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UNIVALENCE PROPERTIES FOR A GENERAL INTEGRAL OPERATOR

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Abstract. We consider the univalence function classes $\mathcal{T}$, $\mathcal{T}_2$, $\mathcal{T}_{2,\mu}$, and $\mathcal{S}(p)$. For these classes we shall study some univalence properties for a general integral operator. Furthermore we shall extend some known univalence criteria, i.e., Becker-type criteria.

1. Introduction

Let $\mathcal{U} = \{z \in \mathbb{C}, |z| < 1\}$ be the unit disk and $\mathcal{A}$ denotes the class of the functions f of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \ z \in \mathcal{U},$$

which are analytic in the open disk, $\mathcal{U}$ and satisfy the condition $f(0) = f'(0) - 1 = 0$. Consider $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent functions in } \mathcal{U}\}$.

Let $\mathcal{A}_2$ be the subclass of $\mathcal{A}$ consisting of functions is of the form

$$f(z) = z + \sum_{k=3}^{\infty} a_k z^k. \tag{1.1}$$

Let $\mathcal{T}$ be the univalent subclass of $\mathcal{A}$ which satisfies

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1 \quad (z \in \mathcal{U}). \tag{1.2}$$

Let $\mathcal{T}_2$ be the subclass of $\mathcal{T}$ for which $f''(0) = 0$. Let $\mathcal{T}_{2,\mu}$ be the subclass of $\mathcal{T}_2$ consisting of functions is of the form (1.1) which satisfy

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| \le \mu \quad (z \in \mathcal{U}) \tag{1.3}$$

for some μ $(0 < \mu \le 1)$, and let us denote $\mathcal{T}_{2,1} \equiv \mathcal{T}_2$. Furthermore, for some real p with $0 < p \le 2$ we define a subclass $\mathcal{S}(p)$ of $\mathcal{A}$ consisting of all function

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$f(z)$ which satisfy

$$\left|\left(\frac{z}{f(z)}\right)''\right| \le p \quad (z \in \mathcal{U}).$$

In [9], Singh has shown that if $f(z) \in \mathcal{S}(p)$, then $f(z)$ satisfies

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| \le p|z|^2, \quad (z \in \mathcal{U}). \tag{1.4}$$

L. V. Alfors in [1] and J. Becker in [2] has obtained the next univalence criterion:

Theorem 1.1. *Let c be a complex number, $|c| \le 1$, $c \ne -1$. If $f(z) = z + a_2 z^2 + \cdots$ is a regular function in $\mathcal{U}$ and*

$$\left|c\,|z|^2 + \left(1 - |z|^2\right)\frac{zf''(z)}{f'(z)}\right| \le 1$$

for all $z \in \mathcal{U}$, then the function f is regular and univalent in $\mathcal{U}$.

In the paper [7], Pescar need the following theorem:

Theorem 1.2 ([7]). *Let β be a complex number, $\mathbf{Re}\,\beta > 0$, and c a complex number, $|c| \le 1, c \ne -1$ and $h(z) = z + a_2 z^2 + \cdots$, a regular function in $\mathcal{U}$. If*

$$\left|c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right| \le 1$$

for all the $z \in \mathcal{U}$, then the function

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} h'(t)\,dt\right]^{\frac{1}{\beta}} = z + \cdots$$

is regular and univalent in $\mathcal{U}$.

The General Schwarz Lemma. *Let the function $f(z)$ be regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C}; |z| < R\}$, with $|f(z)| < M$ for fixed M. If $f(z)$ has one zero with multiply $\ge m$ for $z = 0$, then*

$$|f(z)| \le \frac{M}{R^m}|z|^m, \ \ z \in \mathcal{U}_R. \tag{1.5}$$

The equality (in the inequality (1.5) for $z \ne 0$) can hold only if $f(z) = e^{i\theta}\frac{M}{R^m}z^m$, where θ is constant.

In the paper [8], Seenivasagan and Breaz consider for $f_i \in \mathcal{A}_2$ $(i = 1, 2, \ldots, n)$ and $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta \in \mathbb{C}$, the integral operator

$$F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1}\prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}} dt\right\}^{\frac{1}{\beta}}. \tag{1.6}$$

When $\alpha_i = \alpha$ for all $i = 1, 2, \ldots, n$, $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ becomes the integral operator $F_{\alpha,\beta}$ considered in [3].

2. Main results

Theorem 2.1. *Let $M_i \geq 1$ for all $i \in \{1,\dots,n\}$, c be a complex number and the functions $f_i \in \mathcal{S}(p_i)$ for $i \in \{1,\dots,n\}$ satisfying the condition* (1.4). *Consider α_i, β be a complex number with the property $\mathbf{Re}\,\beta \geq \sum_{i=1}^n \frac{(1+p_i)M_i+1}{|\alpha_i|}$. If*

$$|c| \leq 1 - \frac{1}{\mathbf{Re}\,\beta}\sum_{i=1}^n \frac{(1+p_i)M_i+1}{|\alpha_i|} \tag{2.1}$$

and

$$|f_i(z)| \leq M_i$$

for all $z \in \mathcal{U}$ and $i \in \{1,\dots,n\}$, then the function $F_{\alpha_1,\alpha_2,\dots,\alpha_n,\beta}$ defined in (1.6) *is univalent.*

Proof. Define a function

$$h(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}} dt,$$

then we have $h(0) = h'(0) - 1 = 0$. Also a simple computation yields

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^n \frac{1}{\alpha_i}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right). \tag{2.2}$$

From equality (2.2), we have

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^n \frac{1}{|\alpha_i|}\left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1\right) = \sum_{i=1}^n \frac{1}{|\alpha_i|}\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)}\right|\left|\frac{f_i(z)}{z}\right| + 1\right). \tag{2.3}$$

From the hypothesis, we have $|f_i(z)| \leq M_i, z \in \mathcal{U}$ and $i \in \{1,\dots,n\}$, then by General Schwarz Lemma, we obtain that

$$|f_i(z)| \leq M_i|z|$$

for all $z \in \mathcal{U}$ and $i \in \{1,\dots,n\}$.

We apply this result in inequality (2.3), we obtain

$$\begin{aligned}
\left|\frac{zh''(z)}{h'(z)}\right| &\leq \sum_{i=1}^n \frac{1}{|\alpha_i|}\left(\left|\frac{z^2f_i'(z)}{(f_i(z))^2}\right| M_i + 1\right) \\
&\leq \sum_{i=1}^n \frac{1}{|\alpha_i|}\left(\left|\frac{z^2f_i'(z)}{(f_i(z))^2} - 1\right| M_i + M_i + 1\right) \\
&= \sum_{i=1}^n \frac{1}{|\alpha_i|}\left(p_iM_i\,|z|^2 + M_i + 1\right) < \sum_{i=1}^n \frac{(1+p_i)M_i+1}{|\alpha_i|}.
\end{aligned}$$

Next, we evaluate the expression:

$$\begin{aligned}&\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right|\\ \leq\ & |c|+\frac{1}{|\beta|}\left|\frac{zh''(z)}{h'(z)}\right|\leq|c|+\frac{1}{|\beta|}\sum_{i=1}^{n}\frac{(1+p_i)M_i+1}{|\alpha_i|}\\ <\ & |c|+\frac{1}{\mathbf{Re}\,\beta}\sum_{i=1}^{n}\frac{(1+p_i)M_i+1}{|\alpha_i|}.\end{aligned}$$

So, from (2.1) we have:

$$\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right|\leq 1.$$

Applying Theorem 1.2, we obtain that $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ is univalent. □

Corollary 2.2. *Let $M\geq 1$, c be a complex number and the functions $f_i\in\mathcal{S}(p)$ for $i\in\{1,\ldots,n\}$ satisfying the condition* (1.4). *Consider α_i,β be a complex numbers with the property $\mathbf{Re}\,\beta\geq\sum_{i=1}^{n}\frac{(1+p)M+1}{|\alpha_i|}$. If*

$$|c|\leq 1-\frac{1}{\mathbf{Re}\,\beta}\sum_{i=1}^{n}\frac{(1+p)M+1}{|\alpha_i|}$$

and

$$|f_i(z)|\leq M$$

for all $z\in\mathcal{U}$, then the function $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ defined in (1.6) *is univalent.*

Proof. We consider in Theorem 2.1 $M_1=M_2=\cdots=M_n=M$. □

Corollary 2.3. *Let $M_i\geq 1$ for $i\in\{1,\ldots,n\}$, c be a complex number and the functions $f_i\in\mathcal{S}(p_i)$, for $i\in\{1,\ldots,n\}$ satisfying the condition* (1.4). *Consider α,β be a complex numbers, $\mathbf{Re}\,\beta\geq\sum_{i=1}^{n}\frac{((1+p_i)M_i+1)}{|\alpha|}$. If*

$$|c|\leq 1-\frac{1}{\mathbf{Re}\,\beta}\sum_{i=1}^{n}\frac{((p_i+1)M_i+1)}{|\alpha|}$$

and

$$|f_i(z)|\leq M_i$$

for all $z\in\mathcal{U}$ and $i\in\{1,\ldots,n\}$, then the function

$$F_{\alpha,\beta}(z)=\left\{\beta\int_0^z t^{\beta-1}\prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha}}dt\right\}^{\frac{1}{\beta}}$$

is univalent.

Proof. In Theorem 2.1 we consider $\alpha_1=\alpha_2=\cdots=\alpha_n=\alpha$. □

Corollary 2.4. *Let $M \geq 1$, c be a complex number and the function $f \in \mathcal{S}(p)$, satisfy the condition* (1.4)*. Consider α, β be a complex numbers with the property $\mathbf{Re}\,\beta \geq \frac{(1+p)M+1}{|\alpha|}$. If*

$$|c| \leq 1 - \frac{(1+p)M+1}{\beta\,|\alpha|}$$

and

$$|f(z)| \leq M$$

for all $z \in \mathcal{U}$, then the function

$$G_{\alpha,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\frac{1}{\alpha}} dt\right\}^{\frac{1}{\beta}}$$

is univalent.

Proof. In Theorem 2.1 we consider $n = 1$. □

Theorem 2.5. *Let $M_i \geq 1$ for all $i \in \{1, \ldots, n\}$, c a complex number and the functions $f_i \in \mathcal{T}_{2,\mu_i}$ for $i \in \{1, \ldots, n\}$ satisfy the condition* (1.3)*. We consider α_i, β be a complex numbers with the property $\mathbf{Re}\,\beta \geq \sum_{i=1}^{n} \frac{(1+\mu_i)M_i+1}{|\alpha_i|}$. If*

$$|c| \leq 1 - \frac{1}{\mathbf{Re}\,\beta}\sum_{i=1}^{n}\frac{(1+\mu_i)M_i+1}{|\alpha_i|} \tag{2.4}$$

and

$$|f_i(z)| \leq M_i$$

for all $z \in \mathcal{U}$ and $i \in \{1, \ldots, n\}$, then the function $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ defined in (1.6) *is univalent.*

Proof. Define a function

$$h(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}} dt,$$

then we have $h(0) = h'(0) - 1 = 0$. After the same steps with the Theorem 2.1 we have:

$$\begin{aligned}
\left|\frac{zh''(z)}{h'(z)}\right| &\leq \sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2}\right| M_i + 1\right) \\
&\leq \sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2} - 1\right| M_i + M_i + 1\right) \\
&= \sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left(\mu_i M_i + M_i + 1\right) < \sum_{i=1}^{n}\frac{(1+\mu_i)M_i+1}{|\alpha_i|}.
\end{aligned}$$

We evaluate the next expression:

$$\begin{aligned}
&\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right|\\
\leq\ & |c|+\frac{1}{|\beta|}\left|\frac{zh''(z)}{h'(z)}\right|\leq|c|+\frac{1}{|\beta|}\sum_{i=1}^{n}\frac{(1+\mu_i)M_i+1}{|\alpha_i|}\\
<\ & |c|+\frac{1}{\mathbf{Re}\,\beta}\sum_{i=1}^{n}\frac{(1+\mu_i)M_i+1}{|\alpha_i|}.
\end{aligned}$$

So, from (2.4) we have:

$$\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right|\leq 1.$$

Applying Theorem 1.2, we obtain that $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ is univalent. □

Corollary 2.6. *Let $M\geq 1$, c be a complex number and the functions $f_i\in\mathcal{T}_{2,\mu_i}$ for $i\in\{1,\ldots,n\}$ satisfy the condition* (1.3). *We consider α_i,β be a complex numbers with the property $\mathbf{Re}\,\beta\geq\sum_{i=1}^{n}\frac{(1+\mu_i)M+1}{|\alpha_i|}$. If*

$$|c|\leq 1-\frac{1}{\mathbf{Re}\,\beta}\sum_{i=1}^{n}\frac{(1+\mu_i)M+1}{|\alpha_i|}$$

and

$$|f_i(z)|\leq M$$

for all $z\in\mathcal{U}$ and $i\in\{1,\ldots,n\}$, then the function $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ defined in (1.6) *is univalent.*

Proof. We consider in Theorem 2.5 $M_1=M_2=\cdots=M_n=M$. □

Corollary 2.7. *Let $M_i\geq 1$ for $i\in\{1,\ldots,n\}$, c be a complex number and the functions $f_i\in\mathcal{T}_{2,\mu_i}$ for $i\in\{1,\ldots,n\}$ satisfy the condition* (1.3). *We consider α,β be a complex numbers with the property $\mathbf{Re}\,\beta\geq\sum_{i=1}^{n}\frac{((1+\mu_i)M_i+1)}{|\alpha|}$. If*

$$|c|\leq 1-\frac{1}{\mathbf{Re}\,\beta}\sum_{i=1}^{n}\frac{((1+\mu_i)M_i+1)}{|\alpha|}$$

and

$$|f_i(z)|\leq M_i$$

for all $z\in\mathcal{U}$ and $i\in\{1,\ldots,n\}$, then the function

$$F_{\alpha,\beta}(z)=\left\{\beta\int_0^z t^{\beta-1}\prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha}}dt\right\}^{\frac{1}{\beta}}$$

is univalent.

Proof. In Theorem 2.5 we consider $\alpha_1=\alpha_2=\cdots=\alpha_n=\alpha$. □

Corollary 2.8. *Let $M \geq 1$, c be a complex number and the function $f \in \mathcal{T}_{2,\mu}$ satisfy the condition* (1.3). *We consider α, β be a complex numbers with the property $\mathbf{Re}\,\beta \geq \frac{(1+\mu)M+1}{|\alpha|}$. If*

$$|c| \leq 1 - \frac{1}{\mathbf{Re}\,\beta|}\frac{(1+\mu)M+1}{|\alpha|}$$

and

$$|f(z)| \leq M$$

for all $z \in \mathcal{U}$, then the function

$$G_{\alpha,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\frac{1}{\alpha}} dt\right\}^{\frac{1}{\beta}}$$

is univalent.

Proof. In Theorem 2.5 we consider $n = 1$. □

Corollary 2.9. *Let $M_i \geq 1$ for all $i \in \{1, \ldots, n\}$, c a complex number and the functions $f_i \in \mathcal{T}$ for $i \in \{1, \ldots, n\}$ satisfy the condition* (1.2). *We consider α_i, β be a complex numbers with the property $\mathbf{Re}\,\beta \geq \sum_{i=1}^{n} \frac{2M_i+1}{|\alpha_i|}$. If*

$$|c| \leq 1 - \frac{1}{\mathbf{Re}\,\beta}\sum_{i=1}^{n}\frac{2M_i+1}{|\alpha_i|}$$

and

$$|f_i(z)| \leq M_i$$

for all $z \in \mathcal{U}$ and $i \in \{1, \ldots, n\}$, then the function $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ defined in (1.6) *is univalent.*

Proof. After the same steps with the Theorem 2.5 we obtain the conclusion of this corollary. □

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