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ON SEPARATIVE REFINEMENT MONOIDS

Huanyin Chen

Abstract. We obtain two new characterizations of separativity of refinement monoids, in terms of comparability-type conditions. As applications, we get several equivalent conditions of separativity for exchange ideals.

A commutative monoid $(M, +, 0)$ has a refinement (or is a refinement monoid) if, for all a, b, c and d in M, the equation $a+b = c+d$ implies the existence of $a_1, b_1, c_1, d_1 \in M$ such that $a = a_1 + d_1, b = b_1 + c_1, c = a_1 + b_1$ and $d = d_1 + c_1$. These equations are represented in the form of a refinement matrix: $\begin{smallmatrix} & a & b \\ c & a_1 & b_1 \\ d & d_1 & c_1 \end{smallmatrix}$. Refinement monoids have been extensively studied in recent years (cf. [4] and [7]). A commutative monoid M is separative if, for all $a, b \in M$, $2a = a+b = 2b$ implies $a = b$. Separativity is a weak form of cancellativity for commutative monoids. Many authors have studied separative refinement monoids from various view-points (see [3-4] and [6-7]). In this article, we get two new characterizations of separative refinement monoids. We prove that every separative refinement monoid can be characterized by a certain sort of comparability. Also we introduce the concept of refinement extensions of a refinement monoid. We see that every separative refinement monoid can be characterized by such refinement extensions. Let I be an ideal of a ring R. We use $FP(I)$ to denote the class of finitely generated projective right R-modules P with $P = PI$ and $V(I)$ to denote the monoid of isomorphism classes of objects from $FP(I)$. Following Ara et al. (see [3]), an exchange ideal I of a ring R is separative if $V(I)$ is a separative refinement monoid, that is, for any $A, B, C \in FP(I)$, $A \oplus A \cong A \oplus B \cong B \oplus B \Longrightarrow A \cong B$. We say that R is a separative ring if R is separative as an ideal of R.

Separativity plays a key role in the direct sum decomposition theory of exchange rings. It seems rather likely that non-separative exchange rings should exist. We say that an exchange ring R satisfies the comparability axiom provided that, for any finitely generated projective right R-modules A and B, either $A \lesssim^{\oplus} B$ or $B \lesssim^{\oplus} A$. In [7, Theorem 3.9], Pardo showed that every

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exchange ring satisfying the comparability axiom is separative. But the converse is not true. For instance, there exist unit-regular rings which do not satisfy the comparability axiom (see [5, Example 8.7]). We will give a new characterization of the separativity for exchange ideals of a ring, in terms of comparability-type conditions. Using refinement extensions of a refinement monoid, we observe that the separativity over exchange ideals possesses nice weak cancellation properties for arbitrary right modules.

Throughout, all monoids are commutative, so we will write + for the monoid operation and 0 for the identity elements of all monoids. Every monoid M will be endowed with the preordering $\leq$ defined by $a \leq b$ in M if and only if there exists some $c \in M$ such that $a + c = b$. A monoid M has an order-unit u if $u \in M$ is an element such that every element of M is bounded above by a positive multiple of u. A subclass I of a monoid M is an o-ideal provided that $(\forall x, y \in I \iff x+y \in I)$. All rings in this article are associative with identities and all modules are right unitary modules. Let A and B be right R-modules. The symbol $A \lesssim^{\oplus} B$ means that A is isomorphic to a direct summand of B and nA means that the direct sum of n copies of A. We always use $\mathbb{N}$ to denote the set of all positive integers.

For refinement monoids M, we note that separativity can be reduced to the statement $(\forall a, b, c \in M)(a + c = b + c$ with $c \leq a, b \Longrightarrow a = b)$. In general, this property is weaker than separativity. This follows very easily from [3, Lemma 2.1].

Theorem 1. *Let M be a refinement monoid. Then the following are equivalent*:

(1) *M is separative.*
(2) $(\forall a, b \in M)(a = 2b$ *and* $3a = 3b \Longrightarrow a \leq b$ *or* $b \leq a)$.
(3) $(\forall a, b, c \in M)(c + a = c + b$ *with* $c \leq a, b \Longrightarrow a \leq b$ *or* $b \leq a)$.

Proof. (1) $\Rightarrow$ (2) is clear by [3, Lemma 2.1].

(2) $\Rightarrow$ (3) Given $c + a = c + b$ with $c \leq a, b$ in M, then there exist $e, f \in M$ such that $a = c + e$ and $b = c + f$; hence, $2a = a + (c + e) = (b + c) + e = b + a$. Likewise, we have $2b = a + b$. This implies that $2a = 2b$. Furthermore, we get $3a = a + 2b = (a + b) + b = 3b$. So either $a \leq b$ or $b \leq a$.

(3) $\Rightarrow$ (1) Suppose that $c + a = c + b$ with $c \leq a, b$. It will suffice to show that $a = b$. In view of [3, Lemma 2.7], we have a refinement matrix over M : $\begin{array}{c} \\ a \\ c \end{array}\!\!\begin{array}{c} \begin{matrix} b & c \end{matrix} \\ \begin{pmatrix} d_1 & a_1 \\ b_1 & c_1 \end{pmatrix} \end{array}$, where $c_1 \leq a_1, b_1$. From $a_1 + c_1 = b_1 + c_1$ with $c_1 \leq a_1, b_1$, we deduce that $a_1 \leq b_1$ or $b_1 \leq a_1$. If $a_1 \leq b_1$, then $b_1 = a_1 + e$. Thus $c = c_1 + b_1 = c_1 + a_1 + e = c + e$. As $c \leq a, b$, we have $a = a + e$ and $b = b + e$; hence, $a = a + e = a_1 + d_1 + e = b_1 + d_1 = b$. The proof of the case $b_1 \leq a_1$ is just symmetric from the case $a_1 \leq b_1$. By the note above, we obtain the result. $\square$

Corollary 2. *Let M be a refinement monoid. Then the following are equivalent*:

(1) *M is separative.*
(2) $(\forall a, b \in M)(\forall n \in \mathbb{N})(na = nb$ *and* $(n+1)a = (n+1)b \Longrightarrow a \leq b$ *or* $b \leq a)$.
(3) $(\forall a, b \in M)(2a = a + b = 2b \Longrightarrow a \leq b$ *or* $b \leq a)$.

Proof. (1) $\Rightarrow$ (2) is obvious by [3, Lemma 2.1].

(2) $\Rightarrow$ (3) Suppose that $2a = a + b = 2b$ in M. Then $2a = 2b$ and $3a = a + (a+b) = 2a + b = 3b$. Furthermore, we get $4a = 3b + a = 2b + (a+b) = 4b$. Similarly, we deduce that $na = nb$ and $(n+1)a = (n+1)b$ $(n \geq 2)$. So $a \leq b$ or $b \leq a$.

(3) $\Rightarrow$ (1) Given $c + a = c + b$ with $c \leq a, b$ in M, then we have $e, f \in M$ such that $a = c + e$ and $b = c + f$. It is easy to check that $a + a = a + (c + e) = (b + c) + e = b + a$. Similarly, $b + b = a + b$. Hence $a \leq b$ or $b \leq a$. By virtue of Theorem 1, the result follows. □

We say that M is an ordered-separative monoid provided that $(\forall a, b \in M)(a + b = 2b \Longrightarrow a \leq b)$. In [8, Theorem 4.1], Wehrung proved that if M is separative, then M is order-separative. We note that 'separativity' used in [8] differs from that in this paper. Wehrung's 'separativity' satisfies an additional condition: $(\forall a, b, c \in M)(a + c \leq b + c$ with $c \propto b \Rightarrow a \leq b)$. A natural problem asked whether the converse is true. In general, order-separativity does not imply general separativity. Let M be the monoid generated by three elements a, b and c such that $2a = 0, a + b = c, a + c = b$ and $b + c = 2b$. Then $M = \{0, a, b, c, 2b, 3b, 4b, \ldots\}$ defined by the following addition:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	$2b$	$2b$
c	c	b	$2b$	$2b$

and $a + mb = mb, c + mb = (m+1)b (m \geq 2)$. As $2b = b + c = 2c$, M is not a separative monoid. But one checks that M is an ordered-separative monoid. In [4, Proposition 9.5], Brookfield proved that every refinement order-separative monoid is separative. Now we generalize Brookfield's result as follows.

Corollary 3. *Let M be a refinement monoid. If $(\forall a, b \in M)(a + b = 2b \Longrightarrow a \leq b$ or $b \leq a)$, then M is separative.*

Proof. It is obvious from Corollary 2. □

The converse of Corollary 3 is not true. Let $\{0, \infty\}$ be the monoid such that $\infty + \infty = \infty$, and let $\mathbb{R}^{++}$ the subgroup of strictly positive real numbers. Let M be the monoid obtained from $\{0, \infty\} \times \mathbb{R}^{++}$ by adding a zero element. Since $\{0, \infty\}$ and $\mathbb{R}^{++}$ are separative refinement subgroups, we prove that M is a separative refinement monoid. Choose $a = (0, 1)$ and $b = (\infty, 1)$. Then $a + b = 2b$, while $a \nleq b$ and $b \nleq a$.

Following Ara (cf. [1-2]), an ideal I of a ring R is an exchange ideal provided that for every $x \in I$ there exist an idempotent $e \in I$ and elements $r, s \in I$ such that $e = xr = x + s - xs$. Let I be an exchange ideal of a ring R, and let $e \in R$ be an idempotent. By [1, Lemma 1.1], one easily checks that eIe is an exchange ring.

Lemma 4. *Let I be an exchange ideal of a ring R. Then for all right R-modules A, B, C, D such that $A \oplus B \cong C \oplus D$ and $A \in FP(I)$, there are right R-modules $A_1 \cong A_2, B_1 \cong B_2, C_1 \cong C_2, D_1 \cong D_2$ such that $A = A_1 \oplus D_1, B = B_1 \oplus C_1, C = A_2 \oplus B_2$, and $D = D_2 \oplus C_2$.*

Proof. Suppose that $\psi : A \oplus B \cong C \oplus D$. Then $A \oplus B = \psi^{-1}(C) \oplus \psi^{-1}(D)$. As $A \in FP(I)$, there is a right R-module E such that $A \oplus E \cong nR$ for some $n \in \mathbb{N}$. Let $e : nR \to A$ be the projection onto A. Then $A \cong e(nR)$, whence $\mathrm{End}_R(A) \cong eM_n(R)e$. As $A = AI$, we have $e(nR) = e(nR)I \subseteq nI$. Set $e = (\alpha_1, \ldots, \alpha_1) \in M_n(R)$. Then $e(1, 0, \ldots, 0)^T \in nI$; hence, $\alpha_1 \in nI$. Likewise, we have $\alpha_2, \ldots, \alpha_n \in nI$. It follows that $e \in M_n(I)$. Since I is an exchange ideal of R, $M_n(I)$ is also an exchange ideal of $M_n(R)$, and then $\mathrm{End}_R(A)$ is an exchange ring. Thus A has the finitely exchange property. So we can find $B_1 \subseteq \psi^{-1}(C)$ and $B_2 \subseteq \psi^{-1}(D)$ such that $A \oplus B = A \oplus B_1 \oplus B_2$. So $B \cong B_1 \oplus B_2$. As $B_1 \subseteq \psi^{-1}(C) \subseteq B_1 \oplus (A \oplus B_2)$, we get $\psi^{-1}(C) = \psi^{-1}(C) \bigcap (B_1 \oplus A \oplus B_2) = B_1 \oplus \psi^{-1}(C) \bigcap (A \oplus B_2)$. Let $C_1 = \psi(C) \bigcap (A \oplus B_2)$. Then $C \cong \psi^{-1}(C) = B_1 \oplus C_1$, Likewise, we have a right R-module D_1 such that $D \cong \psi^{-1}(D) = B_2 \oplus D_1$. In addition, $A \oplus (B_1 \oplus B_2) = A \oplus B = \psi^{-1}(C) \oplus \psi^{-1}(D) = (B_1 \oplus C_1) \oplus (B_2 \oplus D_1)$; hence, $A \cong C_1 \oplus D_1$. Therefore we complete the proof. □

Theorem 5. *Let I be an exchange ideal of a ring R. Then the following are equivalent*:

(1) *I is separative.*
(2) *For any $A, B \in FP(I)$, $C \oplus A \cong C \oplus B$ with $C \precsim^{\oplus} A, B \Longrightarrow A \precsim^{\oplus} B$ or $B \precsim^{\oplus} A$.*
(3) *For any $A, B \in FP(I)$, $2A = 2B$ and $3A = 3B \Longrightarrow A \precsim^{\oplus} B$ or $B \precsim^{\oplus} A$.*
(4) *For any $A, B \in FP(I)$, $2A \cong A \oplus B \cong 2B \Longrightarrow A \precsim^{\oplus} B$ or $B \precsim^{\oplus} A$.*

Proof. In view of Lemma 4, $V(I)$ is a refinement monoid. Applying Theorem 1 and Corollary 2 to $V(I)$, we obtain the result. □

Corollary 6. *Let I be an exchange ideal of a ring R, and let $m, n \geq 2$ with $gcd(m, n) = 1$. Then the following are equivalent*:

(1) *I is separative.*
(2) *For any $A, B \in FP(I)$, $mA \cong mB$ and $nA \cong nB \Longrightarrow A \precsim^{\oplus} B$ or $B \precsim^{\oplus} A$.*

Proof. (1) ⇒ (2) Since I is separative, $V(I)$ is a separative monoid. According to [4, Proposition 8.10], we get $A \cong B$, as desired.

(2) $\Rightarrow$ (1) Given $2A \cong A \oplus B \cong 2B$ with $A, B \in FP(I)$, then $mA \cong mB$ and $nA \cong nB$ by an easy induction. So either $A \lesssim^{\oplus} B$ or $B \lesssim^{\oplus} A$. In view of Theorem 5, I is separative. $\square$

So far, we have been investigating separativity only in a refinement monoid. Let M be a refinement submonoid of a monoid N. We say that N is a refinement extension of M in case the following hold:

(1) M is an o-ideal of N.

(2) $(\forall b, c, d \in N)(\forall a \in M)(a + b = c + d \Longrightarrow$ there exists a refinement matrix over N : $\begin{smallmatrix} & a & b \\ c & a_1 & b_1 \\ d & d_1 & c_1 \end{smallmatrix}$.

We write $\{0, 1, \infty\}$ for the monoid such that $1 + 1 = 1 + \infty = \infty + \infty = \infty$. Since the equation $1 + 1 = \infty + \infty$ can not be refined, $\{0, 1, \infty\}$ is not a refinement monoid. The monoid $\{0, \infty\}$ is a refinement submonoid of $\{0, 1, \infty\}$. Obviously, the condition (1) is equivalent to the statement: $(\forall a \in M)(b \leq a$ in $N \Longrightarrow b \in M)$. Although the condition (2) is satisfied in this case, we claim that $\{0, 1, \infty\}$ is not a refinement extension of $\{0, \infty\}$. This is clear from $1 \leq \infty$ and $1 \notin \{0, \infty\}$. Let $\mathbb{Z}^+$ be the monoid of non-negative integers, and let $A = \{0, 2, 3, 4, \ldots\}$ be the submonoid of $\mathbb{Z}^+$ obtained by deleting the number 1. Let $N = A \times \{0, \infty\}$ and $M = 0 \times \{0, \infty\}$. Then N is a refinement extension of the refinement M, while N is not a refinement monoid. As $(2, 0) \not\leq (3, 0)$ in N, we see that $(2, 0) + (4, 0) = (3, 0) + (3, 0)$ has no a refinement matrix over N. Now we observe that separativity can be partially extended to refinement extensions of a refinement monoid.

Theorem 7. *Let N be a refinement extension of a refinement monoid M. Then the following are equivalent*:

(1) *M is separative.*

(2) $(\forall a, b \in N)(\forall c \in M)(c + a = c + b$ *with* $c \leq a, b \Longrightarrow a = b)$.

(3) $(\forall a, b \in N)(\forall c \in M)(2c + a = 2c + b \Longrightarrow c + a = c + b)$.

Proof. (1) $\Rightarrow$ (2) Suppose that $c \in M, a, b \in N$ such that $c + a = c + b$ with $c \leq a, b$. Since N is a refinement extension of M, we have a refinement matrix over N : $\begin{smallmatrix} & c & a \\ c & c_1 & a_1 \\ b & b_1 & d_1 \end{smallmatrix}$. From $c_1 \leq c \leq a = a_1 + d_1$ in N, there exists some $e \in N$ such that $c_1 + e = a_1 + d_1$. As $c_1 \leq c$ and $c \in M$, we deduce that $c_1 \in M$. Furthermore, we get a refinement matrix over N : $\begin{smallmatrix} & c_1 & e \\ a_1 & a_1' & e' \\ d_1 & d_1' & f \end{smallmatrix}$. Hence $c_1 = a_1' + d_1'$ and $a_1' \leq a_1, d_1' \leq d_1$. So we can write that $d_1 = d_1' + d_1''$ for some $d_1'' \in N$. Thus there is a refinement matrix over N : $\begin{smallmatrix} & c & a \\ c & a_1' & a_1 + d_1' \\ b & b_1 + d_1' & d_1'' \end{smallmatrix}$. Let $c_2 = a_1', a_2 = a_1 + d_1', b_2 = b_1 + d_1'$ and $d_2 = d_1''$. We get a refinement matrix over N :

$$(*) \qquad \begin{array}{cc} & \begin{array}{cc} c & a \end{array} \\ \begin{array}{c} c \\ b \end{array} & \begin{pmatrix} c_2 & a_2 \\ b_2 & d_2 \end{pmatrix} \end{array}$$

with $c_2 \leq a_2$. In addition, we see that $c_2 \leq c_1$ and $b_1 \leq b_2$. As $c \leq b$, we apply the argument above to the refinement matrix $(*)$ and get a new refinement matrix over N : $\begin{array}{cc} & \begin{matrix} c & a \end{matrix} \\ \begin{matrix} c \\ b \end{matrix} & \begin{pmatrix} c_3 & a_3 \\ b_3 & d_3 \end{pmatrix} \end{array}$ with $c_3 \leq b_3$. Furthermore, we have $c_3 \leq c_2 \leq a_2 \leq a_3$. Thus $c_3 + a_3 = c_3 + b_3 = c \in M$ with $c_3 \leq a_3, b_3$. Clearly, $c_3, a_3, b_3 \in M$. As M is a separative monoid, it follows that $a_3 = b_3$. Therefore $a = a_3 + d_3 = b_3 + d_3 = b$, as desired.

(2) $\Rightarrow$ (3) Suppose that $c \in M$ and $a, b \in N$ such that $2c + a = 2c + b$. Then $c + (c + a) = c + (c + b)$ with $c \leq c + a, c + b$; hence, $c + a = c + b$.

(3) $\Rightarrow$ (1) is trivial by [3, Lemma 2.1]. □

Let a and b be elements in a monoid. The notation $a \propto b$ means that $a \leq nb$ for some $n \in \mathbb{N}$.

Corollary 8. *Let N be a refinement extension of a refinement monoid M. Then the following are equivalent*:

(1) *M is separative.*

(2) $(\forall a, b \in N)(\forall c \in M)(c + a = c + b$ *with* $c \propto a, b \Longrightarrow a = b)$.

Proof. (1) $\Rightarrow$ (2) Suppose that $c + a = c + b$ and $c \in M, c \propto a, b$. Then we may choose $n \in \mathbb{N}$ such that $c \leq na, nb$. So there exists $d \in N$ such that $c + d = a + (n-1)a$. Since N is a refinement extension of M, we have a refinement matrix over N : $\begin{array}{cc} & \begin{matrix} c & d \end{matrix} \\ \begin{matrix} a \\ (n-1)a \end{matrix} & \begin{pmatrix} c_1 & d_1 \\ a_1 & e_1 \end{pmatrix} \end{array}$. This infers that $a_1 + e_1 = a + (n-2)a$. It follows by $a_1 \leq c \in M$ that $a_1 \in M$. Similarly, we have a refinement matrix over N : $\begin{array}{cc} & \begin{matrix} a_1 & e_1 \end{matrix} \\ \begin{matrix} a \\ (n-2)a \end{matrix} & \begin{pmatrix} c_2 & d_2 \\ a_2 & e_2 \end{pmatrix} \end{array}$. Furthermore, we have a refinement matrix over N : $\begin{array}{cc} & \begin{matrix} a_{n-2} & e_{n-2} \end{matrix} \\ \begin{matrix} a \\ a \end{matrix} & \begin{pmatrix} c_{n-1} & d_{n-1} \\ a_{n-1} & e_{n-1} \end{pmatrix} \end{array}$. Hence $c = c_1 + a_1 = c_1 + (c_2 + a_2) = \cdots = c_1 + c_2 + \cdots + c_{n-1} + a_{n-1}$. Set $c_n = a_{n-1}$. Then $c = c_1 + c_2 + \cdots + c_n$ with $c_1, \ldots, c_n \leq a$. As $c_1 \leq c \in M$, we see that $c_1 \in M$. Similarly, we prove that $c_1 = c_{11} + \cdots + c_{1m_1}$ with $c_{1j} \leq b$ $(j = 1, \ldots, m_1)$. Analogously, we have $c_{ij} \in M$ such that $c_i = c_{i1} + \cdots + c_{im_i} (i = 2, \ldots, n)$. As a result, $(\sum\limits_{1 \leq i \leq n, 1 \leq j \leq m_i} c_{ij}) + a = (\sum\limits_{1 \leq i \leq n, 1 \leq j \leq m_i} c_{ij}) + b$ with all $c_{ij} \in M, c_{ij} \leq a, b$. By using Theorem 7 repeatedly, we get $a = b$, as required.

(2) $\Rightarrow$ (1) is obvious by [3, Lemma 2.1]. □

Recall that a right R-module P is a R-progenerator in case there exist $m, n \in \mathbb{N}$ and modules P' and R' such that $mR \cong P \oplus P'$ and $nP \cong R \oplus R'$. Let I be a separative exchange ideal of a ring R, and let C be a finitely generated projective right R-module with $C = CI$. If A and B are any R-progenerators such that $C \oplus A \cong C \oplus B$, we claim that $A \cong B$. This is an immediate consequence of Corollary 8.

Theorem 9. *Let I be an exchange ideal of a ring R. Then the following are equivalent*:

(1) *I is separative.*

(2) *For any* $C \in FP(I)$, $C \oplus A \cong C \oplus B$ *with* $C \lesssim^{\oplus} A, B \Longrightarrow A \cong B$ *for any right* R*-modules* A *and* B.

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{M}_R$ denote the class of all right R-modules, and let $W(R)$ be the monoid of isomorphism classes of objects from $\mathcal{M}_R$. Then $V(I)$ is a submonoid of $W(R)$. Suppose that $C \oplus A \cong C \oplus B$ with $C \in FP(I)$ and $A, B \in \mathcal{M}_R$. According to Lemma 4, we have a refinement matrix $\begin{smallmatrix} & C & A \\ C & C_1 & A_1 \\ B & B_1 & D_1 \end{smallmatrix}$ over $W(R)$. This means that $W(R)$ is a refinement extension of the refinement monoid $V(I)$. It follows by Theorem 7 that $A \cong B$.

(2) $\Rightarrow$ (1) For any $A, B, C \in FP(I)$, $C \oplus A \cong C \oplus B$ with $C \lesssim^{\oplus} A, B \Longrightarrow A \cong B$, and therefore the result follows from [3, Lemma 2.1]. □

Corollary 10. *Let* A *be a finitely generated projective right module over a separative exchange ring* R. *If* A *and* B *are any right* R*-modules such that* $C \oplus A \cong C \oplus B$ *with* $C \lesssim^{\oplus} A, B$, *then* $A \cong B$.

Proof. It is obvious by Theorem 9. □

Theorem 11. *Let* N *be a refinement extension of a refinement monoid* M. *If* N *contains an order-unit* u, *then the following are equivalent*:

(1) M *is separative.*

(2) $(\forall a, b \in N)(\forall c \in M)(c + a = c + b \leq u$ *with* $c \leq a, b \Longrightarrow a \leq b$ *or* $b \leq a)$.

(3) $(\forall a, b, c \in M)(c + a = c + b \leq u$ *with* $c \leq a, b \Longrightarrow a \leq b$ *or* $b \leq a)$.

Proof. (1) $\Rightarrow$ (2) is obvious by Theorem 7.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Given $c+a = c+b$ with $c \leq a, b$ in M, then we can find some $n \in \mathbb{N}$ such that $c \leq nu$ in N. Since N is a refinement extension of M, by induction, the refinement property also holds for the sum nu. So there exist $c_1, \dots, c_n \leq u$ such that $c = c_1 + \cdots + c_n$. Hence $c_1 + (c_2 + \cdots + c_n + a) = c_1 + (c_2 + \cdots + c_n + b)$. Let $a_1 = c_2 + \cdots + c_n + a$ and $b_1 = c_2 + \cdots + c_n + b$. Then $c_1 + a_1 = c_1 + b_1$ with $c_1 \in M$ and $c_1 \leq a_1, b_1$. By the proof of Theorem 7, we have a refinement matrix over N : $\begin{smallmatrix} & c_1 & a_1 \\ c_1 & c_1' & a_1' \\ b_1 & b_1' & d_1 \end{smallmatrix}$, where $c_1' \leq a_1', b_1'$. It follows from $c_1' + a_1' = c_1' + b_1' = c_1 \leq u$ with $c_1' \leq a_1', b_1'$ that either $a_1' \leq b_1'$ or $b_1' \leq a_1'$. If $a_1' \leq b_1'$, then $b_1' = a_1' + e$. As a result, we get $c_1 = c_1' + b_1' = c_1' + a_1' + e = c_1 + e$. Since $c_1 \leq a_1, b_1$, we see that $a_1 = a_1 + e$ and $b_1 = b_1 + e$, whence $a_1 = a_1 + e = a_1' + d_1 + e = b_1' + d_1 = b_1$. Similarly, we deduce that $a_1 = b_1$ if $b_1' \leq a_1'$. This means that $c_2 + (c_3 + \cdots + c_n + a) = c_2 + (c_3 + \cdots + c_n + b)$. By iteration of this process, we get $a = b$. Therefore, M is separative, which concludes the proof. □

Corollary 12. *Let* N *be a refinement extension of a refinement monoid* M. *If* N *contains an order-unit* u, *then the following are equivalent*:

(1) M *is separative.*

(2) $(\forall a, b \in N)(\forall c \in M)(c + a = c + b \leq u$ *with* $c \propto a, b \Longrightarrow a \leq b$ *or* $b \leq a)$.

Proof. (2) $\Rightarrow$ (1) follows from Theorem 11.

(1) $\Rightarrow$ (2) Suppose that $c + a = c + b \leq u$ and $c \in M, c \propto a, b$. Then we may choose $n \in \mathbb{N}$ such that $c \leq na, nb$. Thus we have $d \in N$ such that $c + d = a + (n-1)a$. Analogously to Corollary 8, there are refinement matrices over N :

$$\begin{array}{c} \\ a \\ (n-1)a \end{array}\begin{array}{c} \begin{array}{cc} c & d \end{array} \\ \begin{pmatrix} c_1 & d_1 \\ a_1 & e_1 \end{pmatrix} \end{array}, \quad \begin{array}{c} \\ a \\ (n-2)a \end{array}\begin{array}{c} \begin{array}{cc} a_1 & e_1 \end{array} \\ \begin{pmatrix} c_2 & d_2 \\ a_2 & e_2 \end{pmatrix} \end{array}, \ \ldots, \ \begin{array}{c} \\ a \\ a \end{array}\begin{array}{c} \begin{array}{cc} a_{n-2} & e_{n-2} \end{array} \\ \begin{pmatrix} c_{n-1} & d_{n-1} \\ a_{n-1} & e_{n-1} \end{pmatrix} \end{array}.$$

Let $c_n = a_{n-1}$. Then $c = c_1 + c_2 + \cdots c_n$ with $c_1, \ldots, c_n \leq a$. Similarly, we prove that $c_1 = c_{11} + \cdots + c_{1m_1}$ with $c_{1j} \leq a, b(j = 1, \ldots, m_1)$. Analogously, we have $c_{ij} \in M$ such that $c_i = c_{i1} + \cdots + c_{im_i}$ and $c_{ij} \leq a, b(i = 2, \ldots, n, j = 1, \ldots, m_i)$. This implies that $(\sum\limits_{1 \leq i \leq n, 1 \leq j \leq m_i} c_{ij}) + a = (\sum\limits_{1 \leq i \leq n, 1 \leq j \leq m_i} c_{ij}) + b$ with all $c_{ij} \in M, c_{ij} \leq a, b$. Using Theorem 7 repeatedly, we conclude that $a = b$, as desired. □

Theorem 13. *Let N be a refinement extension of a refinement monoid M. If N contains an order-unit u, then the following are equivalent*:

(1) *M is separative.*

(2) $(\forall a, b \in M)(2a = a + b = 2b \leq u \Longrightarrow a = b$.

(3) $(\forall a, b \in M)(2a = a + b = 2b \leq u \Longrightarrow a \leq b$ *or* $b \leq a)$.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial.

(3) $\Rightarrow$ (1) Given $c + a = c + b \leq u$ with $c \leq a, b$ and $a, b, c \in M$, it follows by [3, Lemma 2.7] that there is a refinement matrix over M : $\begin{smallmatrix} \\ c \\ b \end{smallmatrix}\begin{smallmatrix} c \quad a \\ \begin{pmatrix} c_1 & a_1 \\ b_1 & d \end{pmatrix} \end{smallmatrix}$ with $c_1 \leq a_1, b_1$. So we can find $x, y \in M$ such that $a_1 = c_1 + x$ and $b_1 = c_1 + y$, and then $2c_1 + x = c_1 + a_1 = c = c_1 + b_1 = 2c_1 + y$. This implies that $2(c_1 + x) = (c_1 + x) + (c_1 + y) = 2(c_1 + y) = a_1 + b_1 \leq a + c \leq u$. By hypothesis, we get $a_1 \leq b_1$ or $b_1 \leq a_1$. As a result, $a = a_1 + d \leq b_1 + d = b$ or $b = b_1 + d \leq a_1 + d = a$, and therefore the proof is true by Theorem 11. □

Corollary 14. *Let I be an exchange ideal of a ring R. Then the following are equivalent*:

(1) *I is separative.*

(2) *For any $A, B, C \in FP(I)$, $A \oplus C \cong B \oplus C \lesssim^{\oplus} R$ with $C \lesssim^{\oplus} A, B \Longrightarrow A \lesssim^{\oplus} B$ or $B \lesssim^{\oplus} A$.*

(2) *For any $A, B \in FP(I)$, $2A \cong A \oplus B \cong 2B \lesssim^{\oplus} R \Longrightarrow A \lesssim^{\oplus} B$ or $B \lesssim^{\oplus} A$.*

Proof. In view of Lemma 4, $V(I)$ is a refinement monoid. Let $FP(R)$ denote the class of finitely generated projective right R-modules, and let $V(R)$ be the monoid of isomorphism classes of objects from $FP(R)$. Then $V(I)$ is a submonoid of $V(R)$. Furthermore, we prove that $V(R)$ is a refinement extension of the refinement monoid $V(I)$ and $V(R)$ contains an order-unit $[R]$. Therefore we complete the proof by Theorem 11 and Theorem 13. □

Recall that a ring R is regular provided that for every $a \in R$ there exists $x \in R$ such that $a = axa$. We say that $a \in R$ is one-sided unit-regular if there exists a right or left invertible $u \in R$ such that $a = aua$. We write $r(a)$ and $\ell(a)$ for the right and left annihilators of $a \in R$. In [3, Proposition 6.2], Ara et al. proved that a regular ring R is separative if and only if each $a \in R$ satisfying $Rr(a) = \ell(a)R = R(1-a)R$ is unit-regular. We generalize this result as follows.

Corollary 15. *Let I be an ideal of a regular ring R. Then the following are equivalent*:

(1) *I is separative.*
(2) *Each $a \in R$ satisfying $RaR \bigcap R(1-a)R \subseteq Rr(a) \bigcap \ell(a)R \bigcap I$ is one-sided unit-regular.*
(3) *Each $a \in R$ satisfying $Rr(a) = \ell(a)R = R(1-a)R \subseteq I$ is one-sided unit-regular.*

Proof. (1)⇒(2) Suppose $RaR \bigcap R(1-a)R \subseteq Rr(a) \bigcap \ell(a)R \bigcap I$. Then $R = \big(r(a) \oplus r(1-a)\big) \oplus B$ for a right R-module B, and so $aR = ar(1-a) + aB = ar(1-a) \oplus aB$. Assume that $a = aca$ for a $c \in R$. Then $R = r(a) \oplus r(1-a) \oplus B = (1-ac)R \oplus r(1-a) \oplus aB$. This yields $r(a) \oplus B \cong (1-ac)R \oplus aB$ with $B \cong aB$. Let $\varphi : aB \to (1-a)aB$ given by $\varphi(ar) = (1-a)ar$ for any $r \in B$. It is easy to verify that φ is a right R-module isomorphism, and so $B \cong a(1-a)B = a(1-a)R$. As $a(1-a)R \subseteq RaR \bigcap R(1-a)R \subseteq Rr(a)$, it follows by [5, Corollary 2.23] that $B \lesssim^{\oplus} mr(a)$ for some $m \in \mathbb{N}$. As $a(1-a)R \subseteq RaR \bigcap R(1-a)R \subseteq \ell(a)R = R(1-ac)R$. By [5, Corollary 2.23] again, $B \lesssim^{\oplus} n(1-ac)R$ for some $n \in \mathbb{N}$. Since $a(1-a) \in I$, we see that $B \in FP(I)$. Let $\mathcal{M}_R$ denote the class of all right R-modules, and let $W(R)$ be the monoid of isomorphism classes of objects from $\mathcal{M}_R$. Analogously to Theorem 9, we prove that $W(R)$ is a refinement extension of the refinement monoid $V(I)$. By Corollary 8, we get $r(a) \cong (1-ac)R \cong R/aR$. This implies that $a \in R$ is unit-regular, as required.

(2) ⇒ (3) is obvious.

(3)⇒(1) Suppose $A \oplus C \cong B \oplus C \lesssim^{\oplus} R$ and $C \lesssim^{\oplus} A, B$ for some $A, B, C \in FP(I)$. Write $R = A_1 \oplus C_1 \oplus D = A_2 \oplus C_2 \oplus D$, where $A_1 \cong A$, $C_1 \cong C \cong C_2$ and $A_2 \cong B$. Let $a \in R$ induce an endomorphism of R_R, which is zero on A_1, an isomorphism from C_1 onto C_2, and the identity on D. Then $(1-a)R = (1-a)(A_1 \oplus C_1)$; hence, $a \in 1+I$. Let $\varphi : A_1 \oplus C_1 \to (1-a)(A_1 \oplus C_1)$ be a right R-module given by $\varphi(x) = (1-a)x$ for any $x \in A_1 \oplus C_1$. Since $(1-a)(A_1 \oplus C_1)$ is a projective right R-module, $(1-a)R \lesssim^{\oplus} A_1 \oplus C_1 \lesssim^{\oplus} 2A_1 = 2r(a)$. By [5, Corollary 2.23], $(1-a)R \subseteq Rr(a)$. This yields $R(1-a)R = Rr(a)$. Assume that $a = aca$ for a $c \in R$. Then $(1-a)R \lesssim^{\oplus} A_1 \oplus C_1 \cong A_2 \oplus C_2 \lesssim^{\oplus} 2A_2 = 2(R/aR) \cong 2(1-ac)R$. Using [5, Corollary 2.23] again, $(1-a)R \subseteq$

$R(1-ac)R = \ell(a)R$, and then $R(1-a)R = \ell(a)R$. By assumption, $a \in R$ is one-sided unit-regular. This shows that $r(a) \lesssim^{\oplus} R/aR$ or $R/aR \lesssim^{\oplus} r(a)$. Thus we have either $A \lesssim^{\oplus} B$ or $B \lesssim^{\oplus} A$. According to Corollary 14, we complete the proof. □

As is well known, every one-sided unit-regular ring is separative. It follows from Corollary 15 that a regular ring R is separative if and only if each $a \in R$ satisfying $RaR(1-a)R \subseteq Rr(a)\ell(a)R$ is one-sided unit-regular.

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Department of Mathematics
Hangzhou Normal University
Hangzhou 310036, China
E-mail address: huanyinchen@yahoo.cn