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# COMMUTING STRUCTURE JACOBI OPERATOR FOR HOPF HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS

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Abstract. In this paper we give a non-existence theorem for Hopf real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ satisfying the condition that the structure Jacobi operator $R_\xi$ commutes with the 3-structure tensors $\phi_i$, $i = 1, 2, 3$.

## 0. Introduction

In the geometry of real hypersurfaces in complex space forms $M_n(c)$ Kimura [6] has proved that Hopf real hypersurfaces in a complex projective space $P_n(\mathbb{C})$ with constant principal curvatures are locally congruent to of type (A), a tube over a totally geodesic $P_k(\mathbb{C})$, of type (B), a tube over a complex quadric $Q_{n-1}$, $\cot^2 2r = n-2$, of type (C), a tube over $P_1(\mathbb{C}) \times P_{(n-1)/2}(\mathbb{C})$, $\cot^2 2r = \frac{1}{n-2}$ and $n$ is odd, of type (D), a tube over a complex two-plane Grassmannian $G_2(\mathbb{C}^5)$, $\cot^2 2r = \frac{3}{5}$ and $n = 9$, of type (E), a tube over a Hermitian symmetric space $SO(10)/U(5)$, $\cot^2 2r = \frac{5}{9}$ and $n = 15$.

The notion of Hopf real hypersurfaces means that the structure vector $\xi$ defined by $\xi = -JN$ satisfies $A\xi = \alpha\xi$, where $J$ denotes a Kaehler structure of $P_n(\mathbb{C})$, $N$ and $A$ a unit normal and the shape operator of $M$ in $P_n(\mathbb{C})$.

A Jacobi field along geodesics of a given Riemannian manifold $(M, g)$ is an important role in the study of differential geometry. It satisfies an well-known differential equation which inspires Jacobi operators. The Jacobi operator is defined by $(R_X(Y))(p) = (R(Y, X)X)(p)$, where $R$ denotes the curvature tensor of $M$ and $X, Y$ denote tangent vector fields on $M$. Then we see that $R_X$ is a self-adjoint endomorphism on the tangent space of $M$ and is related to the differential equation, so called Jacobi equation, which is given by $\nabla_{\gamma'}(\nabla_{\gamma'} Y) +$

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$R(Y,\gamma')\gamma' = 0$ along a geodesic $\gamma$ on $M$, where $\gamma'$ denotes the velocity vector along $\gamma$ on $M$.

When we study a real hypersurface $M$ in a complex space form $M_n(x)$, $c \neq 0$, we will call $R_\xi$ the *structure Jacobi operator* on $M$ with respect to the structure vector $\xi$ and will denote it by $R_\xi$, where $R_\xi$ is defined by $R_\xi(X) = R(X,\xi)\xi$ for the curvature tensor $R$ and any tangent vector field $X$ on $M$.

For a commuting problem in quaternionic space forms Berndt [2] has introduced the notion of normal Jacobi operator $\bar{R}(X,N)N \in \text{End}\, T_xM$, $x \in M$ for real hypersurfaces $M$ in quaternionic projective space $\mathbb{Q}P^m$ or in quaternionic hyperbolic space $\mathbb{Q}H^m$, where $\bar{R}$ denotes the curvature tensor of a quaternionic projective space $\mathbb{Q}P^m$ or of a quaternionic hyperbolic space $\mathbb{Q}H^m$. He [2] also has shown that the curvature adaptedness, that is, the normal Jacobi operator commutes with the shape operator $A$, is equivalent to the fact that the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant by the shape operator $A$ of $M$, where $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

Now let us consider a complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ which consists of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The ambient space $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure $J$ and a quaternionic Kähler structure $\mathfrak{J}$ not containing $J$. So, in $G_2(\mathbb{C}^{m+2})$ we have two natural geometrical conditions for real hypersurfaces that $[\xi] = \text{Span}\{\xi\}$ or $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. By using such kinds of geometric conditions Berndt and Suh [3] have proved the following:

**Theorem A.** *Let $M$ be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of $M$ if and only if*

(A) *$M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *$m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

If the structure vector field $\xi$ of a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator, $M$ is said to be a *Hopf hypersurface*. In such a case the integral curves of the structure vector field $\xi$ are geodesics (see Berndt and Suh [4]). Moreover, the flow generated by the integral curves of the structure vector field $\xi$ for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be the *geodesic Reeb flow*. Moreover, we say that the Reeb vector field is Killing, that is, $\mathcal{L}_\xi g = 0$ for the Lie derivative along the direction of the structure vector field $\xi$, where $g$ denotes the Riemannian metric induced from $G_2(\mathbb{C}^{m+2})$. Then this is equivalent to the fact that the structure tensor $\phi$ commutes with the shape operator $A$ of $M$ in $G_2(\mathbb{C}^{m+2})$. This condition also has the geometric meaning that the flow of Reeb vector field is isometric. Moreover, Berndt and

Suh [4] have proved that real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with isometric Reeb flow is of a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Now we introduce a structure Jacobi operator $R_\xi$ defined in such a way that

$$R_\xi(X) = R(X,\xi)\xi$$

for the curvature tensor $R(X,Y)Z$ of $M$ in $G_2(\mathbb{C}^{m+2})$, where $\xi$ denotes the structure vector, $X$, $Y$ and $Z$ any tangent vector fields of $M$ in $G_2(\mathbb{C}^{m+2})$. Then we put the structure vector $\xi = -JN$ into the curvature tensor $R$ of a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$, where $N$ denotes a unit normal vector of $M$ in $G_2(\mathbb{C}^{m+2})$. Then we calculate the structure Jacobi operator $R_\xi$ in such a way that

$$\begin{aligned} R_\xi(X) &= R(X,\xi)\xi \\ &= X - \eta(X)\xi - \sum_{\nu=1}^{3}\{\eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu + 3g(\phi_\nu X,\xi)\phi_\nu\xi \\ &\qquad + \eta_\nu(\xi)\phi_\nu\phi X\} + \eta(A\xi)AX - \eta(AX)A\xi \end{aligned}$$

for any tangent vector field $X$ on $M$ in $G_2(\mathbb{C}^{m+2})$.

Recently, some geometric properties for such a structure Jacobi operator $R_\xi$ of a real hypersurface in complex projective space $P_n(\mathbb{C})$ or in complex hyperbolic space $H_n(\mathbb{C})$ have been studied by many authors (see [5], [8] and [9]). Among them commuting and parallel properties of such a structure Jacobi operator $R_\xi$ were studied by Ki, Pérez, Santos and Suh [5]. Moreover, $\mathfrak{D}$-parallelity or Lie $\xi$-parallelity of such a structure Jacobi operator are studied by Pérez, Santos and Suh (see [8] and [10]).

On the other hand, the structure Jacobi operator $R_\xi$ is said to be *commuting* if the structure Jacobi operator $R_\xi$ commutes with the 3-structure tensors $\phi_i$, that is, $R_\xi \circ \phi_i = \phi_i \circ R_\xi$, $i = 1,2,3$.

In this paper we give a non-existence theorem and a corollary for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying the condition that the structure Jacobi operator $R_\xi$ commutes with the 3-structure tensors $\phi_i$ as follows:

**Theorem.** *Any Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, do not exist if the structure Jacobi operator commutes with the 3-structure tensors and the $\mathfrak{D}^\perp$ component of the structure vector $\xi$ is invariant by the shape operator.*

**Corollary.** *Any Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, do not exist if the structure Jacobi operator commutes with the 3-structure tensors and the $\mathfrak{D}^\perp$ component of the structure vector $\xi$ has the geodesic Reeb flow.*

## 1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [3] and [4]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+$

2) acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space $G/K$, which we equip with the unique analytic structure for which the natural action of $G$ on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of $G$ and $K$, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form $B$ of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_o G_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since $B$ is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of $B$ this inner product can be extended to a $G$-invariant Riemannian metric $g$ on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize $g$ such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight. Since $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight we will assume $m \geq 2$ from now on. Note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{s}u(m) \oplus \mathfrak{s}u(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure $J$ and the $\mathfrak{s}u(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If $J_1$ is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1 J$, and $JJ_1$ is a symmetric endomorphism with $(JJ_1)^2 = I$ and $tr(JJ_1) = 0$.

A canonical local basis $J_1, J_2, J_3$ of $\mathfrak{J}$ consists of three local almost Hermitian structures $J_\nu$ in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $J_1, J_2, J_3$ of $\mathfrak{J}$ three local one-forms $q_1, q_2, q_3$ such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2} \tag{1.1}$$

for all vector fields $X$ on $G_2(\mathbb{C}^{m+2})$.

Let $p \in G_2(\mathbb{C}^{m+2})$ and $W$ a subspace of $T_p G_2(\mathbb{C}^{m+2})$. We say that $W$ is a quaternionic subspace of $T_p G_2(\mathbb{C}^{m+2})$ if $JW \subset W$ for all $J \in \mathfrak{J}_p$. And we say that $W$ is a totally complex subspace of $T_p G_2(\mathbb{C}^{m+2})$ if there exists a one-dimensional subspace $\mathfrak{V}$ of $\mathfrak{J}_p$ such that $JW \subset W$ for all $J \in \mathfrak{V}$ and $JW \perp W$ for all $J \in \mathfrak{V}^\perp \subset \mathfrak{J}_p$. Here, the orthogonal complement of $\mathfrak{V}$ in $\mathfrak{J}_p$ is taken with respect to the bundle metric and orientation on $\mathfrak{J}$ for which any local oriented orthonormal frame field of $\mathfrak{J}$ is a canonical local basis of $\mathfrak{J}$. A quaternionic (resp. totally complex) submanifold of $G_2(\mathbb{C}^{m+2})$ is a submanifold all of whose tangent spaces are quaternionic (resp. totally complex) subspaces of the corresponding tangent spaces of $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}(1.2)\quad &\bar{R}(X,Y)Z\\ &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ\\ &\quad + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\}\\ &\quad + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned}$$

where $J_1, J_2, J_3$ is any canonical local basis of $\mathfrak{J}$.

## 2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

Now in this section we want to derive the structure Jacobi operator from the curvature tensor of complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ given in (1.2) and the equation of Gauss. Moreover, in this section we derive some basic formulae from the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [13], [14] and [15]).

Let $M$ be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on $M$ will also be denoted by $g$, and $\nabla$ denotes the Riemannian connection of $(M, g)$. Let $N$ be a local unit normal field of $M$ and $A$ the shape operator of $M$ with respect to $N$. The Kähler structure $J$ of $G_2(\mathbb{C}^{m+2})$ induces on $M$ an almost contact metric structure $(\phi, \xi, \eta, g)$. Furthermore, let $J_1, J_2, J_3$ be a canonical local basis of $\mathfrak{J}$. Then each $J_\nu$ induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on $M$. Using the above expression for $\bar{R}$, the Codazzi equation becomes

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\\ &\quad + \sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\big\}\\ &\quad + \sum_{\nu=1}^{3}\big\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\big\}\\ &\quad + \sum_{\nu=1}^{3}\big\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\big\}\xi_\nu\ .\end{aligned}$$

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned}&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \quad \phi\xi_\nu = \phi_\nu\xi,\\ (2.1)\qquad &\eta_\nu(\phi X) = \eta(\phi_\nu X), \quad \phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu,\\ &\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}.\end{aligned}$$

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{2.2}$$

for any tangent vector $X$ of a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$, where $N$ denotes a normal vector of $M$ in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (1.1) and (2.1) we have that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X \xi = \phi AX, \tag{2.3}$$

$$\nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.4}$$

$$\begin{aligned}(\nabla_X \phi_\nu)Y = & -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX \\ & - g(AX,Y)\xi_\nu.\end{aligned} \tag{2.5}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, it follows that

$$\phi\phi_\nu X = \phi_\nu \phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{2.6}$$

Then by (1.2),(2.2) and the equation of Gauss, the Riemannian curvature tensor $R$ of $M$ in $G_2(\mathbb{C}^{n+2})$ is given by

$$\begin{aligned}R(X,Y)Z = \; & g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X \\ & - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ & + \sum_{\nu=1}^{3}\{g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y)\phi_\nu Z\} \\ & + \sum_{\nu=1}^{3}\{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X - g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\} \\ & - \sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\} \\ & - \sum_{\nu=1}^{3}\{\eta(X)g(\phi_\nu\phi Y,Z) - \eta(Y)g(\phi_\nu\phi X,Z)\}\xi_\nu \\ & + g(AY,Z)AX - g(AX,Z)AY.\end{aligned} \tag{2.7}$$

Now let us define a structure Jacobi operator $R_\xi$ from the curvature tensor $R$ in (2.7) in such a way that

$$\begin{aligned}R_\xi X & = R(X,\xi)\xi \\ & = X - \eta(X)\xi - \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu \\ & \qquad + 3g(\phi_\nu X,\xi)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X\Big\} + \alpha AX - \eta(AX)A\xi\,,\end{aligned}$$

where $\alpha = \eta(A\xi)$.

## 3. Hopf hypersurfaces with commuting structure Jacobi operator

Hereafter, unless otherwise stated, the structure Jacobi operator is said to be *commuting* if the structure Jacobi operator $R_\xi$ commutes with the structure tensors $\phi_i$, that is, $R_\xi \circ \phi_i = \phi_i \circ R_\xi$ for $i = 1, 2, 3$.

Let $M$ be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting structure Jacobi operator, that is, $R_\xi \circ \phi_i = \phi_i \circ R_\xi$ for any $i = 1, 2, 3$. Then it follows that

$$\begin{aligned} R_\xi(\phi_i X) = \phi_i X - \eta(\phi_i X)\xi - \sum_{\nu=1}^{3}\{\eta_\nu(\phi_i X)\xi_\nu - \eta(\phi_i X)\eta_\nu(\xi)\xi_\nu \\ + 3g(\phi_\nu\phi_i X, \xi)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi\phi_i X\} + \alpha A\phi_i X - \eta(A\phi_i X)A\xi, \end{aligned}$$

and

$$\begin{aligned} \phi_i R_\xi(X) = \phi_i X - \eta(X)\phi_i\xi - \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_i\xi_\nu - \eta(X)\eta_\nu(\xi)\phi_i\xi_\nu \\ + 3g(\phi_\nu X, \xi)\phi_i\phi_\nu\xi + \eta_\nu(\xi)\phi_i\phi_\nu\phi X\} + \alpha\phi_i AX - \eta(AX)\phi_i A\xi. \end{aligned}$$

For any $i = 1, 2, 3$ the commuting structure Jacobi operator $R_\xi \circ \phi_i = \phi_i \circ R_\xi$ gives

$$\begin{aligned} &\eta(\phi_i X)\xi + \sum_{\nu=1}^{3}\{\eta_\nu(\phi_i X)\xi_\nu - \eta(\phi_i X)\eta_\nu(\xi)\xi_\nu \\ &\quad + 3g(\phi_\nu\phi_i X, \xi)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi\phi_i X\} - \alpha A\phi_i X + \eta(A\phi_i X)A\xi \\ &= \eta(X)\phi_i\xi + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_i\xi_\nu - \eta(X)\eta_\nu(\xi)\phi_i\xi_\nu \\ &\quad + 3g(\phi_\nu X, \xi)\phi_i\phi_\nu\xi + \eta_\nu(\xi)\phi_i\phi_\nu\phi X\} - \alpha\phi_i AX + \eta(AX)\phi_i A\xi. \end{aligned} \tag{3.1}$$

Since we have assumed that $M$ is Hopf, (3.1) gives for any $i = 1, 2, 3$

$$\begin{aligned} &\eta(\phi_i X)\xi + \sum_{\nu=1}^{3}\{\eta_\nu(\phi_i X)\xi_\nu - \eta(\phi_i X)\eta_\nu(\xi)\xi_\nu \\ &\quad + 3g(\phi_\nu\phi_i X, \xi)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi\phi_i X\} - \alpha A\phi_i X + \alpha^2\eta(\phi_i X)\xi \\ &= \eta(X)\phi_i\xi + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_i\xi_\nu - \eta(X)\eta_\nu(\xi)\phi_i\xi_\nu \\ &\quad + 3g(\phi_\nu X, \xi)\phi_i\phi_\nu\xi + \eta_\nu(\xi)\phi_i\phi_\nu\phi X\} - \alpha\phi_i AX + \alpha^2\eta(X)\phi_i\xi. \end{aligned} \tag{3.2}$$

Moreover, by putting $X = \xi$ we have the following

$$\sum_{\nu=1}^{3}\{\eta_\nu(\phi_i\xi)\xi_\nu + 3g(\phi_\nu\phi_i\xi, \xi)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi\phi_i\xi\} - \alpha A\phi_i\xi - \phi_i\xi = 0. \tag{3.3}$$

Now we prove the following:

**Proposition 3.1.** *Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting structure Jacobi operator. If the $\mathfrak{D}^\perp$ component of the structure vector $\xi$ is invariant by the shape operator, then $\xi$ belongs to either the distribution $\mathfrak{D}$ or to the distribution $\mathfrak{D}^\perp$.*

*Proof.* First, if $\alpha = 0$, by the lemma due to Pérez and Suh [10], we have $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$.

Secondly, we assume that $\alpha \neq 0$. Now let us put $i = 1$ in (3.2). Then we have the following

$$\begin{aligned}
\text{Left} &= \eta(\phi_1 X)\xi + \sum_{\nu=1}^{3}\{\eta_\nu(\phi_1 X)\xi_\nu - \eta(\phi_1 X)\eta_\nu(\xi)\xi_\nu + 3g(\phi_\nu\phi_1 X,\xi)\phi_\nu\xi \\
&\quad + \eta_\nu(\xi)\phi_\nu\phi\phi_1 X\} - \eta(A\xi)A\phi_1 X + \alpha^2\eta(\phi_1 X)\xi \\
&= \eta(\phi_1 X)\xi + \eta_2(\phi_1 X)\xi_2 + \eta_3(\phi_1 X)\xi_3 - \eta(\phi_1 X)\eta_1(\xi)\xi_1 \\
&\quad + 3\{g(\phi_1^2 X,\xi)\phi_1\xi + g(\phi_2\phi_1 X,\xi)\phi_2\xi + g(\phi_3\phi_1 X,\xi)\phi_3\xi\} \\
&\quad + \eta_1(\xi)\phi_1\phi\phi_1 X - \eta(A\xi)A\phi_1 X + \alpha^2\eta(\phi_1 X)\xi \\
&= \eta(\phi_1 X)\xi - \eta_3(X)\xi_2 + \eta_2(X)\xi_3 - \eta(\phi_1 X)\eta_1(\xi)\xi_1 \\
&\quad + 3\{g(-X + \eta_1(X)\xi_1,\xi)\phi_1\xi + g(-\phi_3 X,\xi)\phi_2\xi + g(\phi_2 X,\xi)\phi_3\xi\} \\
&\quad + \eta_1(\xi)\{-\phi X + \eta_1(\phi X)\xi_1 + \eta_1(X)\phi_1\xi\} - \alpha A\phi_1 X + \alpha^2\eta(\phi_1 X)\xi,
\end{aligned}$$

$$\begin{aligned}
\text{Right} &= \eta(X)\phi_1\xi + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_1\xi_\nu - \eta(X)\eta_\nu(\xi)\phi_1\xi_\nu + 3g(\phi_\nu X,\xi)\phi_1\phi_\nu\xi \\
&\quad + \eta_\nu(\xi)\phi_1\phi_\nu\phi X\} - \alpha\phi_1 AX + \alpha^2\eta(X)\phi_1\xi \\
&= \eta(X)\phi_1\xi + \eta_2(X)\phi_1\xi_2 + \eta_3(X)\phi_1\xi_3 + 3\{g(\phi_1 X,\xi)\phi_1^2\xi + g(\phi_2 X,\xi)\phi_1\phi_2\xi \\
&\quad + g(\phi_3 X,\xi)\phi_1\phi_3\xi\} + \eta_1(\xi)\phi_1^2\phi X - \alpha\phi_1 AX + \alpha^2\eta(X)\phi_1\xi \\
&= \eta(X)\phi_1\xi + \eta_2(X)\xi_3 - \eta_3(X)\xi_2 \\
&\quad + 3\{g(\phi_1 X,\xi)\phi_1^2\xi + g(\phi_2 X,\xi)\phi_3\xi - g(\phi_3 X,\xi)\phi_2\xi\} \\
&\quad + \eta_1(\xi)\{-\phi X + \eta_1(\phi X)\xi_1\} - \alpha\phi_1 AX + \alpha^2\eta(X)\phi_1\xi.
\end{aligned}$$

Then we have

$$\begin{aligned}
&(4+\alpha^2)\eta_1(\phi X)\xi - (4+\alpha^2)\eta(X)\phi_1\xi + 4\eta_1(X)\eta(\xi_1)\phi_1\xi \\
&\qquad - 4\eta_1(\phi X)\eta(\xi_1)\xi_1 - \alpha(A\phi_1 - \phi_1 A)X = 0.
\end{aligned}$$

Let us put $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit $X_0$ in $\mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$. By putting $X = \xi_1$ in the above equation we have

$$\phi_1 A\xi_1 = \alpha\eta(\xi_1)\phi_1\xi.$$

And by applying $\phi_1$ to the above equation, we know the following

$$A\xi_1 = \alpha\eta(X_0)\eta(\xi_1)X_0 + \eta_1(A\xi_1)\xi_1. \tag{3.4}$$

Then the assumption of our proposition gives $\eta(X_0)\eta(\xi_1) = 0$, from which we have our assertion. □

On the other hand, in this section we give another proposition with the assumption related to Proposition 3.1. We take an inner product (3.4) with $\xi$. Then it follows that

$$\alpha\eta_1(\xi) = \alpha\eta^2(X_0)\eta(\xi_1) + \eta_1(A\xi_1)\eta(\xi_1). \tag{3.5}$$

When $\eta(\xi_1) = 0$, then $\xi$ belongs to the distribution $\mathfrak{D}$. If $\eta(\xi_1) \neq 0$, then (3.5) gives $\eta_1(A\xi_1) = \alpha\{1 - \eta^2(X_0)\} = \alpha\eta^2(\xi_1)$.

Therefore,

$$\begin{aligned} A\xi_1 &= \alpha\eta(X_0)\eta(\xi_1)X_0 + \alpha\eta^2(\xi_1)\xi_1 \\ &= \alpha\eta(\xi_1)\{\eta(X_0)X_0 + \eta(\xi_1)\xi_1\} \\ &= \alpha\eta(\xi_1)\xi. \end{aligned} \tag{3.6}$$

From these formulas we have the following:

**Lemma 3.2.** *Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting structure Jacobi operator. Then $q_2(\xi_1) = 0$ and $q_3(\xi_1) = 0$.*

*Proof.* Form (3.4) and (3.6) we have $g(\phi A\xi_1, \xi_3) = 0$, because we have put

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1.$$

On the other hand, we know

$$\begin{aligned} g(\phi A\xi_1, \xi_3) &= g(\nabla_{\xi_1}\xi, \xi_3) = -g(\xi, \nabla_{\xi_1}\xi_3) \\ &= -g(\xi, q_2(\xi_1)\xi_1 - q_1(\xi_1)\xi_2 + \phi_3 A\xi_1) \\ &= -q_2(\xi_1)\eta(\xi_1) - g(\xi, \phi_3 A\xi_1). \end{aligned}$$

Therefore, it follows that

$$2g(\phi A\xi_1, \xi_3) = -q_2(\xi_1)\eta(\xi_1) = 0.$$

If $\eta(\xi_1) \neq 0$, then $q_2(\xi_1) = 0$. Similarly, we get also $q_3(\xi_1) = 0$. □

By Lemmas 3.1 and 3.2 we assert the following:

**Lemma 3.3.** *Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If the structure Jacobi operator $R_\xi$ commutes with the 3-structure tensors $\phi_i$, $i = 1, 2, 3$. Then the $\mathfrak{D}^\perp$ component of the structure vector $\xi$ is invariant by the shape operator if and only if the $\mathfrak{D}^\perp$ component of the structure vector $\xi$ has a geodesic Reeb flow.*

*Proof.* By Lemma 3.2, we have the following

$$\begin{aligned} \nabla_{\xi_1}\xi_1 &= q_3(\xi_1)\xi_2 - q_2(\xi_1)\xi_3 + \phi_1 A\xi_1 \\ &= \phi_1 A\xi_1. \end{aligned} \tag{3.7}$$

If we assume that the $\mathfrak{D}^\perp$-component of the structure vector $\xi$ has a geodesic Reeb flow, then $\phi_1 A\xi_1 = 0$. From this, applying $\phi_1$, we know that the $\mathfrak{D}^\perp$ component of the structure vector $\xi$ is principal.

Conversely, by the assumption and the above expression $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$, the structure vector $\xi_1$ is invariant by the shape operator $A$, that is, $A\xi_1 = g(A\xi_1, \xi_1)\xi_1$. From this, together with (3.7), we have

$$\nabla_{\xi_1}\xi_1 = \phi_1 A\xi_1 = g(A\xi_1, \xi_1)\phi_1\xi_1 = 0\,.$$

So, we have our assertion. □

Now we assert the following:

**Proposition 3.4.** *Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting structure Jacobi operator. If the $\mathfrak{D}^\perp$ component of the structure vector $\xi$ has a geodesic Reeb flow, then the Reeb vector $\xi$ belongs to either the distribution $\mathfrak{D}$ or to the distribution $\mathfrak{D}^\perp$.*

## 4. Commuting structure Jacobi operator for $\xi \in \mathfrak{D}^\perp$

In this section we consider the case that $\xi \in \mathfrak{D}^\perp$. Then from (3.2) we assert the following:

**Lemma 4.1.** *Let $M$ be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting structure Jacobi operator. If $\xi \in \mathfrak{D}^\perp$, then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

*Proof.* Since we have assumed $\xi \in \mathfrak{D}^\perp$, we may put $\xi = \xi_1$. Then from (3.2) we know that

$$\phi_1\phi\phi_i X - \alpha A\phi_i X = \phi_i\phi_1\phi X - \alpha\phi_i AX, \tag{4.1}$$

where we used $\eta(\phi_i X) = 0$ and $\eta_\nu(\phi_i X) = 0$ for any $X \in \mathfrak{D}$ and any $i = 1, 2, 3$. From this, taking an inner product with $\xi$, we have

$$\alpha g(AX, \phi_i\xi) = 0 \tag{4.2}$$

for any $X \in \mathfrak{D}$.

**CASE I.** $\alpha = 0$.

Then for any $X \in \mathfrak{D}$ and $i = 1, 2, 3$, by (4.1) we have the following

$$\phi_1\phi\phi_i X = \phi_i\phi_1\phi X. \tag{4.3}$$

From this, putting $i = 2$, we have

$$\phi_1\phi\phi_2 X = \phi_2\phi_1\phi X$$

for any $X \in \mathfrak{D}$. Then from (2.1), (2.6) it follows that we have

$$\phi_3\phi X = 0.$$

By applying $\phi_3$ to the above equation we have

$$\phi X = 0.$$

Then from this, by applying $\phi$, we have

$$X = 0,$$

which gives a contradiction. Similarly, by putting $i = 3$ in (4.3) we can make a contradiction.

**CASE II.** $\alpha \neq 0$
From (4.2) we have for $i = 1, 2, 3$

$$g(AX, \phi_i \xi) = 0.$$

That is, $g(AX, \xi_2) = 0$ and $g(AX, \xi_3) = 0$.
On the other hand, $g(AX, \xi_1) = g(AX, \xi) = \alpha g(X, \xi) = 0$ for any $X \in \mathfrak{D}$.
Therefore, $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. □

Now it remains to check whether such kind of hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfy commuting structure Jacobi operator or not? Then we recall a proposition given by Berndt and Suh [3] as follows:

**Proposition A.** *Let $M$ be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and $\xi$ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then $M$ has three (if $r = \pi/2\sqrt{8}$) of four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r)\ ,\ \beta = \sqrt{2}\cot(\sqrt{2}r)\ ,\ \gamma = -\sqrt{2}\tan(\sqrt{2}r)\ ,\ \mu = 0$$

*with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are*

$$m(\alpha) = 1\ ,\ m(\beta) = 2\ ,\ m(\gamma) = 2m - 2 = m(\mu),$$

*and the corresponding eigenspaces are*

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1, \\ T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3, \\ T_\gamma &= \{X | X \perp \mathbb{H}\xi, JX = J_1 X\}, \\ T_\mu &= \{X | X \perp \mathbb{H}\xi,\ JX = -J_1 X\}, \end{aligned}$$

*where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$ respectively denotes real, complex and quaternionic span of the structure vector $\xi$ and $\mathbb{C}^\perp\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.*

By putting $i = 2$ and $i = 3$ in (3.3) we have

$$\alpha A\xi_3 + 2\xi_3 = 0,\ \alpha A\xi_2 + 2\xi_2 = 0.$$

That is,

$$(\alpha\beta + 2)\xi_3 = 0,\ (\alpha\beta + 2)\xi_2 = 0.$$

Then

$$\alpha\beta + 2 = 0. \tag{4.4}$$

On the other hand, from Lemma 14 given in Berndt and Suh [3] we have

$$\beta^2 - \alpha\beta - 2 = 0. \tag{4.5}$$

From (4.4) and (4.5) we have

$$\beta^2 = \alpha\beta + 2 = 0.$$

But the principal curvature $\beta = \sqrt{2}\cot(\sqrt{2}r)$ given in Proposition A is never vanishing for any $r \in (0, \pi/\sqrt{8})$. So this makes a contradiction.

## 5. Commuting structure Jacobi operator for $\xi \in \mathfrak{D}$

In this section, we also consider the commuting structure Jacobi operator. That is, the structure Jacobi operator $R_\xi$ satisfies $R_\xi \circ \phi_i = \phi_i \circ R_\xi$ for $i = 1, 2, 3$.

Now we consider for the case that $\xi \in \mathfrak{D}$. Then using $\xi \in \mathfrak{D}$ in (3.2) we have the following:

**Lemma 5.1.** *Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting structure Jacobi operator. If $\xi \in \mathfrak{D}$, then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

*Proof.* Now let us put $X \in \mathfrak{D}$ in (3.2) and use $\xi \in \mathfrak{D}$, we have

$$\begin{aligned} &\eta(\phi_i X)\xi - \alpha A\phi_i X + \alpha^2 \eta(\phi_i X)\xi + 3\sum_{\nu=1}^{3} g(\phi_\nu \phi_i X, \xi)\phi_\nu \xi \\ &= \eta(X)\phi_i \xi - \alpha \phi_i A X + \alpha^2 \eta(X)\phi_i \xi + 3\sum_{\nu=1}^{3} g(\phi_\nu X, \xi)\phi_i \phi_\nu \xi \end{aligned} \tag{5.1}$$

for any $X \in \mathfrak{D}$.

**CASE I.** $\alpha = 0$

(5.1) implies the following for any $X \in \mathfrak{D}$

$$\eta(\phi_i X)\xi + 3\sum_{\nu=1}^{3} g(\phi_\nu \phi_i X, \xi)\phi_\nu \xi = \eta(X)\phi_i \xi + 3\sum_{\nu=1}^{3} g(\phi_\nu X, \xi)\phi_i \phi_\nu \xi.$$

Now by putting $X = \xi$ into the above equation we have

$$3\sum_{\nu=1}^{3} g(\phi_\nu \phi_i \xi, \xi)\phi_\nu \xi = \phi_i \xi.$$

Then $\phi_i \xi = 0$. So this makes a contradiction.

**CASE II.** $\alpha \neq 0$

Let us denote a distribution $\mathfrak{D}$ in $T_x M$, $x \in M$, in such a way the $\mathfrak{D} = [\xi] \oplus [\ \phi_1\xi,\ \phi_2\xi,\ \phi_3\xi\ ] \oplus \mathfrak{D}_0$, where the subdistribution $\mathfrak{D}_0$ is defined by

$$\mathfrak{D}_0 = \{X \in \mathfrak{D} \mid X \perp \xi,\ \phi_1\xi,\ \phi_2\xi, \phi_3\xi\}.$$

Subcase II-1: $g(A\xi, \xi_\nu) = 0$, $\nu = 1, 2, 3$.

If fact, we know that

$$g(A\xi, \xi_\nu) = \alpha g(\xi, \xi_\nu) = 0, \quad \nu = 1, 2, 3.$$

Subcase II-2: $g(A\phi_i\xi, \xi_\nu) = 0$, $i$, $\nu = 1, 2, 3$.

In fact, by putting $X = \xi$ in (5.1), we have

$$\alpha A\phi_i\xi + 4\phi_i\xi = 0.$$

From this, taking an inner product with $\xi_\nu$, we have

$$0 = \alpha g(A\phi_i\xi, \xi_\nu).$$

Since $\alpha$ is non-vanishing, $g(A\phi_i\xi, \xi_\nu) = 0$.

Subcase II-3: $g(AX, \xi_i) = 0$ for any $X \in \mathfrak{D}_0$ and $i = 1, 2, 3$.

In fact, by substituting any $X \in \mathfrak{D}_0$ in (5.1), we have

$$\alpha(A\phi_i X - \phi_i AX) = 0,$$

where we have used $\phi_i X \in \mathfrak{D}_0$ .

Since $\alpha$ is non-vanishing, we have for any $X \in \mathfrak{D}_0$

$$A\phi_i X - \phi_i AX = 0. \tag{5.2}$$

Then by replacing $X$ with $\phi_i X$, we have

$$\begin{aligned} 0 &= A\phi_i^2 X - \phi_i A\phi_i X \\ &= A(-X + \eta_i(X)\xi_i) - \phi_i^2 AX \\ &= -AX - (-AX + \eta_i(AX)\xi_i) \\ &= -\eta_i(AX)\xi_i. \end{aligned}$$

Therefore, $g(AX, \xi_i) = 0$ for any $X \in \mathfrak{D}_0$ and any $i = 1, 2, 3$. Summing up all cases mentioned above, we have our assertion. □

Now by virtue of Lemma 5.1 we are able to use Theorem A [3] in the introduction. In order to introduce the geometrical structure of such a tube we recall the following:

**Proposition B.** *Let $M$ be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and $\xi$ is tangent to $\mathfrak{D}$. Then the quaternionic dimension $m$ of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and $M$ has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r)\ ,\ \beta = 2\cot(2r)\ ,\ \gamma = 0\ ,\ \lambda = \cot(r)\ ,\ \mu = -\tan(r)$$

*with some $r \in (0, \pi/4)$. The corresponding multiplicities are*

$$m(\alpha) = 1\ ,\ m(\beta) = 3 = m(\gamma)\ ,\ m(\lambda) = 4n - 4 = m(\mu)$$

*and the corresponding eigenspaces are*

$$T_\alpha = \mathbb{R}\xi\ ,\ T_\beta = \mathfrak{J}J\xi\ ,\ T_\gamma = \mathfrak{J}\xi\ ,\ T_\lambda\ ,\ T_\mu,$$

*where*

$$T_\lambda \oplus T_\mu = (\mathbb{HC}\xi)^\perp \ , \ \mathfrak{J}T_\lambda = T_\lambda \ , \ \mathfrak{J}T_\mu = T_\mu \ , \ JT_\lambda = T_\mu.$$

In such a Proposition B, by using $\xi \in \mathfrak{D}$ and $A\phi_i\xi = \lambda\phi_i\xi = 0$ in (3.3), we have the following

$$\begin{aligned} 0 &= -3\sum_{\nu=1}^{3} g(\phi_i\xi, \phi_\nu\xi)\phi_\nu\xi - \alpha A\phi_i\xi - \phi_i\xi \\ &= -3\phi_i\xi - \phi_i\xi \\ &= -4\phi_i\xi. \end{aligned}$$

That is, $\phi_i\xi = 0$. So this makes a contradiction.

Summing up the results given in sections 4 and 5, together with Propositions 3.1 and 3.4, we complete the proof of our Theorem and Corollary in the introduction.

*Remark* 5.1. It is known that a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ in Theorem A has commuting shape operator on the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp$, that is, $A\circ\phi = \phi\circ A$. So naturally it is Hopf. But in section 4 we have asserted that such a hypersurface does not satisfy $R_\xi\circ\phi_i = \phi_i\circ R_\xi$ for $i = 1, 2, 3$.

*Remark* 5.2. A tube over a totally real totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$, does not have commuting shape operator on the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp$. In section 5 we have proved that such a hypersurface is Hopf, but can not satisfy $R_\xi\circ\phi_i = \phi_i\circ R_\xi$ for $i = 1, 2, 3$.

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