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ON THE REFLEXIVE SOLUTIONS OF THE MATRIX EQUATION $AXB + CYD = E$

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Abstract. A matrix $P \in \mathbb{C}^{n\times n}$ is called a generalized reflection matrix if $P^* = P$ and $P^2 = I$. An $n \times n$ complex matrix A is said to be a reflexive (anti-reflexive) matrix with respect to the generalized reflection matrix P if $A = PAP$ ($A = -PAP$). It is well-known that the reflexive and anti-reflexive matrices with respect to the generalized reflection matrix P have many special properties and widely used in engineering and scientific computations. In this paper, we give new necessary and sufficient conditions for the existence of the reflexive (anti-reflexive) solutions to the linear matrix equation $AXB + CYD = E$ and derive representation of the general reflexive (anti-reflexive) solutions to this matrix equation. By using the obtained results, we investigate the reflexive (anti-reflexive) solutions of some special cases of this matrix equation.

1. Introduction

Throughout the paper, we denote the complex n-vector space by $\mathbb{C}^n$ and the set of $m \times n$ complex matrices by $\mathbb{C}^{m\times n}$. The symbols A^t and A^* stand for the transpose and the conjugate transpose of a complex matrix A, respectively. We denote a reflexive inverse of a matrix A by A^+ which satisfies simultaneously $AA^+A = A$ and $A^+AA^+ = A^+$. We denote by I_n the $n \times n$ identity matrix. We also write it as I, when the dimension of this matrix is clear. Moreover, given a matrix A, define $L_A = I - A^+A$ and $R_A = I - AA^+$, where A^+ is any arbitrary but fixed reflexive inverse of the matrix A.

Let P be a generalized reflection matrix of size n, that is, $P^* = P$ and $P^2 = I$. The following two special classes of subspaces in $\mathbb{C}^{n\times n}$

$$\mathbb{C}_r^{n\times n}(P) = \{A \in \mathbb{C}^{n\times n} : A = PAP\},$$
$$\mathbb{C}_a^{n\times n}(P) = \{A \in \mathbb{C}^{n\times n} : A = -PAP\},$$

are proposed by Chen [3], Chen and Sameh [5]. The matrices A in $\mathbb{C}_r^{n\times n}(P)$ and B in $\mathbb{C}_a^{n\times n}(P)$ are, respectively, said to be the reflexive and anti-reflexive

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matrices with respect to the generalized reflection matrix P. The reflexive and anti-reflexive matrices with respect to the generalized reflection matrix P have many special properties and widely used in engineering and scientific computations [3, 4, 5]. We know that matrix equation is one of the topics of very active research in the computational mathematics, and a large number of papers have presented various methods for solving several matrix equations [8, 11, 12, 25, 26, 27, 28]. The matrix equation

$$AX - YB = C,$$

with the unknown matrices X and Y has been investigated by Baksalary and Kala [1], Flanders and Wimmer [20], and Roth [22]. In [9], Cvetković-Ilić considered Re-nnd solutions of the matrix equation

$$AXB = C,$$

with respect to X. In [23], Wang considered the system of matrix equations

$$A_1XB_1 = C_1 \qquad \text{and} \qquad A_2XB_2 = C_2,$$

over an arbitrary regular ring with identity and derived the necessary and sufficient conditions for the existence and the expression of the general solution to the system. Also Wang in [24], considered the system of four linear matrix equations over an arbitrary von Neumann regular ring with identity. For the matrix equation

$$AXB + CYD = E, \tag{1.1}$$

solvability conditions and general solutions have been derived [2, 6, 29] by using generalized inverses and the general singular value decomposition (GSVD) of the matrices. In [13], several iterative algorithms are proposed to solve (1.1) over reflexive and anti-reflexive matrices. Dehghan and Hajarian [10] proposed an iterative method for solving the generalized coupled Sylvester matrix equations over reflexive matrices. Zhou and Duan [30, 31] established the solution of the several generalized Sylvester matrix equations. On the solutions of matrix equations, Ding and Chen presented the hierarchical gradient-iterative (HGI) algorithms for general matrix equations [14, 19] and hierarchical least-squares-iterative (HLSI) algorithms for generalized coupled Sylvester matrix equations and general coupled matrix equations [15, 16]. The HGI algorithms [14, 19] and HLSI algorithms [16, 18, 19] for solving general (coupled) matrix equations are an innovational and computationally efficient numerical ones and were proposed based on the hierarchical identification principle [15, 17] which regards the unknown matrix as the system parameter matrix to be identified.

The reflexive and anti-reflexive matrices are two classes of important matrices and have practical applications in information theory, linear system theory, linear estimate theory and numerical analysis. We know that the reflexive and anti-reflexive solutions of the matrix equation (1.1) have not been concerned yet. In this paper, we will discuss about the reflexive and anti-reflexive solutions of (1.1).

This paper is organized as follows: In Section 2 we first review some structure properties of the generalized reflection matrix P and the subspaces $\mathbb{C}_r^{n\times n}(P)$ and $\mathbb{C}_a^{n\times n}(P)$ of $\mathbb{C}^{n\times n}$. Then we present the necessary and sufficient conditions for the existence of the reflexive (anti-reflexive) solutions and give the representation of the reflexive (anti-reflexive) solutions to the matrix equation (1.1). In Section 3 some special cases of the matrix equation (1.1) will be considered.

2. Main results

In this section we first review some structure properties of the generalized reflection matrix P and the subspaces $\mathbb{C}_r^{n\times n}(P)$ and $\mathbb{C}_a^{n\times n}(P)$ of $\mathbb{C}^{n\times n}$. Then we give the necessary and sufficient conditions for the existence and the expression of the reflexive and anti-reflexive solutions to the matrix equation (1.1).

Let $P \in \mathbb{C}^{n\times n}$ be a generalized reflection matrix. We can express the matrix P by the following form [7, 21]:

$$P = U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^*, \tag{2.1}$$

where $U = (U_1, U_2)$ is an unitary matrix and $U_1 \in \mathbb{C}^{n\times r}$, $U_2 \in \mathbb{C}^{n\times(n-r)}$.

Lemma 2.1 ([21])**.** *The matrix $A \in \mathbb{C}_r^{n\times n}(P)$ if and only if A can be expressed as*

$$A = U \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix} U^*, \tag{2.2}$$

where $A_1 \in \mathbb{C}^{r\times r}$, $A_4 \in \mathbb{C}^{(n-r)\times(n-r)}$ and U, U^* are as in (2.1)*.*

Lemma 2.2 ([21])**.** *The matrix $A \in \mathbb{C}_a^{n\times n}(P)$ if and only if A can be expressed as*

$$A = U \begin{pmatrix} 0 & A_2 \\ A_3 & 0 \end{pmatrix} U^*, \tag{2.3}$$

where $A_2 \in \mathbb{C}^{r\times(n-r)}$, $A_3 \in \mathbb{C}^{(n-r)\times r}$ and U, U^* are as in (2.1)*.*

Without loss of generality, we assume that matrices A, B, C, D, $E \in \mathbb{C}^{n\times n}$ have the following decompositions:

$$\begin{cases} A = U \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} U^*, \ B = U \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} U^*, \ C = U \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} U^*, \\ D = U \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} U^*, \ E = U \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix} U^* \text{ and } E' = U^*EU, \end{cases} \tag{2.4}$$

where $A_1, B_1, C_1, D_1, E_1 \in \mathbb{C}^{r\times r}$ and $A_4, B_4, C_4, D_4, E_4 \in \mathbb{C}^{(n-r)\times(n-r)}$.

The following theorems represent the general conditions for the existence and the expression of the reflexive and anti-reflexive solutions of the matrix equation (1.1), respectively.

Theorem 2.1. *Given $A, B, C, D, E \in \mathbb{C}^{n\times n}$ and a generalized reflection matrix P of size n. Then the following conditions are equivalent*:

(1) *The matrix equation $AXB + CYD = E$ has the reflexive solutions $X,\ Y \in \mathbb{C}_r^{n\times n}(P)$.*

(2) *The following matrix equation has a solution*:

$$A'X_1B' + A''X_4B'' + C'Y_1D' + C''Y_4D'' = E', \tag{2.5}$$

where

$$\begin{cases} A' = (A_1^t, A_3^t)^t,\ B' = (B_1, B_2),\ A'' = (A_2^t, A_4^t)^t,\ B'' = (B_3, B_4), \\ C' = (C_1^t, C_3^t)^t,\ D' = (D_1, D_2),\ C'' = (C_2^t, C_4^t)^t,\ D'' = (D_3, D_4). \end{cases} \tag{2.6}$$

(3) *The following system of matrix equations has a solution*:

$$\begin{cases} A_1X_1B_1 + A_2X_4B_3 + C_1Y_1D_1 + C_2Y_4D_3 = E_1, \\ A_1X_1B_2 + A_2X_4B_4 + C_1Y_1D_2 + C_2Y_4D_4 = E_2, \\ A_3X_1B_1 + A_4X_4B_3 + C_3Y_1D_1 + C_4Y_4D_3 = E_3, \\ A_3X_1B_2 + A_4X_4B_4 + C_3Y_1D_2 + C_4Y_4D_4 = E_4. \end{cases} \tag{2.7}$$

In that case, the reflexive solutions of the matrix equation $AXB + CYD = E$ can be expressed by the following

$$X = U\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}U^* \quad \text{and} \quad Y = U\begin{pmatrix} Y_1 & 0 \\ 0 & Y_4 \end{pmatrix}U^*.$$

Proof. First we show that (2) $\Leftrightarrow$ (3): Substituting (2.6) into (2.5), gives us the system of matrix equations (2.7). This implies that (2) $\Leftrightarrow$ (3).

(1) $\Leftrightarrow$ (3): Suppose that the matrix equation (1.1) has the reflexive solutions $X \in \mathbb{C}_r^{n\times n}(P)$ and $Y \in \mathbb{C}_r^{n\times n}(P)$. By Lemma 2.1, there exist $X_1, Y_1 \in \mathbb{C}^{r\times r}$ and $X_4, Y_4 \in \mathbb{C}^{(n-r)\times(n-r)}$ such that:

$$X = U\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}U^* \quad \text{and} \quad Y = U\begin{pmatrix} Y_1 & 0 \\ 0 & Y_4 \end{pmatrix}U^*.$$

Now using the decompositions (2.4), from $AXB + CYD = E$, we can get

$$\begin{pmatrix} A_1X_1B_1 + A_2X_4B_3 + C_1Y_1D_1 + C_2Y_4D_3 & A_1X_1B_2 + A_2X_4B_4 + C_1Y_1D_2 + C_2Y_4D_4 \\ A_3X_1B_1 + A_4X_4B_3 + C_3Y_1D_1 + C_4Y_4D_3 & A_3X_1B_2 + A_4X_4B_4 + C_3Y_1D_2 + C_4Y_4D_4 \end{pmatrix} = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}.$$

If the system of matrix equations (2.7) has a solution, then

$$X = U\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}U^* \in \mathbb{C}_r^{n\times n}(P),\ Y = U\begin{pmatrix} Y_1 & 0 \\ 0 & Y_4 \end{pmatrix}U^* \in \mathbb{C}_r^{n\times n}(P)$$

and $AXB + CYD = E$. $\square$

Similarly to the proof of Theorem 2.1, we can prove the following theorem.

Theorem 2.2. *Given $A, B, C, D, E \in \mathbb{C}^{n\times n}$ and a generalized reflection matrix P of size n. Then the following conditions are equivalent*:

(1) *The matrix equation $AXB + CYD = E$ has the anti-reflexive solutions $X,\ Y \in \mathbb{C}_a^{n\times n}(P)$.*

(2) *The following matrix equation has a solution*:

$$A''X_3B' + A'X_2B'' + C''Y_3D' + C'Y_2D'' = E', \tag{2.8}$$

where

$$\begin{cases} A' = (A_1^t, A_3^t)^t,\ B' = (B_1, B_2),\ A'' = (A_2^t, A_4^t)^t,\ B'' = (B_3, B_4), \\ C' = (C_1^t, C_3^t)^t,\ D' = (D_1, D_2),\ C'' = (C_2^t, C_4^t)^t,\ D'' = (D_3, D_4). \end{cases} \tag{2.9}$$

(3) *The following system of matrix equations has a solution*:

$$\begin{cases} A_2X_3B_1 + A_1X_2B_3 + C_2Y_3D_1 + C_1Y_2D_3 = E_1, \\ A_2X_3B_2 + A_1X_2B_4 + C_2Y_3D_2 + C_1Y_2D_4 = E_2, \\ A_4X_3B_1 + A_3X_2B_3 + C_4Y_3D_1 + C_3Y_2D_3 = E_3, \\ A_4X_3B_2 + A_3X_2B_4 + C_4Y_3D_2 + C_3Y_2D_4 = E_4. \end{cases} \tag{2.10}$$

In that case, the anti-reflexive solutions of the matrix equation $AXB+CYD = E$ can be expressed by the following

$$X = U\begin{pmatrix} 0 & X_2 \\ X_3 & 0 \end{pmatrix}U^* \qquad \textit{and} \qquad Y = U\begin{pmatrix} 0 & Y_2 \\ Y_3 & 0 \end{pmatrix}U^*.$$

The above theorems are very general and they represent just the starting point in the search for the more operative condition in the particular cases.

3. Some special cases

In this section, we will consider the special cases when $B_1 = B_2 = D_1 = D_2 = 0$ or $A_1 = A_3 = C_1 = C_3 = 0$ ($B_1 = B_2 = D_1 = D_2 = 0$ or $A_2 = A_4 = C_2 = C_4 = 0$). We find necessary and sufficient conditions for the existence of the reflexive (anti-reflexive) solutions $X \in \mathbb{C}_r^{n\times n}\ (P)$ and $Y \in \mathbb{C}_r^{n\times n}\ (P)$ ($X \in \mathbb{C}_a^{n\times n}\ (P)$ and $Y \in \mathbb{C}_a^{n\times n}\ (P)$). The following cases are important in applications.

Theorem 3.1. *Let $A, B, C, D, E \in \mathbb{C}^{n\times n}$ be given matrices. If $B_1 = B_2 = D_1 = D_2 = 0$ or $A_1 = A_3 = C_1 = C_3 = 0$, then the following conditions are equivalent*:

(1) *The matrix equation $AXB + CYD = E$ has the reflexive solutions $X \in \mathbb{C}_r^{n\times n}(P)$ and $Y \in \mathbb{C}_r^{n\times n}(P)$*;

(2) $R_MR_{A''}E' = 0,\ R_{A''}E'L_{D''} = 0,\ E'L_{B''}L_N = 0,\ R_{C''}E'L_{B''} = 0$;

(3) $MM^+R_{A''}E'D''^+D'' = R_{A''}E',\ C''C''^+E'L_{B''}N^+N = E'L_{B''}$;

(4) $R_P R_{C''} E' = 0,\ R_{C''} E' L_{B''} = 0,\ R_{A''} E' L_{D''} = 0,\ E' L_{D''} L_Q = 0;$

(5) $PP^+ R_{C''} E' B''^+ B'' = R_{C''} E',\ A'' A''^+ E' L_{D''} Q^+ Q = E' L_{D''}$; *where* $A'' = (A_2^t, A_4^t)^t,\ B'' = (B_3, B_4),\ C'' = (C_2^t, C_4^t)^t,\ D'' = (D_3, D_4),\ M = R_{A''} C'',\ N = D'' L_{B''},\ S = C'' L_M,\ T = R_{D''} N,\ F = N L_T,\ G = R_S C'',\ P = R_{C''} A'',\ Q = B'' L_{D''},\ S_1 = A'' L_P,\ T_1 = R_{B''} Q$ *and* $G_1 = R_{S_1} A''$.

In that case, the reflexive solutions of the matrix equation $AXB + CYD = E$ *can be expressed by the following*

$$X = U \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^* \qquad \text{and} \qquad Y = U \begin{pmatrix} Y_1 & 0 \\ 0 & Y_4 \end{pmatrix} U^*,$$

where

$$\begin{aligned} X_4 &= A''^+(E' - C''Y_4 D'')B''^+ + L_{A''} J + Z R_{B''}, \\ Y_4 &= M^+ R_{A''} E' D''^+ + L_M (V - S^+ S V N N^+) \\ &\quad - L_M S^+ C'' L_G W T N^+ + (W - G^+ G W T T^+) R_{D''}, \end{aligned}$$

or

$$\begin{aligned} X_4 &= P^+ R_{C''} E' B''^+ + L_P (V_1 - S_1^+ S_1 V_1 Q Q^+) \\ &\quad - L_P S_1^+ A'' L_{G_1} W_1 T_1 Q^+ + (W_1 - G_1^+ G_1 W_1 T_1 T_1^+) R_{B''}, \\ Y_4 &= C''^+(E' - C'' X_4 D'') D''^+ + L_{C''} J_1 + Z_1 R_{D''}, \end{aligned}$$

and $X_1, Y_1, J, J_1, V, V_1, W, W_1, Z, Z_1$ *are arbitrary matrices with appropriate dimensions.*

Proof. Let $B_1 = B_2 = D1 = D_2 = 0$ or $A_1 = A_3 = C_1 = C_3 = 0$. Suppose that $X \in \mathbb{C}_r^{n\times n}(P)$ and $Y \in \mathbb{C}_r^{n\times n}(P)$ are the reflexive solutions of the matrix equation (1.1). We can assume that X and Y are represented by

$$X = U \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^* \qquad \text{and} \qquad Y = U \begin{pmatrix} Y_1 & 0 \\ 0 & Y_4 \end{pmatrix} U^*.$$

Now, from (2.5) and (2.6) we can get

$$A'' X_4 B'' + C'' Y_4 D'' = E'. \tag{3.1}$$

It is well-known that [23] the following conditions are equivalent:

(1) The matrix equation $A'' X_4 B'' + C'' Y_4 D'' = E'$ has a solution.

(2) $R_M R_{A''} E' = 0,\ R_{A''} E' L_{D''} = 0,\ E' L_{B''} L_N = 0,\ R_{C''} E' L_{B''} = 0.$

(3) $MM^+ R_{A''} E' D''^+ D'' = R_{A''} E',\ C'' C''^+ E' L_{B''} N^+ N = E' L_{B''}.$

(4) $R_P R_{C''} E' = 0,\ R_{C''} E' L_{B''} = 0,\ R_{A''} E' L_{D''} = 0,\ E' L_{D''} L_Q = 0.$

(5) $PP^+ R_{C''} E' B''^+ B'' = R_{C''} E',\ A'' A''^+ E' L_{D''} Q^+ Q = E' L_{D''}.$

In that case the general solutions of (3.1) are represented by:

$$\begin{aligned} X_4 &= A''^+(E' - C''Y_4 D'')B''^+ + L_{A''} J + Z R_{B''}, \\ Y_4 &= M^+ R_{A''} E' D''^+ + L_M (V - S^+ S V N N^+) \\ &\quad - L_M S^+ C'' L_G W T N^+ + (W - G^+ G W T T^+) R_{D''}, \end{aligned}$$

or

$$\begin{aligned}X_4 &= P^+R_{C''}E'B''^+ + L_P(V_1 - S_1^+S_1V_1QQ^+)\\&\quad - L_PS_1^+A''L_{G_1}W_1T_1Q^+ + (W_1 - G_1^+G_1W_1T_1T_1^+)R_{B''},\\Y_4 &= C''^+(E' - C''X_4D'')D''^+ + L_{C''}J_1 + Z_1R_{D''},\end{aligned}$$

where $J, J_1, V, V_1, W, W_1, Z, Z_1$ are arbitrary matrices with appropriate dimensions. □

Similarly to the proof of Theorem 3.1, we can demonstrate the following theorem.

Theorem 3.2. *Let $A, B, C, D, E \in \mathbb{C}^{n\times n}$ be given matrices. If $B_1 = B_2 = D_1 = D_2 = 0$ or $A_2 = A_4 = C_2 = C_4 = 0$, then the following conditions are equivalent*:

(1) *The matrix equation $AXB + CYD = E$ has the reflexive solutions $X \in \mathbb{C}_a^{n\times n}(P)$ and $Y \in \mathbb{C}_a^{n\times n}(P)$;*

(2) $R_MR_{A'}E' = 0,\ R_{A'}E'L_{D''} = 0,\ E'L_{B''}L_N = 0,\ R_{C'}E'L_{B''} = 0;$

(3) $MM^+R_{A'}E'D''^+D'' = R_{A'}E',\ C'C'^+E'L_{B''}N^+N = E'L_{B''};$

(4) $R_PR_{C'}E' = 0,\ R_{C'}E'L_{B''} = 0,\ R_{A'}E'L_{D''} = 0,\ E'L_{D''}L_Q = 0;$

(5) $PP^+R_{C'}E'B''^+B'' = R_{C'}E',\ A'A'^+E'L_{D''}Q^+Q = E'L_{D''}$; *where* $A' = (A_1^t, A_3^t)^t,\ B'' = (B_3, B_4),\ C' = (C_1^t, C_3^t)^t,\ D'' = (D_3, D_4),\ M = R_{A'}C',\ N = D''L_{B''}, S = C'L_M,\ T = R_{D''}N,\ F = NL_T,\ G = R_SC',\ P = R_{C'}A',\ Q = B''L_{D''},\ S_1 = A'L_P,\ T_1 = R_{B''}Q$ *and* $G_1 = R_{S_1}A'$.

In that case, the anti-reflexive solutions of the matrix equation $AXB + CYD = E$ can be expressed by the following

$$X = U\begin{pmatrix}0 & X_2\\ X_3 & 0\end{pmatrix}U^* \qquad \text{and} \qquad Y = U\begin{pmatrix}0 & Y_2\\ Y_3 & 0\end{pmatrix}U^*,$$

where

$$\begin{aligned}X_2 &= A'^+(E' - C'Y_2D'')B''^+ + L_{A'}J + ZR_{B''},\\Y_2 &= M^+R_{A'}E'D''^+ + L_M(V - S^+SVNN^+)\\&\quad - L_MS^+C'L_GWTN^+ + (W - G^+GWTT^+)R_{D''},\end{aligned}$$

or

$$\begin{aligned}X_2 &= P^+R_{C'}E'B''^+ + L_P(V_1 - S_1^+S_1V_1QQ^+)\\&\quad - L_PS_1^+A'L_{G_1}W_1T_1Q^+ + (W_1 - G_1^+G_1W_1T_1T_1^+)R_{B''},\\Y_2 &= C'^+(E' - C'X_2D'')D''^+ + L_{C'}J_1 + Z_1R_{D''},\end{aligned}$$

and $X_3, Y_3, J, J_1, V, V_1, W, W_1, Z, Z_1$ are arbitrary matrices with appropriate dimensions.

4. Conclusion

In this paper, we have discussed the reflexive and anti-reflexive solutions of the matrix equation $AXB + CYD = E$. We have derived necessary and sufficient conditions for the existence and the expression of the reflexive and anti-reflexive solutions to the matrix equation (1.1). Some special cases of the matrix equation (1.1) have been considered in Section 3. The solvability conditions and explicit formulae for the solutions to the special cases have been also given.

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