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JORDAN θ-DERIVATIONS ON LIE TRIPLE SYSTEMS

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Abstract. In this paper we prove that every Jordan θ-derivation on a Lie triple system is a θ-derivation. Specially, we conclude that every Jordan derivation on a Lie triple system is a derivation.

1. Introduction

The concept of Lie triple system was first introduced by N. Jacobson [2, 3] (see also [4]). We recall that a Lie triple system is a vector space $\mathcal{J}$ over a field $\mathbb{K}$ with a trilinear mapping $\mathcal{J} \times \mathcal{J} \times \mathcal{J} \ni (x,y,z) \mapsto [x,y,z] \in \mathcal{J}$ satisfying the following axioms

(i) $[x,y,z] = -[y,x,z]$,
(ii) $[x,y,z] + [y,z,x] + [z,x,y] = 0$,
(iii) $[u,v,[x,y,z]] = [[u,v,x],y,z] + [x,[u,v,y],z] + [x,y,[u,v,z]]$

for all $u,v,x,y,z \in \mathcal{J}$. It follows from (i) that $[x,x,y] = 0$ for all $x,y \in \mathcal{J}$.

It is clear that every Lie algebra with product $[.,.]$ is a Lie triple system with respect to $[x,y,z] := [[x,y],z]$. Conversely, any Lie triple system $\mathcal{J}$ can be considered as a subspace of a Lie algebra (Bertram [1], Jacobson [3]).

Throughout this paper, let $\mathbb{C}$ be the complex filed and $\mathcal{J}$ be a Lie triple system over $\mathbb{C}$.

Definition 1.1. Let $\theta : \mathcal{J} \to \mathcal{J}$ be a $\mathbb{C}$-linear mapping. A $\mathbb{C}$-linear mapping $D : \mathcal{J} \to \mathcal{J}$ is called a θ-derivation on $\mathcal{J}$ if

$$D([x,y,z]) = [D(x),\theta(y),\theta(z)] + [\theta(x),D(y),\theta(z)] + [\theta(x),\theta(y),D(z)]$$

for all $x,y,z \in \mathcal{J}$. If $\theta = I_{\mathcal{J}}$, a θ-derivation is called a derivation.

Let $u,v \in \mathcal{J}$ and $D_{u,v} : \mathcal{J} \to \mathcal{J}$ be a mapping defined by

$$D_{u,v}(x) := [u,v,x]$$

for all $x \in \mathcal{J}$. It is clear that $D_{u,v}$ is $\mathbb{C}$-linear and we get from (iii) that the mapping $D_{u,v}$ is a derivation on $\mathcal{J}$.

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Definition 1.2. Let $\theta : \mathcal{J} \to \mathcal{J}$ be a $\mathbb{C}$-linear mapping. A $\mathbb{C}$-linear mapping $D : \mathcal{J} \to \mathcal{J}$ is called a Jordan θ-derivation on $\mathcal{J}$ if

$$D([x, y, x]) = [D(x), \theta(y), \theta(x)] + [\theta(x), D(y), \theta(x)] + [\theta(x), \theta(y), D(x)]$$

for all $x, y \in \mathcal{J}$. If $\theta = I_{\mathcal{J}}$, a Jordan θ-derivation is called a Jordan derivation.

In [5], M. Sal Moslehian and Th. M. Rassias have studied the stability of derivations in normed Lie triple systems associated with a Cauchy–Jensen additive mapping.

2. Main results

It is clear that every θ-derivation on a Lie triple system $\mathcal{J}$ is a Jordan θ-derivation. In this section we prove that every Jordan θ-derivation on a Lie triple system $\mathcal{J}$ is a θ-derivation. So we conclude that every Jordan derivation on $\mathcal{J}$ is a derivation.

Throughout this section $D, \theta : \mathcal{J} \to \mathcal{J}$ are $\mathbb{C}$-linear mappings and $A_{D,\theta} : \mathcal{J} \times \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ is a mapping defined by

$$A_{D,\theta}(x, y, z) := [D(x), \theta(y), \theta(z)] + [\theta(x), D(y), \theta(z)] + [\theta(x), \theta(y), D(z)]$$

for all $x, y, z \in \mathcal{J}$. It is clear that the mapping $A_{D,\theta}$ is trilinear and $A_{D,\theta}(x, x, y) = 0$ for all $x, y \in \mathcal{J}$.

Theorem 2.1. *Let $D : \mathcal{J} \to \mathcal{J}$ be a Jordan θ-derivation. Then D is a θ-derivation.*

Proof. Since $D : \mathcal{J} \to \mathcal{J}$ is a Jordan θ-derivation, $A_{D,\theta}(x, y, x) = D([x, y, x])$ for all $x, y \in \mathcal{J}$. Therefore we have

$$\begin{aligned} D([x+z, y, x+z]) &= [D(x) + D(z), \theta(y), \theta(x) + \theta(z)] \\ &\quad + [\theta(x) + \theta(z), D(y), \theta(x) + \theta(z)] \\ &\quad + [\theta(x) + \theta(z), \theta(y), D(x) + D(z)] \\ &= D([x, y, x]) + D([z, y, z]) \\ &\quad + A_{D,\theta}(x, y, z) + A_{D,\theta}(z, y, x) \end{aligned} \tag{2.1}$$

for all $x, y, z \in \mathcal{J}$. On the other hand, we have

$$[x+z, y, x+z] = [x, y, x] + [z, y, z] + [x, y, z] + [z, y, x]$$

for all $x, y, z \in \mathcal{J}$. Therefore

$$\begin{aligned} D([x+z, y, x+z]) &= D([x, y, x]) + D([z, y, z]) \\ &\quad + D([x, y, z]) + D([z, y, x]) \end{aligned} \tag{2.2}$$

for all $x, y, z \in \mathcal{J}$. It follows from (2.1) and (2.2) that

$$D([x, y, z]) + D([z, y, x]) = A_{D,\theta}(x, y, z) + A_{D,\theta}(z, y, x) \tag{2.3}$$

for all $x, y, z \in \mathcal{J}$. Since $[z, y, x] = [x, y, z] - [x, z, y]$, we get that

$$D([x, y, z]) + D([z, y, x]) = 2D([x, y, z]) - D([x, z, y]) \tag{2.4}$$

for all $x, y, z \in \mathcal{J}$. Also

$$\begin{aligned}
&A_{D,\theta}(x,y,z) - A_{D,\theta}(x,z,y)\\
&= [D(x),\theta(y),\theta(z)] + [\theta(x),D(y),\theta(z)] + [\theta(x),\theta(y),D(z)]\\
&\quad - [D(x),\theta(z),\theta(y)] - [\theta(x),D(z),\theta(y)] - [\theta(x),\theta(z),D(y)]\\
&= ([D(x),\theta(y),\theta(z)] + [\theta(z),D(x),\theta(y)])\\
&\quad + ([\theta(x),D(y),\theta(z)] + [\theta(z),\theta(x),D(y)])\\
&\quad + ([\theta(x),\theta(y),D(z)] + [D(z),\theta(x),\theta(y)])\\
&= [\theta(z),\theta(y),D(x)] + [\theta(z),D(y),\theta(x)] + [D(z),\theta(y),\theta(x)]\\
&= A_{D,\theta}(z,y,x)
\end{aligned}$$

for all $x, y, z \in \mathcal{J}$. So

$$A_{D,\theta}(x,y,z) + A_{D,\theta}(z,y,x) = 2A_{D,\theta}(x,y,z) - A_{D,\theta}(x,z,y) \tag{2.5}$$

for all $x, y, z \in \mathcal{J}$. We get from (2.3), (2.4) and (2.5) that

$$2D([x,y,z]) - D([x,z,y]) = 2A_{D,\theta}(x,y,z) - A_{D,\theta}(x,z,y) \tag{2.6}$$

for all $x, y, z \in \mathcal{J}$. Letting $y = z$ in (2.6), we get $D([x,y,y]) = A_{D,\theta}(x,y,y)$ for all $x, y \in \mathcal{J}$. Since $D([x,y+z,y+z]) = A_{D,\theta}(x,y+z,y+z)$ and $[.,.,.], A_{D,\theta}$ are trilinear, we have

$$D([x,y,z]) + D([x,z,y]) = A_{D,\theta}(x,y,z) + A_{D,\theta}(x,z,y) \tag{2.7}$$

for all $x, y, z \in \mathcal{J}$. Adding (2.6) to (2.7), we infer that $D([x,y,z]) = A_{D,\theta}(x,y,z)$ for all $x, y, z \in \mathcal{J}$. So the proof is completed. □

Corollary 2.2. *Every Jordan derivation on a Lie triple system is a derivation.*

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