Bull. Korean Math. Soc. **46** (2009), No. 3, pp. 499–510
DOI 10.4134/BKMS.2009.46.3.499

QUADRATIC FUNCTIONAL EQUATIONS ASSOCIATED WITH BOREL FUNCTIONS AND MODULE ACTIONS

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Abstract. For a Borel function $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the functional equation $\psi(s+t, u+v) + \psi(s-t, u-v) = 2\psi(s,u) + 2\psi(t,v)$, we show that it satisfies the functional equation

$$\psi(s,t) = s(s-t)\psi(1,0) + st\psi(1,1) + t(t-s)\psi(0,1).$$

Using this, we prove the stability of the functional equation

$$f(ax+ay, bz+bw) + f(ax-ay, bz-bw) = 2abf(x,z) + 2abf(y,w)$$

in Banach modules over a unital $C^\star$-algebra.

1. Introduction

Let X and Y be real or complex vector spaces. For a mapping $g : X \to Y$, consider the quadratic functional equation:

$$g(x+y) + g(x-y) = 2g(x) + 2g(y). \tag{1}$$

In 1989, J. Aczél [1] obtained the solution of the equation (1) for the case that Y acts on X. The result also holds when X and Y be arbitrary real or complex vector spaces.

For a mapping $f : X \times X \to Y$, consider the 2-dimensional quadratic functional equation:

$$f(x+y, z+w) + f(x-y, z-w) = 2f(x,z) + 2f(y,w). \tag{2}$$

The quadratic form $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ given by $f(x,y) := ax^2 + bxy + cy^2$ is a solution of the equation (2). In 2007, The authors [2] acquired the general solution and proved the stability of the 2-dimensional quadratic functional equation (2) for the case that X and Y be real vector spaces as follows.

Theorem A. *A mapping $f : X \times X \to Y$ satisfies the equation* (2) *for all $x, y, z, w \in X$ if and only if there exist two symmetric bi-additive mappings $S, T : X \times X \to Y$ and a bi-additive mapping $B : X \times X \to Y$ such that*

$$f(x,y) = S(x,x) + B(x,y) + T(y,y)$$

Received June 20, 2008; Revised November 10, 2008.
2000 *Mathematics Subject Classification.* Primary 39B22, 39B82.
Key words and phrases. stability, functional equation, quadratic form, Borel function.

for all $x, y \in X$.

Assume that $\varphi : X^4 \to [0, \infty)$ is a function satisfying the condition

$$\tilde{\varphi}(x, y, z, w) := \sum_{j=0}^{\infty} \frac{1}{4^{j+1}} \varphi\big(2^j x, 2^j y, 2^j z, 2^j w\big) < \infty \tag{3}$$

for all $x, y, z, w \in X$.

Theorem B. *Let Y be complete and $f : X \times X \to Y$ a mapping such that*

$$\|f(x + y, z + w) + f(x - y, z - w) - 2f(x, z) - 2f(y, w)\| \le \varphi(x, y, z, w)$$

for all $x, y, z, w \in X$. Then there exists a unique mapping $F : X \times X \to Y$ satisfying the equation (2) *such that*

$$\|f(x, y) - F(x, y)\| \le \tilde{\varphi}(x, x, y, y) \tag{4}$$

for all $x, y \in X$. The mapping F is given by $F(x, y) := \lim_{j\to\infty} \frac{1}{4^j} f\big(2^j x, 2^j y\big)$ for all $x, y \in X$.

Remark. The results of Theorem A and Theorem B are also holds for complex vector spaces X and Y.

In this paper, we investigate the stability of the equation (2) with two actions in Banach modules over a unital $C^\star$-algebra.

2. Results

Let A be a unital $C^\star$-algebra with a norm $|\cdot|$, and let ${}_A\mathcal{M}$ and ${}_A\mathcal{N}$ be left Banach A-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively. Put $A_1 := \{a \in A \mid |a| = 1\}$, $A_{in} := \{a \in A \mid a \text{ is invertible in } A\}$, $A_{sa} := \{a \in A \mid a^\star = a\}$, $\mathcal{U}(A) := \{a \in A \mid aa^\star = a^\star a = 1\}$, $A^+ := \{a \in A_{sa} \mid Sp(a) \subset [0, \infty)\}$ and $A_1^+ := A_1 \cap A^+$. Let $\varphi : ({}_A\mathcal{M})^4 \to [0, \infty)$ be a function satisfying the condition (3) for all $x, y, z, w \in {}_A\mathcal{M}$.

Definition. A 2-dimensional quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ is called *A-quadratic* if $F(ax, ay) = a^2 F(x, y)$ for all $a \in A$ and all $x, y \in {}_A\mathcal{M}$.

Theorem 1. *Let $f : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ be a mapping such that*

$$\begin{aligned} &\big\|f(ax + ay, az + aw) + f(ax - ay, az - aw) - 2a^2 f(x, z) - 2a^2 f(y, w)\big\| \\ &\le \varphi(x, y, z, w) \end{aligned} \tag{5}$$

for all $a \in A_1$ and all $x, y, z, w \in {}_A\mathcal{M}$. If $f(tx, ty)$ is continuous in $t \in \mathbb{R}$ for each fixed $x, y \in {}_A\mathcal{M}$, then there exists a unique A-quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying the inequality (4) *for all $x, y \in {}_A\mathcal{M}$.*

Proof. By Theorem B, it follows from the inequality of the statement for $a = 1$ that there exists a unique 2-dimensional quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying the inequality (4) for all $x, y \in {}_A\mathcal{M}$.

Let $x_0, y_0 \in {}_A\mathcal{M}$ be fixed. And let $L : {}_A\mathcal{N} \to \mathbb{R}$ be any continuous linear functional, that is, L is an arbitrary element of the dual space of ${}_A\mathcal{N}$. For $n \in \mathbb{N}$, consider the functions $\psi_n : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi_n(t) := \frac{1}{4^n} L[f(2^n t x_0, 2^n t y_0)]$$

for all $t \in \mathbb{R}$. By the assumption that $f(tx, ty)$ is continuous in $t \in \mathbb{R}$ for each fixed $x, y \in {}_A\mathcal{M}$, the function ψ_n is continuous for all $n \in \mathbb{N}$. Note that $\psi_n(t) = \frac{1}{4^n} L[f(2^n t x_0, 2^n t y_0)] = L\big[\frac{1}{4^n} f(2^n t x_0, 2^n t y_0)\big]$ for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$. By [2], the sequence $\psi_n(t)$ is a Cauchy sequence for all $t \in \mathbb{R}$. Define a function $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(t) := \lim_{n\to\infty} \psi_n(t)$ for all $t \in \mathbb{R}$. Note that $\psi(t) = L[F(tx_0, ty_0)]$ for all $t \in \mathbb{R}$. Thus we get

$$\begin{aligned}
\psi(s+t) + \psi(s-t) &= L\big(F[(s+t)x_0, (s+t)y_0]\big) + L\big(F[(s-t)x_0, (s-t)y_0]\big) \\
&= L\big(F[(s+t)x_0, (s+t)y_0] + F[(s-t)x_0, (s-t)y_0]\big) \\
&= L\big[F(sx_0 + tx_0, sy_0 + ty_0) + F(sx_0 - tx_0, sy_0 - ty_0)\big] \\
&= L\big[2F(sx_0, sy_0) + 2F(tx_0, ty_0)\big] \\
&= 2L[F(sx_0, sy_0)] + 2L[F(tx_0, ty_0)] = 2\psi(s) + 2\psi(t)
\end{aligned}$$

for all $s, t \in \mathbb{R}$. Since ψ is the pointwise limit of continuous functions, it is a Borel function. Thus the function ψ as a measurable quadratic function is continuous (see [7]), so has the form $\psi(t) = t^2\psi(1)$ for all $t \in \mathbb{R}$. Hence we have

$$L[F(tx_0, ty_0)] = \psi(t) = t^2\psi(1) = t^2 L[F(x_0, y_0)] = L\big[t^2 F(x_0, y_0)\big]$$

for all $t \in \mathbb{R}$. Since L is any continuous linear functional, the 2-dimensional quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfies $F(tx_0, ty_0) = t^2 F(x_0, y_0)$ for all $t \in \mathbb{R}$. Therefore we obtain

$$F(tx, ty) = t^2 F(x, y) \tag{6}$$

for all $t \in \mathbb{R}$ and all $x, y \in {}_A\mathcal{M}$.

Let j be an arbitrary positive integer. Replacing x and z by $2^j x$ and $2^j z$, respectively, and letting $y = w = 0$ in the inequality (5), we gain

$$\big\|f\big(2^j ax, 2^j az\big) - a^2 f\big(2^j x, 2^j z\big) - a^2 f(0,0)\big\| \le \frac{1}{2}\varphi\big(2^j x, 0, 2^j z, 0\big)$$

for all $a \in A_1$ and all $x, z \in {}_A\mathcal{M}$. Note that there is a constant $K > 0$ such that the condition

$$\|av\| \le K|a|\|v\| \tag{7}$$

for each $a \in A$ and each $v \in {}_A\mathcal{N}$ (see [4], Definition 12). For all $a \in A_1$ and all $x, y \in {}_A\mathcal{M}$, we get

$$\begin{aligned}
\frac{1}{4^j}\big\|f\big(2^j ax, 2^j ay\big) - a^2 f\big(2^j x, 2^j z\big)\big\| &\le \frac{1}{2\cdot 4^j}\varphi\big(2^j x, 0, 2^j y, 0\big) + \frac{K|a|^2}{4^j}\|f(0,0)\| \\
&\to 0 \text{ as } j \to \infty.
\end{aligned}$$

Hence we have

$$F(ax, ay) = \lim_{j\to\infty} \frac{1}{4^j} f\big(2^j ax, 2^j ay\big) = a^2 \lim_{j\to\infty} \frac{1}{4^j} f\big(2^j x, 2^j y\big) = a^2 F(x, y)$$

for all $a \in A_1$ and all $x, y \in {}_A\mathcal{M}$. Since $F(ax, ay) = a^2 F(x, y)$ for each $a \in A_1$, by the equation (6), we obtain

$$F(ax, ay) = F\left(|a|\frac{a}{|a|}x, |a|\frac{a}{|a|}y\right) = |a|^2 F\left(\frac{a}{|a|}x, \frac{a}{|a|}y\right) = a^2 F(x, y)$$

for all nonzero $a \in A$ and all $x, y \in {}_A\mathcal{M}$. By the equation (6), we get $F(0x, 0y) = 0^2 F(x, y)$ for all $x, y \in {}_A\mathcal{M}$. Therefore the 2-dimensional quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ is the unique A-quadratic mapping satisfying the inequality (4). □

Corollary 2. *Let $f : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ be a mapping such that*

$$\big\|f(ax + ay, az + aw) + f(ax - ay, az - aw) - 2a^2 f(x, z) - 2a^2 f(y, w)\big\| \le \delta$$

for all $a \in A_1$ and all $x, y, z, w \in {}_A\mathcal{M}$. If $f(tx, ty)$ is continuous in $t \in \mathbb{R}$ for each fixed $x, y \in {}_A\mathcal{M}$, then there exists a unique A-quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying $\|f(x, y) - F(x, y)\| \le \frac{\delta}{3}$ for all $x, y \in {}_A\mathcal{M}$.

Corollary 3. *Let E be a complex Banach space and $f : E \times E \to \mathbb{C}$ a function such that*

$$\big\|f(\lambda x + \lambda y, \lambda z + \lambda w) + f(\lambda x - \lambda y, \lambda z - \lambda w) - 2\lambda^2 f(x, z) - 2\lambda^2 f(y, w)\big\| \le \delta$$

for all $\lambda \in \mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x, y, z, w \in E$. If $f(tx, ty)$ is continuous in $t \in \mathbb{R}$ for each fixed $x, y \in E$, then there exists a unique quadratic mapping $F : E \times E \to \mathbb{C}$ satisfying $F(\lambda x) = \lambda^2 x$ and $\|f(x, y) - F(x, y)\| \le \frac{\delta}{3}$ for all $\lambda \in \mathbb{C}$ and all $x, y \in E$.

Lemma 4. *Let $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function satisfying*

$$\psi(s + t, u + v) + \psi(s - t, u - v) = 2\psi(s, u) + 2\psi(t, v)$$

for all $s, t, u, v \in \mathbb{R}$. If the function ψ is a Borel function, then it satisfies

$$\psi(s, t) = s(s - t)\psi(1, 0) + st\psi(1, 1) + t(t - s)\psi(0, 1)$$

for all $s, t \in \mathbb{R}$.

Proof. By Theorem A, there exist two symmetric bi-additive mappings $S, T : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and a bi-additive mapping $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\psi(s, t) = S(s, s) + B(s, t) + T(t, t)$ for all $s, t \in \mathbb{R}$. By the proof of Theorem A in [2], we gain

$$\begin{aligned} \psi(pu, qv) &= S(pu, pu) + B(pu, qv) + T(qv, qv) \\ &= p^2 S(u, u) + pqB(u, v) + q^2 T(v, v) \\ &= p^2 \psi(u, 0) + pq[\psi(u, v) - \psi(u, 0) - \psi(0, v)] + q^2 \psi(0, v) \\ &= p(p - q)\psi(u, 0) + pq\psi(u, v) + q(q - p)\psi(0, v) \end{aligned} \tag{8}$$

for all $p, q \in \mathbb{Q}$ and all $u, v \in \mathbb{R}$. Letting $p = v = 1$ in the equality (8), we get

$$\psi(u,q) = 1(1-q)\psi(u,0) + q\psi(u,1) + q(q-1)\psi(0,1) \tag{9}$$
$$= \psi(u,0) + q[\psi(u,1) - \psi(u,0) - \psi(0,1)] + q^2\psi(0,1)$$

for all $u \in \mathbb{R}$ and all $q \in \mathbb{Q}$. Putting $u = v = 1$ in the equality (8) again, we have

$$\psi(p,q) = p(p-q)\psi(1,0) + pq\psi(1,1) + q(q-p)\psi(0,1) \tag{10}$$

for all $p, q \in \mathbb{Q}$.

Since the function $v \to \psi(u,v)$ is measurable and satisfies the equation (1), by [7], it is continuous. By the same reasoning, $u \to \psi(u,v)$ is also continuous. Let $s, t \in \mathbb{R}$ be fixed. Since ψ is measurable, by Theorem 7.14.26 in [3], for every $m \in \mathbb{N}$ there is a closed set $F_m \subset [s, s+1]$ such that $\mu([s,s+1] \setminus F_m) < \frac{1}{m}$ and $\psi|_{F_m \times \mathbb{R}}$ is continuous. Since $\mu(F_m) \to 1$, one can choose $u_m \in F_m$ satisfying $u_m \to s$. Take a sequence $\{q_n\}$ in $\mathbb{Q}$ converging to t. By the equality (9), we obtain that

(11)
$$\begin{aligned}\psi(u_m,t) &= \psi\Big(u_m, \lim_{n\to\infty} q_n\Big) = \lim_{n\to\infty} \psi(u_m, q_n)\\ &= \lim_{n\to\infty} \big(\psi(u_m,0) + q_n[\psi(u_m,1) - \psi(u_m,0) - \psi(0,1)] + q_n^2\psi(0,1)\big)\\ &= \psi(u_m,0) + t[\psi(u_m,1) - \psi(u_m,0) - \psi(0,1)] + t^2\psi(0,1)\end{aligned}$$

for all $m \in \mathbb{N}$. For each fixed $m \in \mathbb{N}$, take a sequence $\{p_n\}$ in $\mathbb{Q}$ converging to u_m. By the equalities (10) and (11), we see that

$$\begin{aligned}&\psi(u_m,t)\\ &= \psi\Big(\lim_{n\to\infty} p_n, 0\Big) + t\Big[\psi\Big(\lim_{n\to\infty} p_n, 1\Big) - \psi\Big(\lim_{n\to\infty} p_n, 0\Big) - \psi(0,1)\Big] + t^2\psi(0,1)\\ &= \lim_{n\to\infty} \psi(p_n,0) + t\Big[\lim_{n\to\infty} \psi(p_n,1) - \lim_{n\to\infty} \psi(p_n,0) - \psi(0,1)\Big] + t^2\psi(0,1)\\ &= \lim_{n\to\infty} [(1-t)\psi(p_n,0) + t\psi(p_n,1)] + t(t-1)\psi(0,1)\\ &= (1-t)\lim_{n\to\infty} p_n^2\psi(1,0) + t(t-1)\psi(0,1)\\ &\quad + t\lim_{n\to\infty} [p_n(p_n-1)\psi(1,0) + p_n\psi(1,1) + (1-p_n)\psi(0,1)]\\ &= u_m(u_m-t)\psi(1,0) + u_m t\psi(1,1) + t(t-u_m)\psi(0,1).\end{aligned}$$

Hence we obtain that

$$\begin{aligned}\psi(s,t) &= \psi\Big(\lim_{m\to\infty} u_m, t\Big) = \lim_{m\to\infty} \psi(u_m,t)\\ &= \lim_{m\to\infty} \big[u_m(u_m-t)\psi(1,0) + u_m t\psi(1,1) + t(t-u_m)\psi(0,1)\big]\\ &= s(s-t)\psi(1,0) + st\psi(1,1) + t(t-s)\psi(0,1).\end{aligned}$$

□

Definition. A unital $C^\star$-algebra A is said to have *real rank* 0 (see [5]) if the invertible self-adjoint elements are dense in A_{sa}.

For any element $a \in A$, $a = a_1 + ia_2$, where $a_1 := \frac{a+a^\star}{2}$ and $a_2 := \frac{a-a^\star}{2i}$ are self-adjoint elements, furthermore, $a = a_1^+ - a_1^- + ia_2^+ - ia_2^-$, where a_1^+, a_1^-, a_2^+ and a_2^- are positive elements (see [4], Lemma 38.8).

Theorem 5. *Let A be of real rank 0 and let $f : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ be a mapping such that*

$$(12) \quad \big\|f(ax+ay, bz+bw) + f(ax-ay, bz-bw) - 2abf(x,z) - 2ab(y,w)\big\| \le \varphi(x,y,z,w)$$

for all $a, b \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, y, z, w \in {}_A\mathcal{M}$. For each fixed $x, y \in {}_A\mathcal{M}$, let the sequence $\{\frac{1}{4^j} f(2^j ax, 2^j by)\}$ converge uniformly on $A_1 \times A_1$. If $f(ax, by)$ is continuous in $(a,b) \in (A_1 \cup \mathbb{R})^2$ for each fixed $x, y \in {}_A\mathcal{M}$, then there exists a unique 2-dimensional quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying the inequality (4) such that

$$F(ax, ay) = a^2 F(x,y) + \big[\, \big(|a_2^-| - |a_2^+|\big)\, a_1 + \big(|a_1^-| - |a_1^+|\big)\, ia_2 \,\big]\, [F(x,0) + F(0,y)]$$

for all $a \in A$ and all $x, y \in {}_A\mathcal{M}$.

Proof. By Theorem B, there exists a unique 2-dimensional quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying the equation (2) and the inequality (4) on ${}_A\mathcal{M} \times {}_A\mathcal{M}$.

Let $x_0, y_0 \in {}_A\mathcal{M}$ be fixed. And let L be an arbitrary element of the dual space of ${}_A\mathcal{N}$. For $n \in \mathbb{N}$, consider the functions $\psi_n : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$\psi_n(s,t) := \frac{1}{4^n} L[f(2^n s x_0, 2^n t y_0)]$$

for all $s, t \in \mathbb{R}$. By the assumption that $f(ax, by)$ is continuous in $(a,b) \in (A_1 \cup \mathbb{R})^2$ for each fixed $x, y \in {}_A\mathcal{M}$, the function ψ_n is continuous for all $n \in \mathbb{N}$. Note that $\psi_n(s,t) = \frac{1}{4^n} L[f(2^n s x_0, 2^n t y_0)] = L\big[\frac{1}{4^n} f(2^n s x_0, 2^n t y_0)\big]$ for all $n \in \mathbb{N}$ and all $s, t \in \mathbb{R}$. By [2], the sequence $\psi_n(s,t)$ is a Cauchy sequence for all $s, t \in \mathbb{R}$. Define a function $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\psi(s,t) := \lim_{n\to\infty} \psi_n(s,t)$ for all $s, t \in \mathbb{R}$. Note that $\psi(s,t) = L[F(sx_0, ty_0)]$ for all $t \in \mathbb{R}$. Thus we have

$$\begin{aligned}
&\psi(s_1+s_2, t_1+t_2) + \psi(s_1-s_2, t_1-t_2) \\
&= L\big(F[(s_1+s_2)x_0, (t_1+t_2)y_0]\big) + L\big(F[(s_1-s_2)x_0, (t_1-t_2)y_0]\big) \\
&= L\big(F[(s_1+s_2)x_0, (t_1+t_2)y_0] + F[(s_1-s_2)x_0, (t_1-t_2)y_0]\big) \\
&= L\big[F(s_1x_0 + s_2x_0, t_1y_0 + t_2y_0) + F(s_1x_0 - s_2x_0, t_1y_0 - t_2y_0)\big] \\
&= L\big[2F(s_1x_0, t_1y_0) + 2F(s_2x_0, t_2y_0)\big] \\
&= 2L[F(s_1x_0, t_1y_0)] + 2L[F(s_2x_0, t_2y_0)] = 2\psi(s_1,t_1) + 2\psi(s_2,t_2)
\end{aligned}$$

for all $s_1, s_2, t_1, t_2 \in \mathbb{R}$. Since ψ is the pointwise limit of continuous functions, it is a Borel function. By Lemma 4, we gain

$$\psi(s,t) = s(s-t)\psi(1,0) + st\psi(1,1) + t(t-s)\psi(0,1)$$

for all $s, t \in \mathbb{R}$. Hence we get

$$\begin{aligned} L[F(sx_0, ty_0)] &= \psi(s,t) = s(s-t)\psi(1,0) + st\psi(1,1) + t(t-s)\psi(0,1) \\ &= s(s-t)L[F(x_0,0)] + stL[F(x_0,y_0)] + t(t-s)L[F(0,y_0)] \\ &= L\big[s(s-t)F(x_0,0) + stF(x_0,y_0) + t(t-s)F(0,y_0)\big] \end{aligned}$$

for all $s, t \in \mathbb{R}$. Since L is any continuous linear functional, the 2-dimensional quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfies

$$F(sx_0, ty_0) = s(s-t)F(x_0,0) + stF(x_0,y_0) + t(t-s)F(0,y_0)$$

for all $s, t \in \mathbb{R}$. Therefore we obtain

$$F(sx, ty) = s(s-t)F(x,0) + stF(x,y) + t(t-s)F(0,y) \tag{13}$$

for all $s, t \in \mathbb{R}$ and all $x, y \in {}_A\mathcal{M}$.

Let j be an arbitrary positive integer. Replacing x and z by $2^j x$ and $2^j z$, respectively, and letting $y = w = 0$ in the inequality (12), we get

$$\big\|f\big(2^j ax, 2^j bz\big) - abf\big(2^j x, 2^j z\big) - abf(0,0)\big\| \le \frac{1}{2}\varphi\big(2^j x, 0, 2^j z, 0\big)$$

for all $a, b \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, z \in {}_A\mathcal{M}$. By the condition (7), for all $a, b \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, y \in {}_A\mathcal{M}$, we have

$$\begin{aligned} \frac{1}{4^j}\big\|f\big(2^j ax, 2^j by\big) - abf\big(2^j x, 2^j z\big)\big\| &\le \frac{1}{2\cdot 4^j}\varphi\big(2^j x, 0, 2^j y, 0\big) + \frac{K|a||b|}{4^j}\|f(0,0)\| \\ &\to 0 \text{ as } j \to \infty. \end{aligned}$$

Hence we obtain that

$$F(ax, by) = \lim_{j\to\infty} \frac{1}{4^j} f\big(2^j ax, 2^j by\big) = ab \lim_{j\to\infty} \frac{1}{4^j} f\big(2^j x, 2^j y\big) = abF(x,y) \tag{14}$$

for all $a, b \in \big(A_1^+ \cap A_{in}\big) \cup \{i\}$ and all $x, y \in {}_A\mathcal{M}$.

Let $c, d \in A_1^+ \setminus A_{in}$. Since $A_{in} \cap A_{sa}$ is dense in A_{sa}, there exist two sequences $\{c_j\}$ and $\{d_j\}$ in $A_{in} \cap A_{sa}$ such that $c_j \to c$ and $d_j \to d$ as $j \to \infty$. Put $p_j := \frac{1}{|c_j|}c_j$ and $q_j := \frac{1}{|d_j|}d_j$. Then $p_j \to c$ and $q_j \to d$ as $j \to \infty$. Set $a_j := \sqrt{p_j{}^\star p_j}$ and $b_j := \sqrt{q_j{}^\star q_j}$. Then $a_j \to c$ and $b_j \to d$ as $j \to \infty$ and $a_j, b_j \in A_1^+ \cap A_{in}$. Since $\{\frac{1}{4^j} f\big(2^j ax, 2^j by\big)\}$ is uniformly converges on $A_1 \times A_1$ and $f(ax, by)$ is continuous in $a, b \in A_1$, we see that $F(ax, by)$ is also continuous in $a, b \in A_1$ for each $x, y \in {}_A\mathcal{M}$. In fact, we gain

$$\begin{aligned} \lim_{(a,b)\to(c,d)} F(ax, by) &= \lim_{(a,b)\to(c,d)} \lim_{j\to\infty} \frac{1}{4^j} f\big(2^j ax, 2^j by\big) \\ &= \lim_{j\to\infty} \lim_{(a,b)\to(c,d)} \frac{1}{4^j} f\big(2^j ax, 2^j by\big) \\ &= \lim_{j\to\infty} \frac{1}{4^j} f\big(2^j cx, 2^j dy\big) = F(cx, dy) \end{aligned}$$

for all $x, y \in {}_A\mathcal{M}$. Thus we get

$$\lim_{j\to\infty} F(a_jx, b_jy) = F\Big(\lim_{j\to\infty} a_jx, \lim_{j\to\infty} b_jy\Big) = F(cx, dy) \tag{15}$$

for all $x, y \in {}_A\mathcal{M}$. By the equality (14), we have

$$\begin{aligned}\|F(a_jx, b_jy) - cdF(x, y)\| &= \|a_jb_jF(x, y) - cdF(x, y)\| \\ &\to \|cdF(x, y) - cdF(x, y)\| = 0\end{aligned}$$

as $j \to \infty$ for all $x, y \in {}_A\mathcal{M}$. By the equality (15) and the above convergence, we see that

$$\begin{aligned}&\|F(cx, dy) - cdF(x, y)\| \\ &\le \|F(cx, dy) - F(a_jx, b_jy)\| + \|F(a_jx, b_jy) - cdF(x, y)\| \to 0 \quad \text{as} \quad j \to \infty\end{aligned}$$

for all $x, y \in {}_A\mathcal{M}$. By the equality (14) and the above convergence, we obtain

$$F(ax, by) = abF(x, y) \tag{16}$$

for all $a, b \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{M}$.

By Theorem A, there exist two symmetric bi-additive mappings $S, T : X \times X \to Y$ and a bi-additive mapping $B : X \times X \to Y$ such that $f(x, y) = S(x, x) + B(x, y) + T(y, y)$ for all $x, y \in X$. Thus we see that

$$\begin{aligned}
&F(ax, ay) = S(ax, ax) + B(ax, ay) + T(ay, ay) \\
&= S\big(a_1^+x - a_1^-x + ia_2^+x - ia_2^-x, a_1^+x - a_1^-x + ia_2^+x - ia_2^-x\big) \\
&\quad + B\big(a_1^+x - a_1^-x + ia_2^+x - ia_2^-x, a_1^+y - a_1^-y + ia_2^+y - ia_2^-y\big) \\
&\quad + T\big(a_1^+y - a_1^-y + ia_2^+y - ia_2^-y, a_1^+y - a_1^-y + ia_2^+y - ia_2^-y\big) \\
&= S\big(a_1^+x, a_1^+x\big) - S\big(a_1^+x, a_1^-x\big) + S\big(a_1^+x, ia_2^+x\big) - S\big(a_1^+x, ia_2^-x\big) \\
&\quad - S\big(a_1^-x, a_1^+x\big) + S\big(a_1^-x, a_1^-x\big) - S\big(a_1^-x, ia_2^+x\big) + S\big(a_1^-x, ia_2^-x\big) \\
&\quad + S\big(ia_2^+x, a_1^+x\big) - S\big(ia_2^+x, a_1^-x\big) + S\big(ia_2^+x, ia_2^+x\big) - S\big(ia_2^+x, ia_2^-x\big) \\
&\quad - S\big(ia_2^-x, a_1^+x\big) + S\big(ia_2^-x, a_1^-x\big) - S\big(ia_2^-x, ia_2^+x\big) + S\big(ia_2^-x, ia_2^-x\big) \\
&\quad + B\big(a_1^+x, a_1^+y\big) - B\big(a_1^+x, a_1^-y\big) + B\big(a_1^+x, ia_2^+y\big) - B\big(a_1^+x, ia_2^-y\big) \\
&\quad - B\big(a_1^-x, a_1^+y\big) + B\big(a_1^-x, a_1^-y\big) - B\big(a_1^-x, ia_2^+y\big) + B\big(a_1^-x, ia_2^-y\big) \\
&\quad + B\big(ia_2^+x, a_1^+y\big) - B\big(ia_2^+x, a_1^-y\big) + B\big(ia_2^+x, ia_2^+y\big) - B\big(ia_2^+x, ia_2^-y\big) \\
&\quad - B\big(ia_2^-x, a_1^+y\big) + B\big(ia_2^-x, a_1^-y\big) - B\big(ia_2^-x, ia_2^+y\big) + B\big(ia_2^-x, ia_2^-y\big) \\
&\quad + T\big(a_1^+y, a_1^+y\big) - T\big(a_1^+y, a_1^-y\big) + T\big(a_1^+y, ia_2^+y\big) - T\big(a_1^+y, ia_2^-y\big) \\
&\quad - T\big(a_1^-y, a_1^+y\big) + T\big(a_1^-y, a_1^-y\big) - T\big(a_1^-y, ia_2^+y\big) + T\big(a_1^-y, ia_2^-y\big) \\
&\quad + T\big(ia_2^+y, a_1^+y\big) - T\big(ia_2^+y, a_1^-y\big) + T\big(ia_2^+y, ia_2^+y\big) - T\big(ia_2^+y, ia_2^-y\big) \\
&\quad - T\big(ia_2^-y, a_1^+y\big) + T\big(ia_2^-y, a_1^-y\big) - T\big(ia_2^-y, ia_2^+y\big) + T\big(ia_2^-y, ia_2^-y\big) \\
&= \big[S\big(a_1^+x, a_1^+x\big) + B\big(a_1^+x, a_1^+y\big) + T\big(a_1^+y, a_1^+y\big)\big]
\end{aligned}$$

$$
\begin{aligned}
&- \left[S\left(a_1^+ x, a_1^- x\right) + B\left(a_1^+ x, a_1^- y\right) + T\left(a_1^+ y, a_1^- y\right)\right] \\
&+ \left[S\left(a_1^+ x, ia_2^+ x\right) + B\left(a_1^+ x, ia_2^+ y\right) + T\left(a_1^+ y, ia_2^+ y\right)\right] \\
&- \left[S\left(a_1^+ x, ia_2^- x\right) + B\left(a_1^+ x, ia_2^- y\right) + T\left(a_1^+ y, ia_2^- y\right)\right] \\
&- \left[S\left(a_1^- x, a_1^+ x\right) + B\left(a_1^- x, a_1^+ y\right) + T\left(a_1^- y, a_1^+ y\right)\right] \\
&+ \left[S\left(a_1^- x, a_1^- x\right) + B\left(a_1^- x, a_1^- y\right) + T\left(a_1^- y, a_1^- y\right)\right] \\
&- \left[S\left(a_1^- x, ia_2^+ x\right) + B\left(a_1^- x, ia_2^+ y\right) + T\left(a_1^- y, ia_2^+ y\right)\right] \\
&+ \left[S\left(a_1^- x, ia_2^- x\right) + B\left(a_1^- x, ia_2^- y\right) + T\left(a_1^- y, ia_2^- y\right)\right] \\
&+ \left[S\left(ia_2^+ x, a_1^+ x\right) + B\left(ia_2^+ x, a_1^+ y\right) + T\left(ia_2^+ y, a_1^+ y\right)\right] \\
&- \left[S\left(ia_2^+ x, a_1^- x\right) + B\left(ia_2^+ x, a_1^- y\right) + T\left(ia_2^+ y, a_1^- y\right)\right] \\
&+ \left[S\left(ia_2^+ x, ia_2^+ x\right) + B\left(ia_2^+ x, ia_2^+ y\right) + T\left(ia_2^+ y, ia_2^+ y\right)\right] \\
&- \left[S\left(ia_2^+ x, ia_2^- x\right) + B\left(ia_2^+ x, ia_2^- y\right) + T\left(ia_2^+ y, ia_2^- y\right)\right] \\
&- \left[S\left(ia_2^- x, a_1^+ x\right) + B\left(ia_2^- x, a_1^+ y\right) + T\left(ia_2^- y, a_1^+ y\right)\right] \\
&+ \left[S\left(ia_2^- x, a_1^- x\right) + B\left(ia_2^- x, a_1^- y\right) + T\left(ia_2^- y, a_1^- y\right)\right] \\
&- \left[S\left(ia_2^- x, ia_2^+ x\right) + B\left(ia_2^- x, ia_2^+ y\right) + T\left(ia_2^- y, ia_2^+ y\right)\right] \\
&+ \left[S\left(ia_2^- x, ia_2^- x\right) + B\left(ia_2^- x, ia_2^- y\right) + T\left(ia_2^- y, ia_2^- y\right)\right] \\
= &F\left(a_1^+ x, a_1^+ y\right) - F\left(a_1^+ x, a_1^- y\right) + F\left(a_1^+ x, ia_2^+ y\right) - F\left(a_1^+ x, ia_2^- y\right) \\
&- F\left(a_1^- x, a_1^+ y\right) + F\left(a_1^- x, a_1^- y\right) - F\left(a_1^- x, ia_2^+ y\right) + F\left(a_1^- x, ia_2^- y\right) \\
&+ F\left(ia_2^+ x, a_1^+ y\right) - F\left(ia_2^+ x, a_1^- y\right) + F\left(ia_2^+ x, ia_2^+ y\right) - F\left(ia_2^+ x, ia_2^- y\right) \\
&- F\left(ia_2^- x, a_1^+ y\right) + F\left(ia_2^- x, a_1^- y\right) - F\left(ia_2^- x, ia_2^+ y\right) + F\left(ia_2^- x, ia_2^- y\right)
\end{aligned}
$$

for all $a \in A$ and all $x, y \in {}_A\mathcal{M}$. By the equation (13) and the equality (16), we have

$$
\begin{aligned}
F(px, qy) = &F\left(|p|\,\frac{p}{|p|}\,x, |q|\,\frac{q}{|q|}\,y\right) \\
&= |p|(|p| - |q|)\,F\left(\frac{p}{|p|}\,x, 0\right) + |p|\,|q|\,F\left(\frac{p}{|p|}\,x, \frac{q}{|q|}\,y\right) \\
&\quad + |q|(|q| - |p|)\,F\left(0, \frac{q}{|q|}\,y\right) \\
&= (|p| - |q|)\,pF(x,0) + pqF(x,y) + (|q| - |p|)\,qF(0,y) \\
&= pqF(x,y) + (|p| - |q|)\left[pF(x,0) - qF(0,y)\right]
\end{aligned}
$$

for all $p, q \in \left\{a_1^+, a_1^-, a_2^+, a_2^-\right\}$ and all $x, y \in {}_A\mathcal{M}$. Note that $a_1^+ a_1^- = a_1^- a_1^+ = a_2^+ a_2^- = a_2^- a_2^+ = 0$. Hence we obtain that

$$
\begin{aligned}
&F(ax, ay) \\
= &\left(a_1^+\right)^2 F(x,y) + ia_1^+\, a_2^+\, F(x,y) + \left(\left|a_1^+\right| - \left|a_2^+\right|\right)\left[a_1^+ F(x,0) - ia_2^+ F(0,y)\right]
\end{aligned}
$$

$$
\begin{aligned}
&- ia_1^+ a_2^- F(x,y) - \big(|a_1^+| - |a_2^-|\big)\left[a_1^+ F(x,0) - ia_2^- F(0,y)\right] + \big(a_1^-\big)^2 F(x,y) \\
&- ia_1^- a_2^+ F(x,y) - \big(|a_1^-| - |a_2^+|\big)\left[a_1^- F(x,0) - ia_2^+ F(0,y)\right] \\
&+ ia_1^- a_2^- F(x,y) + \big(|a_1^-| - |a_2^-|\big)\left[a_1^- F(x,0) - ia_2^- F(0,y)\right] \\
&+ ia_2^+ a_1^+ F(x,y) + \big(|a_2^+| - |a_1^+|\big)\left[ia_2^+ F(x,0) - a_1^+ F(0,y)\right] \\
&- ia_2^+ a_1^- F(x,y) - \big(|a_2^+| - |a_1^-|\big)\left[ia_2^+ F(x,0) - a_1^- F(0,y)\right] \\
&- \big(a_2^+\big)^2 F(x,y) - ia_2^- a_1^+ F(x,y) - \big(|a_2^-| - |a_1^+|\big)\left[ia_2^- F(x,0) - a_1^+ F(0,y)\right] \\
&+ ia_2^- a_1^- F(x,y) + \big(|a_2^-| - |a_1^-|\big)\left[ia_2^- F(x,0) - a_1^- F(0,y)\right] - \big(a_2^-\big)^2 F(x,y) \\
= &\Big[\big(a_1^+\big)^2 + ia_1^+ a_2^+ - ia_1^+ a_2^- + \big(a_1^-\big)^2 - ia_1^- a_2^+ + ia_1^- a_2^- \\
&+ ia_2^+ a_1^+ - ia_2^+ a_1^- - \big(a_2^+\big)^2 - ia_2^- a_1^+ + ia_2^- a_1^- - \big(a_2^-\big)^2\Big] F(x,y) \\
&+ \left[\,\big(|a_2^-| - |a_2^+|\big)\, a_1 + \big(|a_1^-| - |a_1^+|\big)\, ia_2\,\right]\left[F(x,0) + F(0,y)\right] \\
= &\big(a_1^+ - a_1^- + ia_2^+ - ia_2^-\big)^2 F(x,y) \\
&+ \left[\,\big(|a_2^-| - |a_2^+|\big)\, a_1 + \big(|a_1^-| - |a_1^+|\big)\, ia_2\,\right]\left[F(x,0) + F(0,y)\right] \\
= &a^2 F(x,y) + \left[\,\big(|a_2^-| - |a_2^+|\big)\, a_1 + \big(|a_1^-| - |a_1^+|\big)\, ia_2\,\right]\left[F(x,0) + F(0,y)\right]
\end{aligned}
$$

for all $a \in A$ and all $x, y \in {}_A\mathcal{M}$. □

Theorem 6. *Let A be of real rank 0 and commutative. Let $D := \{a \in A \mid Sp(a) \subset \mathbb{C} \setminus [0,\infty)\}$, $E := \{a \in A \mid Sp(a) \subset \mathbb{C} \setminus (-\infty, 0]\}$ and let $D \cup E$ be dense in A_{in}. Let $f : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ be a mapping satisfying the inequality* (5) *for all $a \in \exp(\mathcal{U}(A)) \cup \{1\}$ and all $x, y, z, w \in {}_A\mathcal{M}$. For each fixed $x, y \in {}_A\mathcal{M}$, let the sequence $\{\frac{1}{4^j} f\big(2^j ax, 2^j ay\big)\}$ converge uniformly on A_1. If $f(ax, ay)$ is continuous in $a \in A_1 \cup \mathbb{R}$ for each fixed $x, y \in {}_A\mathcal{M}$, then there exists a unique A-quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ such that the inequality* (4) *for all $x, y \in {}_A\mathcal{M}$.*

Proof. Since f satisfies the inequality (5) for $a = 1$ and all $x, y, z, w \in {}_A\mathcal{M}$, by the same reasoning as in the proof of Theorem B, there exists a unique 2-dimensional quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying the inequality (4) for all $x, y \in {}_A\mathcal{M}$. By a similar method to the proof of Theorem 1, the quadratic mapping F satisfies $F(ax, ay) = a^2 F(x,y)$ for all $a \in \exp(\mathcal{U}(A)) \cup \{1\}$ and all $x, y \in {}_A\mathcal{M}$.

For every element $a \in D$, there is a positive integer m greater than 2 such that $\left|\frac{1+\log a}{m}\right| < 1 - \frac{2}{m}$. By [6], there are unitary elements $u_1, \ldots, u_m \in \mathcal{U}(A)$ such that $1 + \log a = u_1 + \cdots + u_m$. Then we get

$$
\begin{aligned}
F(eax, eay) &= F\big(e^{1+\log a} x, e^{1+\log a} y\big) = F\big(e^{u_1 + \cdots + u_m} x, e^{u_1 + \cdots + u_m} y\big) \\
&= F\big(e^{u_1} \cdots e^{u_m} x, e^{u_1} \cdots e^{u_m} y\big) = e^{2u_1} \cdots e^{2u_m} F(x,y) \\
&= \big(e^{u_1 + \cdots + u_m}\big)^2 F(x,y) = \big(e^{1+\log a}\big)^2 F(x,y) = e^2 a^2 F(x,y)
\end{aligned}
$$

for all $a \in D$ and all $x, y \in {}_A\mathcal{M} \setminus \{0\}$. Since $1^\star = 11^\star = (1^\star)^\star 1^\star = (11^\star)^\star = (1^\star)^\star = 1$, we have $11^\star = 1^\star 1 = 1$. So 1 is unitary. Thus $e = e^1 \in \exp(\mathcal{U}(A))$. Hence we have $e^2 F(ax, ay) = F(eax, eay) = e^2 a^2 F(x, y)$ for all $a \in D$ and all $x, y \in {}_A\mathcal{M} \setminus \{0\}$. Therefore we obtain $F(ax, ay) = a^2 F(x, y)$ for all $a \in D$ and all $x, y \in {}_A\mathcal{M}$. By the same process as the above argument, one can see that $F(ax, ay) = a^2 F(x, y)$ for all $a \in E$ and all $x, y \in {}_A\mathcal{M}$.

By the same reasoning as in the proof of Theorem 1 with the assumption that $f(ax, ay)$ is continuous in $a \in \mathbb{R}$ for each fixed $x, y \in {}_A\mathcal{M}$, we obtain that F satisfies the equation (6) for all $t \in \mathbb{R}$ and all $x, y \in {}_A\mathcal{M}$. Since $D \cup E$ is dense in A_{in} for $a \in A_{in}$, there exists a sequence $\{d_j\}$ in $(D \cup E) \setminus \{0\}$ such that $d_j \to a$ as $j \to \infty$. By the assumption that $f(ax, ay)$ is continuous in $a \in A_1 \cup \mathbb{R}$ for each fixed $x, y \in {}_A\mathcal{M}$, we have

$$
\begin{aligned}
(17) \quad F(ax, ay) &= F\left(\lim_{j\to\infty} |d_j| \frac{1}{|d_j|} d_j x, \lim_{j\to\infty} |d_j| \frac{1}{|d_j|} d_j y \right) \\
&= \lim_{j\to\infty} F\left(|d_j| \frac{1}{|d_j|} d_j x, |d_j| \frac{1}{|d_j|} d_j y \right) \\
&= \lim_{j\to\infty} |d_j|^2 F\left(\frac{1}{|d_j|} d_j x, \frac{1}{|d_j|} d_j y \right) \\
&= \lim_{j\to\infty} |d_j|^2 \left(\frac{1}{|d_j|} d_j \right)^2 F(x, y) = \lim_{j\to\infty} d_j^2 F(x, y) = a^2 F(x, y)
\end{aligned}
$$

for all $a \in A_{in}$ and all $x, y \in {}_A\mathcal{M}$.

Since $\left\{ \frac{1}{4^j} f\big(2^j ax, 2^j ay\big) \right\}$ is uniformly converges on A_1 and $f(ax, ay)$ is continuous in $a \in A_1$, we see that $F(ax, ay)$ is also continuous in $a \in A_1$ for each $x, y \in {}_A\mathcal{M}$. Let $b \in A \setminus (A_{in} \cup \{0\})$. Since $A_{in} \cap A_{sa}$ is dense in A_{sa}, there exists a sequence $\{b_j\}$ in $A_{in} \cap A_{sa}$ such that $b_j \to b$ as $j \to \infty$. Put $a_j := \frac{1}{|b_j|} b_j$. Then $a_j \to \frac{1}{|b|} b$ as $j \to \infty$ and $a_j \in A_1 \setminus A_{in}$. By the continuity of F, we have

$$
(18) \qquad \lim_{j\to\infty} F(a_j x, a_j y) = F\Big(\lim_{j\to\infty} a_j x, \lim_{j\to\infty} a_j y \Big) = F\left(\frac{1}{|b|} bx, \frac{1}{|b|} by \right)
$$

for all $x, y \in {}_A\mathcal{M}$. By the equality (17), we obtain that

$$
\begin{aligned}
\left\| F(a_j x, a_j y) - \left(\frac{1}{|b|} b \right)^2 F(x, y) \right\| &= \left\| {a_j}^2 F(x, y) - \left(\frac{1}{|b|} b \right)^2 F(x, y) \right\| \\
&\to \left\| \left(\frac{1}{|b|} b \right)^2 F(x, y) - \left(\frac{1}{|b|} b \right)^2 F(x, y) \right\| = 0
\end{aligned}
$$

as $j \to \infty$. By the equality (18) and the above convergence, we see that

$$
\begin{aligned}
&\left\| F\left(\frac{1}{|b|} bx, \frac{1}{|b|} by \right) - \left(\frac{1}{|b|} b \right)^2 F(x, y) \right\| \\
&\le \left\| F\left(\frac{1}{|b|} bx, \frac{1}{|b|} by \right) - F(a_j x, a_j y) \right\| + \left\| F(a_j x, a_j y) - \left(\frac{1}{|b|} b \right)^2 F(x, y) \right\| \to 0
\end{aligned}
$$

as $j \to \infty$ for all $x, y \in {}_A\mathcal{M}$. By the above convergence and the equation (6), we have

$$F(bx, by) = F\left(|b|\frac{1}{|b|}bx, |b|\frac{1}{|b|}by\right) = |b|^2\left(\frac{1}{|b|}b\right)^2 F(x, y) = b^2 F(x, y)$$

for all $x, y \in {}_A\mathcal{M}$. By the equality (17) and the above equality, $F(ax, ay) = a^2 F(x, y)$ for all $a \in A \setminus \{0\}$ and all $x, y \in {}_A\mathcal{M}$. By the equation (6), we get $F(0x, 0y) = 0^2 F(x, y)$ for all $x, y \in {}_A\mathcal{M}$. Therefore $F(ax, ay) = a^2 F(x, y)$ for all $a \in A$ and all $x, y \in {}_A\mathcal{M}$. □

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