta \ge \frac{3}{2\sigma} - \frac{1}{8\sqrt{1+2\kappa}\sigma} \ge \frac{11}{8\sigma} \ge \frac{1}{\sigma},$$

and hence $v+\tilde{\alpha}d_x \ge \frac{1}{\sigma}e$. By the same way, $v_i+\tilde{\alpha}[d_s]_i \ge v_{\min}-2\sqrt{1+2\kappa}\tilde{\alpha}\delta \ge \frac{1}{\sigma}$, for $i = 1, \ldots, n$, and hence $v + \tilde{\alpha}d_s \ge \frac{1}{\sigma}e$. Thus we will use $\tilde{\alpha}$ as the default step size in the algorithm.

Lemma 4.7 (Lemma 12 in [12]). *Let $h(t)$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and attain its (global) minimum at $t^* > 0$. If $h''(t)$ is increasing for $t \in [0, t^*]$, then $h(t) \le \frac{th'(0)}{2}$ for $0 \le t \le t^*$.*

In the following we obtain the bound for the decrease of the proximity function value.

Lemma 4.8 (Lemma 4.8 in [3]). *For $\alpha \le \tilde{\alpha}$ we have $f(\alpha) \le -\alpha\delta^2$.*

Proof. Define the univariate function h as follows:

$$\begin{aligned}h(0) = f_1(0) = 0,\ h'(0) = f_1'(0) = -2\delta^2, \\ h''(\alpha) = 2(1+2\kappa)\delta^2\psi''(v_{\min} - 2\alpha\sqrt{1+2\kappa}\delta).\end{aligned}$$

By Lemma 4.3, $f_1''(\alpha) \le h''(\alpha)$. So we have $f_1'(\alpha) \le h'(\alpha)$ and $f_1(\alpha) \le h(\alpha)$. By the definition of $h(\alpha)$ and $\psi''(t) > 1$, $h''(\alpha) \ge 2(1+\kappa)\delta^2$. This implies that $h(\alpha)$ is strongly convex and hence $h(\alpha)$ attains its global minimum for some $\alpha^* > 0$. By taking $\alpha \le \bar{\alpha}$, with $\bar{\alpha}$ as defined in Lemma 4.5, using the

fundamental theorem of calculus, and Lemma 4.4, we have

$$\begin{aligned} h^{'}(\alpha) &= h^{'}(0) + \int_0^{\alpha} h^{''}(\xi)d\xi \\ &= -2\delta^2 + 2(1+2\kappa)\delta^2 \int_0^{\alpha} \psi''(v_{\min} - 2\xi\sqrt{1+2\kappa}\delta)d\xi \\ &= -2\delta^2 - \sqrt{1+2\kappa}\delta(\psi^{'}(v_{\min} - 2\alpha\sqrt{1+2\kappa}\delta) - \psi^{'}(v_{\min})) \\ &\leq -2\delta^2 + \sqrt{1+2\kappa}\delta\frac{2\delta}{\sqrt{1+2\kappa}} = 0. \end{aligned}$$

Since $h^{'''}(\alpha) = -4(1+2\kappa)^{\frac{3}{2}}\ \delta^3\ \psi^{'''}(v_{\min} - 2\alpha\sqrt{1+2\kappa}\delta)$ and $\psi^{'''} < 0$, $h^{''}(\alpha)$ is increasing in α. By Lemma 4.7, we have $f_1(\alpha) \leq h(\alpha) \leq \frac{1}{2}\alpha h^{'}(0) = -\alpha\delta^2$. Since $f(\alpha) \leq f_1(\alpha)$, the proof is completed. □

In the following theorem we obtain the upper bound for the difference $f(\alpha)$ between the new and old proximity measures in the algorithm.

Theorem 4.9. *Let $\tilde{\alpha}$ be the step size as defined in* (20)*. Then we have*

$$f(\tilde{\alpha}) \leq -\frac{\Psi(v)^{\frac{p}{p+1}}}{96(1+2\kappa)\sigma}, \tag{21}$$

where $p \in [0,1]$, σ in (11).

Proof. Using Lemma 4.8, (19) and Lemma 3.4, we have

$$f(\tilde{\alpha}) \leq -\tilde{\alpha}\delta^2 = -\frac{\delta^2}{16(1+2\kappa)\sigma\delta} = -\frac{\delta}{16(1+2\kappa)\sigma} \leq -\frac{\Psi(v)^{\frac{p}{p+1}}}{96(1+2\kappa)\sigma}.$$ □

5. Complexity results

In this section we compute the total number of iterations for the algorithm to get an ε-approximate solution. We cite the following technical lemma to obtain iteration bounds. For the proof the reader can refer [12].

Lemma 5.1 (Lemma A.2 in [2])**.** *Let $t_0, t_1, \ldots, t_K$ be a sequence of positive numbers such that $t_{k+1} \leq t_k - \beta t_k^{1-\gamma},\ k = 0, 1, \ldots, K-1$, where $\beta > 0$ and $0 < \gamma \leq 1$. Then $K \leq \lfloor \frac{t_0^{\gamma}}{\beta\gamma} \rfloor$.*

We define the value of $\Psi(v)$ after the μ-update as Ψ_0 and the subsequent values in the same outer iteration are denoted as Ψ_k, $k = 1,\ 2, \ldots$. Let K be the total number of inner iterations in the outer iteration. Then by the definition of K, we have

$$\Psi_{K-1} > \tau,\ 0 \leq \Psi_K \leq \tau.$$

In the following lemma, we obtain the upper bound for the total number of inner iterations which we needed to return to the τ-neighborhood again.

Lemma 5.2. *Let K be the total number of inner iterations in an outer iteration. Then we have*

$$K \leq 96(1+2\kappa)\sigma(p+1)\Psi_0^{\frac{1}{p+1}},$$

where Ψ_0 denotes the value of $\Psi(v)$ after the μ-update.

Proof. Using Theorem 4.9, we have $f(\tilde{\alpha}) \leq -\frac{\Psi(v)^{\frac{p}{p+1}}}{96(1+2\kappa)\sigma}$. This implies that $\Psi_{k+1} \leq \Psi_k - \beta {\Psi_k}^{1-\gamma}$, $k = 0, 1, 2, \ldots, K-1$, where $\beta = \frac{1}{96(1+2\kappa)\sigma}, \gamma = \frac{1}{p+1}$. Hence by Lemma 5.1, K is bounded above by $K \leq \frac{\Psi_0^{\gamma}}{\beta\gamma} = 96(1+2\kappa)\sigma(p+1)\Psi_0^{\frac{1}{p+1}}$. □

From (10), we have

$$\Psi_0 \leq L := \frac{n}{(p+1)(1-\theta)^{\frac{p+1}{2}}}\left(1 + \frac{1}{n}\sqrt{2(\tau^2+n\tau)}\right)^{p+1}.$$

From Lemma 5.2, we have

$$K \leq 96(1+2\kappa)\sigma(p+1)^{\frac{p}{p+1}}\frac{n^{\frac{1}{p+1}}}{\sqrt{1-\theta}}\left(1 + \frac{1}{n}\sqrt{2(\tau^2+n\tau)}\right).$$

The upper bound for the total number of iterations is obtained by multiplying the number K by the number of central path parameter updates. If the central path parameter μ has the initial value μ^0 and is updated by multiplying $1-\theta$, with $0 < \theta < 1$, then after at most $\lceil \frac{1}{\theta} \log \frac{n\mu^0}{\epsilon} \rceil$ iterations we have $n\mu \leq \epsilon$. Thus the total number of iterations is bounded above by $96(1+2\kappa)\sigma(p+1)^{\frac{p}{p+1}}\frac{n^{\frac{1}{p+1}}}{\theta\sqrt{1-\theta}}(1 + \frac{1}{n}\sqrt{2(\tau^2+n\tau)})\log\frac{n\mu^0}{\epsilon}$. So we obtain the main result.

Theorem 5.3. *Let a $P_*(\kappa)$ linear complementarity problem be given, where $\kappa \geq 0$. Assume that a strictly feasible starting point (x^0, s^0) is available with $\Psi(x^0, s^0, \mu^0) \leq \tau$ for some $\mu^0 > 0$ and $L \geq 9$ and $\sigma = 1 + 2\log(L+1)$. Then the total number of iterations to get an ε-approximate solution is bounded above by*

$$\left\lceil 96(1+2\kappa)\sigma(p+1)^{\frac{p}{p+1}}\frac{n^{\frac{1}{p+1}}}{\sqrt{1-\theta}}\left(1 + \frac{1}{n}\sqrt{2(\tau^2+n\tau)}\right)\right\rceil \left\lceil \frac{1}{\theta}\log\frac{n\mu^0}{\epsilon}\right\rceil.$$

Remark 5.4. By taking $\tau = O(n)$, $\theta = \Theta(1)$, and $\sigma = O(\log n)$, the algorithm has $O((1+2\kappa)n^{\frac{1}{p+1}}\log n \log\frac{n}{\varepsilon})$ iteration complexity. If $p = 1$, then we have $O((1+2\kappa)\sqrt{n}\log n\log\frac{n}{\varepsilon})$ complexity which is so far the best known complexity for large-update.

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