# THE GENERALIZED RIEMANN PROBLEM FOR FIRST ORDER QUASILINEAR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS I 

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#### Abstract

In this paper, we consider a generalized Riemann problem of the first order hyperbolic conservation laws. For the case that excludes the centered wave, we prove that the generalized Riemann problem admits a unique piecewise smooth solution $u=u(t, x)$, and this solution has a structure similar to the similarity solution $u=U\left(\frac{x}{t}\right)$ of the corresponding Riemann problem in the neighborhood of the origin provided that the coefficients of the system and the initial conditions are sufficiently smooth.


## 1. Introduction

Consider the first order quasilinear hyperbolic systems of conservation laws

$$
\begin{equation*}
\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0 \tag{1.1}
\end{equation*}
$$

where $u=\left(u_{1}, \ldots, u_{n}\right)^{T}$ is an unknown vector function of $(t, x), x \in \mathbb{R}, t>0$, and $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is a smooth function of $u$. Assume that the system (1.1) is strictly hyperbolic on the domain under consideration, i.e., $A(u)=\nabla_{u} f(u)$ has $n$ real distinct eigenvalues:

$$
\begin{equation*}
\lambda_{1}(u)<\lambda_{2}(u)<\cdots<\lambda_{n}(u) \tag{1.2}
\end{equation*}
$$

Let $l_{i}(u)=\left(l_{i 1}(u), \ldots, l_{i n}(u)\right)$ and $r_{i}(u)=\left(r_{i 1}(u), \ldots, r_{i n}(u)\right)^{T}$ be the left eigenvector and right eigenvector corresponding to the eigenvalue $\lambda_{i}(u), i=$ $1, \ldots, n$, respectively. Without loss of generality, we may assume that

$$
\begin{equation*}
l_{i}(u) \cdot r_{j}(u)=\delta_{i j}, \quad(i, j=1, \ldots, n) \tag{1.3}
\end{equation*}
$$

where $\delta_{i j}$ is the Kronecker's symbol. Obviously, $\lambda_{i}(u), l_{i}(u)$ and $r_{i}(u)(i=$ $1, \ldots, n)$ have the same regularity as $A(u)$.

[^0]We prescribe the following piecewise constant initial data:

$$
t=0: \quad u= \begin{cases}\widehat{u}_{l}, & x \leq 0  \tag{1.4}\\ \widehat{u}_{r}, & x \geq 0\end{cases}
$$

where $\widehat{u}_{l}$ and $\widehat{u}_{r}$ are constant vectors satisfying:

$$
\begin{equation*}
\widehat{u}_{l} \neq \widehat{u}_{r} . \tag{1.5}
\end{equation*}
$$

We first give the following hypothesis:
$\left(\mathbf{H}_{\mathbf{1}}\right)$ The Riemann problem (1.1), (1.4) admits a similarity solution $u=$ $U\left(\frac{x}{t}\right)$, which is composed of $n+1$ constant states $\widehat{u}_{0}=\widehat{u}_{l}, \widehat{u}_{1}, \ldots, \widehat{u}_{n-1}, \widehat{u}_{n}=$ $\widehat{u}_{r}$ and $n$ waves through the origin (containing shock wave, rarefaction wave or contact discontinuity), the states $\widehat{u}_{i-1}$ and $\widehat{u}_{i}$ are connected by the $i$-th wave $(i=1, \ldots, n)$.

For a general quasilinear hyperbolic systems of conservation laws, under the assumption that every eigenvalue $\lambda_{i}(u)$ is either genuinely nonlinear in the sense of P. D. Lax:

$$
\begin{equation*}
\nabla \lambda_{i}(u) \cdot r_{i}(u) \neq 0 \tag{1.6}
\end{equation*}
$$

or linearly degenerate in the sense of P. D. Lax:

$$
\begin{equation*}
\nabla \lambda_{i}(u) \cdot r_{i}(u) \equiv 0 \tag{1.7}
\end{equation*}
$$

P. D. Lax [8] proved that the Riemann problem (1.1), (1.4) admits a unique similarity solution $u=U\left(\frac{x}{t}\right)$ provided $\left|\widehat{u}_{r}-\widehat{u}_{l}\right|$ is sufficiently small, which is composed of $n$ small amplitude waves. In this paper, we only consider a similarity solution $u=U\left(\frac{x}{t}\right)$ given by $\left(\mathrm{H}_{1}\right)$, regardless of its uniqueness, also disregarding whether its $n$ waves having small amplitude or not.

In this paper, we consider the system (1.1) with the following discontinuous initial data:

$$
t=0: \quad u= \begin{cases}\widehat{u}_{l}(x), & x \leq 0  \tag{1.8}\\ \widehat{u}_{r}(x), & x \geq 0\end{cases}
$$

where $\widehat{u}_{l}(x)$ and $\widehat{u}_{r}(x)$ are given smooth vector functions defined on $x \leq 0$ and $x \geq 0$ satisfying

$$
\widehat{u}_{l}(0)=\widehat{u}_{l}, \quad \widehat{u}_{r}(0)=\widehat{u}_{r},
$$

respectively. Since the generalized Riemann problem (1.1), (1.8) may be regarded as a perturbation of the corresponding Riemann problem (1.1), (1.4), we naturally study the following local problem:

In which condition, the generalized Riemann problem (1.1), (1.8) admits a unique piecewise smooth solution $u=u(t, x)$ which possesses a similar structure in a neighborhood of the origin as the solution of the corresponding Riemann problem (1.1), (1.4). Namely, the solution still contains $n$ waves through the origin, for any $i(i=1, \ldots, n)$, the type of the $i$-th wave is same as the $i$-th wave of the similarity solution $u=U\left(\frac{x}{t}\right)$; the $i$-th wave coincides with the $i$-th
wave of $u=U\left(\frac{x}{t}\right)$ at the origin. Moreover, the $i$-th wave links two known states $\widehat{u}_{i-1}$ and $\widehat{u}_{i}$.

Tikhonov and Samarsky [20] discussed the problem in the case of a single equation $(n=1)$. The earliest studies for the case of systems were as follows: one-dimensional isentropic flow systems $(n=2)$ was discussed in [2], $\mathrm{Gu}, \mathrm{Li}$ and How $[3,4,5,6]$ discussed the general reducible systems $(n=2)$. Furthermore, in $[1,10,11]$ one-dimensional gas dynamics systems $(n=3)$ was studied. All the above articles were devoted to investigation of arbitrary discontinuity $\left|\widehat{u}_{r}-\widehat{u}_{l}\right|$ of the initial data. For the general first order quasilinear hyperbolic systems of conservation laws, Li and $\mathrm{Yu}[12,13,14,15,16,17]$ have shown that the problem admits a unique local solution when $\left|\widehat{u}_{r}-\widehat{u}_{l}\right|$ is sufficiently small for the corresponding similarity solution $u=U\left(\frac{x}{t}\right)$ with small amplitude, provided that all the eigenvalues are genuinely nonlinear or linearly degenerate in the sense of P. D. Lax. Li [9] thought the result was still valid for the case where the discontinuity $\left|\widehat{u}_{r}-\widehat{u}_{l}\right|$ is arbitrary and $n$ waves are composed of shocks and contact discontinuities, while not giving the proof. In this paper, we shall give a complete proof for that case. For the case that includes centered waves, we deal with it in a forthcoming paper. For more related results, see the monographs $[7,19]$.

## 2. Main results

Suppose that we prescribe a similarity solution $u=U\left(\frac{x}{t}\right)$ of the Riemann problem, which is composed of $n+1$ constant states $\widehat{u}_{0}=\widehat{u}_{l}, \widehat{u}_{1}, \ldots, \widehat{u}_{n-1}, \widehat{u}_{n}=$ $\widehat{u}_{r}$ and $n$ waves (see Figure 1), in Figure 1,


Figure 1. Similarity solution of Riemann problem

$$
\begin{equation*}
O \widehat{A}_{k}^{ \pm}: x=\widehat{\sigma}_{k}^{ \pm} t, \quad(k=1, \ldots, n) \tag{2.1}
\end{equation*}
$$

is the right (left) boundary of the $k$-th wave, $\widehat{u}_{k}$ is the constant state between $O \widehat{A}_{k}^{+}$and $O \widehat{A}_{k}^{-}$; the eigenvalues $\widehat{\lambda}_{1}^{(k-1)}, \ldots, \widehat{\lambda}_{k-1}^{(k-1)}$ and $\widehat{\lambda}_{k+1}^{(k)}, \ldots, \widehat{\lambda}_{n}^{(k)}$ labeled on both sides of $O \widehat{A}_{k+1}^{-}$are called "coming characteristics", where

$$
\widehat{\lambda}_{j}^{(i)}=\lambda_{j}\left(\widehat{u}_{i}\right), \quad(i=1, \ldots, n-1 ; j=1, \ldots, n)
$$

Our aim is to investigate in what condition, the generalized Riemann problem (1.1), (1.8) admits a unique piecewise smooth solution that possesses a similar structure (see Figure 2), namely, any wave through the origin

$$
O A_{k}^{ \pm}: x=x_{k}^{ \pm}(t),\left(x_{k}^{ \pm}(0)=0\right) \quad(k=1, \ldots, n)
$$

has the same type (shock wave, contact discontinuity or centered wave) as


Figure 2. Solution of generalized Riemann problem
$O \widehat{A}_{k}^{ \pm}$in the solution of the Riemann problem (1.1), (1.4), and

$$
x_{k}^{ \pm}(0)=\widehat{\sigma}_{k}^{ \pm}, \quad(k=1, \ldots, n)
$$

where $\widehat{\sigma}_{k}^{ \pm}$are given by (2.1). $u_{0}, \ldots, u_{n}$ satisfy the system (1.1) in the classical sense on their respective domains, and

$$
\begin{equation*}
u_{k}(0,0)=\widehat{u}_{k}, \quad(k=0, \ldots, n) . \tag{2.2}
\end{equation*}
$$

For the case of the $k$-th $(1 \leq k \leq n)$ wave being a shock wave or a contact discontinuity, we have

$$
\left\{\begin{array}{l}
\widehat{\sigma}_{k}^{+}=\widehat{\sigma}_{k}^{-} \\
O \widehat{A}_{k}^{+}=O \widehat{A}_{k}^{-}
\end{array}\right.
$$

denoting them $\widehat{\sigma}_{k}$ and $O \widehat{A}_{k}$, respectively. On $O \widehat{A}_{k}$ the following RankineHugoniot condition:

$$
\left(\widehat{u}_{k}-\widehat{u}_{k-1}\right) \widehat{\sigma}_{k}=f\left(\widehat{u}_{k}\right)-f\left(\widehat{u}_{k-1}\right)
$$

must be satisfied and since it must satisfy the entropy condition if $O \widehat{A}_{k}$ is a shock wave, and be the $k$-th characteristic line if $O \widehat{A}_{k}$ is a contact discontinuity, combining (1.2) one yields

$$
\left\{\begin{array}{l}
\lambda_{1}\left(\widehat{u}_{k-1}\right)<\cdots<\lambda_{k-1}\left(\widehat{u}_{k-1}\right)<\widehat{\sigma}_{k} \leq \lambda_{k}\left(\widehat{u}_{k-1}\right),  \tag{2.3}\\
\lambda_{k}\left(\widehat{u}_{k}\right) \leq \widehat{\sigma}_{k}<\lambda_{k+1}\left(\widehat{u}_{k}\right)<\cdots<\lambda_{n}\left(\widehat{u}_{k}\right),
\end{array}\right.
$$

where "=" corresponds to the contact discontinuity; "<" corresponds to the shock wave.

For the corresponding generalized Riemann problem, set

$$
O A_{k}=O A_{k}^{+}=O A_{k}^{-}: x=x_{k}(t)
$$

then $x_{k}(t)$ satisfies

$$
\begin{equation*}
x_{k}^{\prime}(0)=\widehat{\sigma}_{k} . \tag{2.4}
\end{equation*}
$$

On both sides of $O A_{k} u_{k-1}(t, x)$ and $u_{k}(t, x)$ have to satisfy the RankineHugoniot condition

$$
\begin{equation*}
\left(u_{k}(t, x)-u_{k-1}(t, x)\right) \frac{d x_{k}(t, x)}{d t}=f\left(u_{k}(t, x)\right)-f\left(u_{k-1}(t, x)\right) \text { on } x=x_{k}(t) \tag{2.5}
\end{equation*}
$$

and by (2.2), (2.3), noting the continuity and the property of contact discontinuity, at least in a neighborhood of the origin it follows that

$$
\left\{\begin{array}{l}
\lambda_{1}\left(u_{k-1}(t, x)\right)<\cdots<\lambda_{k-1}\left(u_{k-1}(t, x)\right)<x_{k}^{\prime}(t) \leq \lambda_{k}\left(u_{k-1}(t, x)\right),  \tag{2.6}\\
\lambda_{k}\left(u_{k}(t, x)\right) \leq x_{k}^{\prime}(t)<\lambda_{k+1}\left(u_{k}(t, x)\right)<\cdots<\lambda_{n}\left(u_{k}(t, x)\right)
\end{array}\right.
$$

where "=" corresponds to the contact discontinuity; " $<$ " corresponds to the shock wave. By (2.6) we can label the "coming character" $\lambda_{i}^{(k-1)}(i=1, \ldots, k-$ 1) and $\lambda_{i}^{(k)}(i=k+1, \ldots, n)$ on both sides of $O A_{k}$, where

$$
\left\{\begin{array}{l}
\lambda_{i}^{(k-1)}=\lambda_{i}\left(u_{k-1}(t, x)\right), \quad(i=1, \ldots, k-1) \\
\lambda_{i}^{(k)}=\lambda_{i}\left(u_{k}(t, x)\right), \quad(i=k+1, \ldots, n)
\end{array}\right.
$$

Let

$$
\left\{\begin{array}{l}
u_{k-1}=\sum_{i=1}^{n} v_{i}^{k-1} r_{i}\left(\widehat{u}_{k-1}\right), \\
u_{k}=\sum_{i=1}^{n} v_{i}^{k} r_{i}\left(\widehat{u}_{k}\right), \quad(i=1, \ldots, n)
\end{array}\right.
$$

Then it follows from (1.3) that

$$
\left\{\begin{array}{l}
v_{i}^{k-1}=l_{i}\left(\widehat{u}_{k-1}\right) u_{k-1},  \tag{2.7}\\
v_{i}^{k}=l_{i}\left(\widehat{u}_{k}\right) u_{k}, \quad(i=1, \ldots, n) .
\end{array}\right.
$$

We present the following hypothesis:
(H2) The Rankine-Hugoniot condition (2.5) can equivalently be written as the explicit form of those variables $v$ corresponding to "coming characteristics". Precisely speaking, the Rankine-Hugoniot condition on $O A_{k}$ can be written as

$$
\begin{gather*}
\frac{d x_{k}(t, x)}{d t}=F_{k}\left(u_{k-1}, u_{k}\right), x_{k}(0)=0  \tag{2.8}\\
\left\{\begin{array}{l}
v_{i}^{k-1}=g_{i}^{k-1}\left(v_{k}^{k-1}, \ldots, v_{n}^{k-1}, v_{1}^{k}, \ldots, v_{k}^{k}\right), \quad(i=1, \ldots, k-1), \\
v_{j}^{k}=g_{j}^{k}\left(v_{k}^{k-1}, \ldots, v_{n}^{k-1}, v_{1}^{k}, \ldots, v_{k}^{k}\right), \quad(j=k+1, \ldots, n) .
\end{array}\right. \tag{2.9}
\end{gather*}
$$

Remark 2.1. To verify the hypothesis (H2), we only need to use the implicit function theorem.

If $O A_{k}$ is a shock wave, it is easy to prove the hypothesis (H2) is fulfilled provided that

$$
\operatorname{det}\left(r_{1}\left(\widehat{u}_{k-1}\right), \ldots, r_{k-1}\left(\widehat{u}_{k-1}\right), \widehat{u}_{k}-\widehat{u}_{k-1}, r_{k+1}\left(\widehat{u}_{k}\right), \ldots, r_{n}\left(\widehat{u}_{k}\right)\right) \neq 0
$$

If $O A_{k}$ is a contact discontinuity, assume $\lambda_{k}(u)$ is linearly degenerate in the sense of P. D. Lax, then the Rankine-Hugoniot condition on $O A_{k}$ can equivalently be written as

$$
\begin{aligned}
\omega_{i}\left(u_{k}\right) & =\omega_{i}\left(u_{k-1}\right),(i=1, \ldots, k-1, k+1, \ldots, n) \\
\frac{d x_{k}(t)}{d t} & =\lambda_{k}\left(u_{k-1}\right)\left(=\lambda_{k}\left(u_{k}\right)\right)
\end{aligned}
$$

where $\omega_{i}(u)$ are $n-1$ independent Riemann invariants corresponding to $\lambda_{k}(u)$, defined as follows:

$$
\nabla \omega_{i}(u) \cdot r_{k}(u)=0
$$

Obviously, if

$$
\operatorname{det}\left(\begin{array}{cc}
\nabla \omega_{i}\left(\widehat{u}_{k-1}\right) \cdot r_{j}\left(\widehat{u}_{k-1}\right) & \nabla \omega_{i}\left(\widehat{u}_{k}\right) \cdot r_{j}\left(\widehat{u}_{k}\right) \\
(j=1, \ldots, k-1) & (j=k+1, \ldots, n)
\end{array}\right) \neq 0
$$

where $i=1, \ldots, k-1, k+1, \ldots, n$, then (H2) is fulfilled.
Remark 2.2. (2.6) implies that $u_{0}(t, x)$ and $u_{n}(t, x)$ can be respectively obtained by solving the Cauchy problem with initial data $\bar{u}_{l}(x)$ and $\bar{u}_{r}(x)$, hence, if $O A_{k}$ is a shock wave or a contact discontinuity, then the Rankine-Hugoniot condition can be written as

$$
\begin{align*}
& \frac{d x_{1}(t)}{d t}=F_{1}\left(t, x, u_{1}\right), x(0)=0  \tag{2.10}\\
& v_{i}^{1}=g_{i}^{1}\left(t, x, v_{1}^{1}\right),(i=2, \ldots, n) \tag{2.11}
\end{align*}
$$

Likewise, if $O A_{n}$ is a shock wave or a contact discontinuity, then the RankineHugoniot condition can be written as

$$
\begin{gather*}
\frac{d x_{n}(t)}{d t}=F_{n}\left(t, x, u_{n-1}\right), \quad x(0)=0,  \tag{2.12}\\
v_{j}^{n-1}=g_{j}^{n-1}\left(t, x, v_{n}^{n-1}\right), \quad(j=1, \ldots, n-1) . \tag{2.13}
\end{gather*}
$$

In what follows we write two groups of $n(n-1) \times n(n-1)$ matrices $\Theta_{j}(j=$ $1,2, \ldots)$ and $\bar{\Theta}_{j}(j=0,1, \ldots)$, and then obtain the main results.

Let

$$
\left\{\begin{array}{rl}
\tau_{i}^{k} & =\frac{\widehat{\lambda}_{i}^{k}-\widehat{\sigma}_{k}^{+}}{\widehat{\lambda}_{i}^{k}-\widehat{\sigma}_{k+1}^{-}},(i=1, \ldots, k),  \tag{2.14}\\
\tau_{i}^{k} & =\frac{\widehat{\lambda}_{i}^{k}-\widehat{\sigma}_{k+1}^{-}}{\hat{\lambda}_{i}^{k}-\widehat{\sigma}_{k}^{+}},(i=k+1, \ldots, n),
\end{array} \quad(k=1, \ldots, n-1),\right.
$$

where $\widehat{\lambda}_{i}^{k}=\lambda_{i}\left(\widehat{u}_{k}\right), \widehat{\sigma}_{i}^{ \pm}$are given by (2.1). Obviously,

$$
0 \leq \tau_{i}^{k}<1 \quad(i=1, \ldots, n ; k=1, \ldots, n-1)
$$

For $k(1 \leq k \leq n)$ corresponding to the shock wave or the contact discontinuity, let

$$
\left\{\begin{align*}
&\left(\Theta_{j}\right)_{n(k-2)+p, n(k-2)+q}=\left(\bar{\Theta}_{j}\right)_{n(k-2)+p, n(k-2)+q}  \tag{2.15}\\
&=\frac{\partial g_{p}^{k-1}}{\partial v_{q}^{k-1}}\left(\tau_{q}^{k-1}\right)^{j}, \quad(q=k, \ldots, n), \\
&\left(\Theta_{j}\right)_{n(k-2)+p, n(k-1)+q}=\left(\bar{\Theta}_{j}\right)_{n(k-2)+p, n(k-1)+q} \\
&=\frac{\partial g_{p}^{k-1}}{\partial v_{q}^{k}}\left(\tau_{q}^{k}\right)^{j}, \quad(q=1, \ldots, k),  \tag{2.16}\\
&\left(\Theta_{j}\right)_{n(k-2)+p, q}=\left(\bar{\Theta}_{j}\right)_{n(k-2)+p, q}=0, \\
&(q<n(k-2)+k\text { or } q>n(k-1)+k),(p=1, \ldots, k-1), \\
&=\frac{\partial g_{p}^{k}}{\partial v_{q}^{k-1}\left(\tau_{q}^{k-1}\right)^{j}, \quad(q=k, \ldots, n),} \\
&\left\{\begin{aligned}
\left(\Theta_{j}\right)_{n(k-1)+p, n(k-2)+q} & =\left(\bar{\Theta}_{j}\right)_{n(k-1)+p, n(k-2)+q}
\end{aligned}\right. \\
& \begin{array}{rl}
\left(\Theta_{j}\right)_{n(k-1)+p, n(k-1)+q} & =\left(\bar{\Theta}_{j}\right)_{n(k-1)+p, n(k-1)+q} \\
& =\frac{\partial g_{p}^{k}}{\partial v_{q}^{k}}\left(\tau_{q}^{k}\right)^{j}, \quad(q=1, \ldots, k), \\
\left(\Theta_{j}\right)_{n(k-1)+p, q} & =\left(\bar{\Theta}_{j}\right)_{n(k-1)+p, q}=0, \\
(q<n(k-2)+k & \text { or } q>n(k-1)+k),(p=k+1, \ldots, n),
\end{array}
\end{align*}\right.
$$

where the functions on the right side of (2.15) and (2.16) take values on $t=$ $0, x=0, v^{i}=\widehat{v}^{i}(i=1, \ldots, n-1),(2.11),(2.13)$ imply that $\left(\Theta_{j}\right)_{p q}(j=1,2, \ldots)$ and $\left(\bar{\Theta}_{j}\right)_{p q}(j=0,1, \ldots)$ do not have elements not vanishing until $1 \leq p \leq$
$n(n-1), 1 \leq q \leq n(n-1)$, thus we define two groups of $n(n-1) \times n(n-1)$ matrices $\Theta_{j}(j=1,2, \ldots)$ and $\bar{\Theta}_{j}(j=0,1, \ldots)$ depending only on the solution of the Riemann problem.

Let the $n(n-1) \times n(n-1)$ diagonal matrix $\tau$ be

$$
\begin{equation*}
\tau=\operatorname{diag}\left\{\tau_{1}^{1}, \ldots, \tau_{n}^{1}, \ldots, \tau_{1}^{n-1}, \ldots, \tau_{n}^{n-1}\right\} . \tag{2.17}
\end{equation*}
$$

For $N \times N$ matrix $A=\left(a_{i j}\right)$ define the following minimal characterizing number:

$$
\|A\|_{\min }=\inf _{\gamma}\left\|\gamma A \gamma^{-1}\right\|,
$$

where $\gamma=\operatorname{diag}\left\{\gamma_{1}, \ldots, \gamma_{N}\right\}, \gamma_{i} \neq 0(i=1, \ldots, N)$, and

$$
\|A\|=\max _{i=1, \ldots, N} \sum_{j=1}^{N}\left|a_{i j}\right| .
$$

We get the following main theorems:
Theorem 2.1. Under the hypotheses (H1),(H2), if $f(u), \hat{u}_{l}(x), \hat{u}_{r}(x)$ are $C^{m+1}$ functions, then if

$$
\begin{gather*}
\operatorname{det}\left|I-\Theta_{j}\right| \neq 0 \quad(j=1, \ldots, n-1), \\
\left\|\bar{\Theta}_{m}\right\|_{\min }<1, \tag{2.18}
\end{gather*}
$$

the generalized Riemann problem (1.1), (1.8) admits a unique piecewise $C^{m+1}$ local solution $u=u(t, x)$ except the origin, which possesses a similar structure at least in a neighborhood of the origin with the given similarity solution of the Riemann problem (1.1), (1.4).

Remark 2.3. As long as one introduces the reversible transformation $\bar{v}=\gamma v$ of the unknown function, where

$$
\begin{gathered}
\gamma=\operatorname{diag}\left\{\gamma_{1}, \ldots, \gamma_{n(n-1)}\right\}, \gamma_{i} \neq 0 \quad(i=1, \ldots, n(n-1)), \\
v=\left(v_{1}^{1}, \ldots, v_{n}^{1}, \ldots, v_{1}^{n-1}, \ldots, v_{n}^{n-1}\right)^{T}
\end{gathered}
$$

then $\bar{\Theta}_{j}$ is reduced to $\gamma \bar{\Theta}_{j} \gamma^{-1}$, hence in the proof of Theorem 2.1 we can substitute

$$
\left\|\bar{\Theta}_{m}\right\|<1
$$

for (2.18).
Theorem 2.2. Under hypotheses (H1), (H2), if $f(u), \hat{u}_{l}(x), \hat{u}_{r}(x)$ are $C^{\infty}$ functions, then

$$
\operatorname{det}\left|I-\Theta_{j}\right| \neq 0,(j=1,2, \ldots)
$$

if and only if the generalized Riemann problem (1.1), (1.8) admits a unique piecewise $C^{\infty}$ local solution $u=u(t, x)$ except the origin, which possesses a similar structure at least in a neighborhood of the origin with $u=U\left(\frac{x}{t}\right)$.

Remark 2.4. Theorems 2.1, 2.2 remain valid for more general hyperbolic systems of conservation laws

$$
\frac{\partial u}{\partial t}+\frac{\partial f(t, x, u)}{\partial x}=g(t, x, u)
$$

and the system of corresponding Riemann problem is

$$
\frac{\partial u}{\partial t}+\nabla_{u} f(0,0, u) \frac{\partial u}{\partial x}=0
$$

## 3. Proof of main results

We consider the generalized Riemann problem of the following form:

$$
\begin{gather*}
\quad \frac{\partial u}{\partial t}+\frac{\partial f(t, x, u)}{\partial x}=g(t, x, u)  \tag{3.1}\\
t=0: \quad u= \begin{cases}\hat{u}_{l}(x), & x \leq 0 \\
\hat{u}_{r}(x), & x \geq 0\end{cases} \tag{3.2}
\end{gather*}
$$

where $f$ is $C^{m+2}$ with respect to $x$ and $u, C^{m+1}$ with respect to $t$, and $g, \hat{u}_{l}, \hat{u}_{r}$ are $C^{m+1}$ functions of all arguments. Suppose a similarity solution $u=U\left(\frac{x}{t}\right)$ of its corresponding Riemann problem

$$
\begin{gather*}
\frac{\partial u}{\partial t}+\nabla_{u} f(0,0, u) \frac{\partial u}{\partial x}=0  \tag{3.3}\\
t=0: \quad u= \begin{cases}\widehat{u}_{l}=\hat{u}_{l}(0), & x \leq 0 \\
\widehat{u}_{r}=\hat{u}_{r}(0), & x \geq 0\end{cases} \tag{3.4}
\end{gather*}
$$

is composed of $n+1$ piecewise constant states $\widehat{u}_{0}=\widehat{u}_{l}, \widehat{u}_{1}, \ldots, \widehat{u}_{n-1}, \widehat{u}_{n}=\widehat{u}_{r}$ and $n$ shocks or contact discontinuities. We shall prove the generalized Riemann problem (3.1), (3.2) admits a unique piecewise $C^{m+1}$ solution which has a similar structure.

Assume the matrix $\zeta(t, x, u)$ is composed of $n$ left eigenvectors $l_{1}, l_{2}, \ldots, l_{n}$ of $\nabla_{u} f(t, x, u)$, and its every element is a piecewise $C^{m+1}$ function. Moreover, in $A_{k} O A_{k+1}(k=1, \ldots, n-1)$ we can always take

$$
\begin{equation*}
\zeta_{i j}\left(0,0, \widehat{u}_{k}\right)=\delta_{i j}, \quad(i, j=1, \ldots, n) \tag{3.5}
\end{equation*}
$$

Multiplying (3.1) by $\zeta$ from the left, we obtain the characteristic form

$$
\begin{equation*}
\zeta(t, x, u) \frac{\partial u}{\partial t}+\lambda(t, x, u) \frac{\partial u}{\partial x}=\mu(t, x, u) \tag{3.6}
\end{equation*}
$$

where $\zeta, \lambda, \mu \in C^{m+1}$,

$$
\begin{gathered}
\lambda(t, x, u)=\operatorname{diag}\left\{\lambda_{1}(t, x, u), \ldots, \lambda_{n}(t, x, u)\right\} \\
\lambda_{1}(t, x, u)<\lambda_{2}(t, x, u)<\cdots<\lambda_{n}(t, x, u)
\end{gathered}
$$

on the domain under consideration. Then (2.3) implies that $u_{0}(t, x)$ and $u_{n}(t, x)$ can be respectively obtained by solving the Cauchy problem (3.1) with initial data $\widehat{u}_{l}(x)$ and $\widehat{u}_{r}(x)$ in a neighborhood of the origin, set

$$
O A_{k}: x=x_{k}(t), \quad(k=1, \ldots, n) .
$$

To get the solution of the generalized Riemann problem (3.1), (3.2), we only have to solve the free boundary problem on the fan-shaped domain

$$
\bigcup_{k=1}^{n-1} D_{k}(\delta)=\left\{(t, x) \mid 0 \leq t \leq \delta, x_{k}(t) \leq x \leq x_{k+1}(t)\right\}
$$

whose solutions $u_{k}(t, x)$ satisfy equation (3.1) on $D_{k}(\delta)$, and

$$
u_{k}(0,0)=\widehat{u}_{k}, \quad(k=1, \ldots, n-1) .
$$

Furthermore, free boundaries $O A_{k}(k=1, \ldots, n)$ satisfy (2.8), (2.10) and (2.12), and $u_{k-1}, u_{k}$ satisfy the Rankine-Hugoniot conditions (2.9), (2.11) and (2.13) on both sides of $O A_{k}$. Noting (2.7) and (3.5), we now have

$$
u_{k}(t, x)=v^{k}(t, x), \quad(k=1, \ldots, n-1) .
$$

Let

$$
\begin{equation*}
T_{k}(t)=\frac{x_{k+1}(t)-x_{k}(t)}{t}, \quad(0 \leq t \leq \delta), \quad(k=1, \ldots, n-1) \tag{3.7}
\end{equation*}
$$

We introduce the following transformation

$$
\begin{aligned}
& \left\{\begin{array}{l}
\bar{t}=t, \\
\bar{x}=\frac{x-x_{k}(t)}{T_{k}(t)} \quad \text { on } D_{k}(\delta), \quad(k=1, \ldots, n-1 \text { and } k \text { is odd }),
\end{array}\right. \\
& \left\{\begin{array}{l}
\bar{t}=t, \\
\bar{x}=\frac{x-x_{k+1}(t)}{T_{k}(t)} \quad \text { on } D_{k}(\delta), \quad(k=1, \ldots, n-1 \text { and } k \text { is even }) .
\end{array}\right.
\end{aligned}
$$

Thus all $D_{k}(\delta)(k=1, \ldots, n-1)$ are changed to the domain

$$
\bar{D}(\delta)=\{(\bar{t}, \bar{x}) \mid 0 \leq \bar{t} \leq \delta, 0 \leq \bar{x} \leq \bar{t}\} .
$$

Moreover, $O A_{k}(k=1, \ldots, n)$ are respectively mapped onto $\bar{x}=0$ and $\bar{x}=\bar{t}$ for odd $k$ and even $k$. Set

$$
\bar{u}^{k}(\bar{t}, \bar{x})=u^{k}\left(\bar{t}, x_{k}(\bar{t}, \bar{x})\right), \quad(k=1, \ldots, n-1),
$$

where

$$
x_{k}(\bar{t}, \bar{x})= \begin{cases}x_{k}(\bar{t})+\bar{x} T_{k}(\bar{t}) & \text { for odd } k  \tag{3.8}\\ x_{k+1}(\bar{t})-\bar{x} T_{k}(\bar{t}) & \text { for even } k\end{cases}
$$

Then $\bar{u}^{k}(k=1, \ldots, n-1)$ satisfy

$$
\begin{equation*}
\sum_{i=1}^{n} \bar{\zeta}_{l i}^{k}\left(\bar{t}, \bar{x} \mid \bar{u}^{k}\right)\left(\frac{\partial \bar{u}_{i}^{k}}{\partial \bar{t}}+\lambda_{l}^{k}(\bar{t}, \bar{x} \mid \bar{u}) \frac{\partial \bar{u}_{i}^{k}}{\partial \bar{x}}\right)=\bar{\mu}_{l}^{k}\left(\bar{t}, \bar{x} \mid \bar{u}^{k}\right), \quad(l=1, \ldots, n) \tag{3.9}
\end{equation*}
$$

(3.10) $\bar{u}_{r}^{k}=G_{r}^{k}\left(\bar{t} \mid \bar{u}_{k+1}^{k}, \ldots, \bar{u}_{n}^{k}, \bar{u}_{1}^{k+1}, \ldots, \bar{u}_{k+1}^{k+1}\right) \quad$ on $\bar{x}=\bar{t}, \quad(r=1, \ldots, k)$,
(3.11) $\bar{u}_{s}^{k}=G_{s}^{k}\left(\bar{t} \mid \bar{u}_{k}^{k-1}, \ldots, \bar{u}_{n}^{k-1}, \bar{u}_{1}^{k}, \ldots, \bar{u}_{k}^{k}\right) \quad$ on $\bar{x}=0, \quad(s=k+1, \ldots, n)$
for odd $k$, and
$\bar{u}_{r}^{k}=G_{r}^{k}\left(\bar{t} \mid \bar{u}_{k+1}^{k}, \ldots, \bar{u}_{n}^{k}, \bar{u}_{1}^{k+1}, \ldots, \bar{u}_{k+1}^{k+1}\right) \quad$ on $\bar{x}=0, \quad(r=1, \ldots, k)$,
(3.13) $\bar{u}_{s}^{k}=G_{s}^{k}\left(\bar{t} \mid \bar{u}_{k}^{k-1}, \ldots, \bar{u}_{n}^{k-1}, \bar{u}_{1}^{k}, \ldots, \bar{u}_{k}^{k}\right) \quad$ on $\bar{x}=\bar{t}, \quad(s=k+1, \ldots, n)$
for even $k$, where

$$
\begin{array}{r}
\bar{\zeta}_{l i}^{k}\left(\bar{t}, \bar{x} \mid \bar{u}^{k}\right)=\zeta_{l i}\left(\bar{t}, x_{k}(\bar{t}, \bar{x}), \bar{u}^{k}\right), \quad(l, i=1, \ldots, n ; k=1, \ldots, n-1), \\
\lambda_{l}^{k}(\bar{t}, \bar{x} \mid \bar{u})=\left((-1)^{k+1} \lambda_{l}\left(\bar{t}, x_{k}(\bar{t}, \bar{x}), \bar{u}^{k}\right)-\frac{\partial x_{k}(\bar{t}, \bar{x})}{\partial \bar{t}}\right) / T_{k}(\bar{t}) \\
(l=1, \ldots, n ; k=1, \ldots, n-1), \tag{3.16}
\end{array}
$$

and $\bar{u}=\left(\bar{u}^{1}, \ldots, \bar{u}^{n-1}\right)$. As $k$ is odd, we have

$$
\begin{equation*}
G_{r}^{k}\left(\bar{t} \bar{u}_{k+1}^{k}, \ldots, \bar{u}_{n}^{k}, \bar{u}_{1}^{k+1}, \ldots, \bar{u}_{k+1}^{k+1}\right) \tag{3.17}
\end{equation*}
$$

$$
=g_{r}^{k}\left(\bar{t}, x_{k+1}(\bar{t}), \bar{u}_{k+1}^{k}(\bar{t}, \bar{t}), \ldots, \bar{u}_{n}^{k}(\bar{t}, \bar{t}), \bar{u}_{1}^{k+1}(\bar{t}, \bar{t}), \ldots, \bar{u}_{k+1}^{k+1}(\bar{t}, \bar{t})\right),(r=1, \ldots, k)
$$

$$
\begin{equation*}
G_{s}^{k}\left(\bar{t} \mid \bar{u}_{k}^{k-1}, \ldots, \bar{u}_{n}^{k-1}, \bar{u}_{1}^{k}, \ldots, \bar{u}_{k}^{k}\right) \tag{3.18}
\end{equation*}
$$

$$
=g_{s}^{k}\left(\bar{t}, x_{k}(\bar{t}), \bar{u}_{k}^{k-1}(\bar{t}, 0), \ldots, \bar{u}_{n}^{k-1}(\bar{t}, 0), \bar{u}_{1}^{k}(\bar{t}, 0), \ldots, \bar{u}_{k}^{k}(\bar{t}, 0)\right),(s=k+1, \ldots, n),
$$

in addition
(3.19)
$\left\{\begin{aligned} \frac{d x_{k+1}(\bar{t})}{d \bar{t}} & =F_{k+1}\left(\bar{t}, x_{k+1}(\bar{t}), \bar{u}_{k+1}^{k}(\bar{t}, \bar{t}), \ldots, \bar{u}_{n}^{k}(\bar{t}, \bar{t}), \bar{u}_{1}^{k+1}(\bar{t}, \bar{t}), \ldots, \bar{u}_{k+1}^{k+1}(\bar{t}, \bar{t})\right), \\ x_{k+1}(0) & =0,\end{aligned}\right.$

$$
\left\{\begin{align*}
\frac{d x_{k}(\bar{t})}{d \bar{t}} & =F_{k}\left(\bar{t}, x_{k}(\bar{t}), \bar{u}_{k}^{k-1}(\bar{t}, 0), \ldots, \bar{u}_{n}^{k-1}(\bar{t}, 0), \bar{u}_{1}^{k}(\bar{t}, 0), \ldots, \bar{u}_{k}^{k}(\bar{t}, 0)\right)  \tag{3.20}\\
x_{k}(0) & =0
\end{align*}\right.
$$

where $g_{i}^{k}(i=1, \ldots, n), F_{k}, F_{k+1}$ are given by (2.8)-(2.9) and (2.10)-(2.13). Likewise for even $k$, we can also obtain similar boundary conditions.

Thus, we acquire a functional boundary value problem in terms of $\bar{u}^{k}$ ( $k=$ $1, \ldots, n-1$ ) on the angular domain $\bar{D}(\delta)$, which is equivalent to the original problem. We next use the method similar to that used in [18] to extend the systems (3.9)-(3.20).

If $\bar{u}(\bar{t}, \bar{x}) \in C^{m+1}$, define operators

$$
A=\frac{\partial}{\partial \bar{t}}+\frac{\partial}{\partial \bar{x}}, B=\frac{\partial}{\partial \bar{t}},
$$

and set

$$
\left\{\begin{align*}
u^{k, 1}(\bar{t}, \bar{x}) & =\left\{A \bar{u}_{1}^{k}, \ldots, A \bar{u}_{k}^{k}, B \bar{u}_{k+1}^{k}, \ldots, B \bar{u}_{n}^{k}\right\},  \tag{3.21}\\
v^{k, 1}(\bar{t}, \bar{x}) & =\left\{B \bar{u}_{1}^{k}, \ldots, B \bar{u}_{k}^{k}, A \bar{u}_{k+1}^{k}, \ldots, A \bar{u}_{n}^{k}\right\}
\end{align*}\right.
$$

for odd $k$. Substituting (3.21) into (3.9), we obtain
$\sum_{r=1}^{k} \bar{\zeta}_{l r}^{k}\left(\lambda_{l}^{k} u_{r}^{k, 1}+\left(1-\lambda_{l}^{k}\right) v_{r}^{k, 1}\right)+\sum_{s=k+1}^{n} \bar{\zeta}_{l s}^{k}\left(\lambda_{l}^{k} v_{s}^{k, 1}+\left(1-\lambda_{l}^{k}\right) u_{s}^{k, 1}\right)=\bar{\mu}_{l}^{k},(l=1, \ldots, n)$,
from which it yields
$v_{l}^{k, 1}=\sum_{i=1}^{n} a_{l i}^{k, 1}\left(\bar{t}, x_{k}(\bar{t}, \bar{x}), \bar{u}(\bar{t}, \bar{x})\right) u_{i}^{k, 1}+b_{l}^{k, 1}\left(\bar{t}, x_{k}(\bar{t}, \bar{x}), \bar{u}(\bar{t}, \bar{x})\right), \quad(l=1, \ldots, n)$.
By (3.5), (3.14), (3.15) we get

$$
\begin{gather*}
\bar{\zeta}_{l i}^{k}\left(0,0 \mid \widehat{u}_{k}\right)=\zeta_{l i}\left(0,0, \widehat{u}_{k}\right)=\delta_{l i}, \quad(l, i=1, \ldots, n),  \tag{3.22}\\
\lambda_{l}^{k}(0,0 \mid \widehat{u})=\frac{\lambda_{l}\left(0,0, \widehat{u}_{k}\right)-F_{k}(0,0, \widehat{u})}{F_{k+1}(0,0, \widehat{u})-F_{k}(0,0, \widehat{u})}, \quad(l=1, \ldots, n),
\end{gather*}
$$

where $\widehat{u}=\left\{\widehat{u}_{1}, \ldots, \widehat{u}_{n}\right\}$. Noting (2.4), we have

$$
\left\{\begin{array}{l}
F_{k}(0,0, \widehat{u})=\widehat{\sigma}_{k}, \\
F_{k+1}(0,0, \widehat{u})=\widehat{\sigma}_{k+1} .
\end{array}\right.
$$

By (2.14) we easily calculate

$$
\begin{gathered}
a_{l i}^{k, 1}(0,0, \bar{u}(0,0))=\tau_{l}^{k} \delta_{l i}, \quad(l, i=1, \ldots, n), \\
b_{l}^{k, 1}(0,0, \bar{u}(0,0))=\gamma_{l}^{k, 1}, \quad(l=1, \ldots, n),
\end{gathered}
$$

where

$$
\left\{\begin{aligned}
\gamma_{r}^{k, 1} & =\left(\frac{F_{k+1}-F_{k}}{F_{k+1}-\lambda_{r}} \mu_{r}\right)(0,0, \hat{u}), \quad(r=1, \ldots, k), \\
\gamma_{s}^{k, 1} & =\left(\frac{F_{k+1}-F_{k}}{\lambda_{s}-F_{k}} \mu_{s}\right)(0,0, \hat{u}), \quad(s=k+1, \ldots, n)
\end{aligned}\right.
$$

Consequently, at the origin we have

$$
v_{l}^{k, 1}=\tau_{l}^{k} u_{l}^{k, 1}+\gamma_{l}^{k, 1}, \quad(l=1, \ldots, n) .
$$

Differentiating the system (3.9) with respect to $\bar{t}$ and combining (3.14)-(3.16) yields

$$
\begin{aligned}
& \sum_{i=1}^{n} \zeta_{l i}^{k, 1}\left(\bar{t}, x_{k}(\bar{t}, \bar{x}), u(\bar{t}, \bar{x})\right)\left(\frac{\partial u_{i}^{k, 1}}{\partial \bar{t}}+\lambda_{l}^{k} \frac{\partial u_{i}^{k, 1}}{\partial \bar{x}}\right) \\
= & \mu_{l}^{k, 1}\left(\bar{t}, x_{k}(\bar{t}, \bar{x}), u(\bar{t}, \bar{x}), u^{k, 1}(\bar{t}, \bar{x})\right) \quad(l=1, \ldots, n) .
\end{aligned}
$$

When $\zeta, \lambda, \mu$ in the system (3.6) are $C^{m+1}$ functions, obviously $\zeta^{k, 1}, \mu^{k, 1}$ are $C^{m}$ functions, where

$$
\left\{\begin{array}{l}
\zeta_{r p}^{k, 1}=\zeta_{r p}^{k}+\sum_{q=k+1}^{n} \zeta_{r q}^{k} a_{q p}^{k, 1}, \zeta_{r s}^{k, 1}=\sum_{q=k+1}^{n} \zeta_{r q}^{k} a_{q s}^{k, 1}, \\
\zeta_{s r}^{k, 1}=\sum_{p=1}^{k} \zeta_{s p}^{k} a_{p r}^{k, 1}, \zeta_{s q}^{k, 1}=\sum_{p=1}^{k} \zeta_{s p}^{k} a_{p q}^{k, 1}+\zeta_{s q}^{k},(r, p=1, \ldots, k ; s, q=k+1, \ldots, n) .
\end{array}\right.
$$

By (3.22) it follows

$$
\zeta_{l i}^{k, 1}(0,0, \widehat{u})=\delta_{l i}, \quad(l, i=1, \ldots, n)
$$

Repeating the process above $m$ times, we obtain a system in terms of $u^{k, j}(j=0, \ldots, m)$, where $u^{k, 0}=\bar{u}^{k}$. On $\bar{D}(\delta), u^{k, j}$ satisfy

$$
\begin{align*}
& \sum_{i=1}^{n} \zeta_{l i}^{k, j}\left(\bar{t}, x_{k}(\bar{t}, \bar{x}), u(\bar{t}, \bar{x})\right)\left(\frac{\partial u_{i}^{k, j}}{\partial \bar{t}}+\lambda_{l}^{k} \frac{\partial u_{i}^{k, j}}{\partial \bar{x}}\right)  \tag{3.23}\\
= & \mu_{l}^{k, j}\left(\bar{t}, x_{k}(\bar{t}, \bar{x}), u^{p, q}(\bar{t}, \bar{x})\right), \quad(p=1, \ldots, n-1 ; q=0, \ldots, j), \quad(l=1, \ldots, n),
\end{align*}
$$

where $\zeta^{k, j}, \mu^{k, j}(j=0, \ldots, m)$ are at least $C^{1}$ functions, and satisfy

$$
\begin{equation*}
\zeta_{l i}^{k, j}(0,0, \widehat{u})=\delta_{l i}, \quad(l, i=1, \ldots, n) \tag{3.24}
\end{equation*}
$$

Likewise, for even $k$, in (3.21) replacing $u^{k, j}$ by $v^{k, j}$, we can derive similar systems, and (3.24) remains valid.

Next, we shall consider the boundary conditions. As $k=1, \ldots, n$ and $k$ is even, $O A_{k}:\left\{(t, x) \mid 0 \leq t \leq \delta, x=x_{k}(t)\right\}$ is transformed into $\{(\bar{t}, \bar{x}) \mid 0 \leq \bar{t} \leq$ $\delta, \bar{x}=\bar{t}\}$, on which we have the boundary condition (3.20) and

$$
\begin{align*}
\bar{u}_{r}^{k-1}= & G_{r}^{k-1}\left(\bar{t} \mid \bar{u}_{r}^{k-1}, \ldots, \bar{u}_{n}^{k-1}, \bar{u}_{1}^{k}, \ldots, \bar{u}_{k}^{k}\right)  \tag{3.25}\\
= & g_{r}^{k-1}\left(\bar{t}, x_{k}(\bar{t}), \bar{u}_{k}^{k-1}(\bar{t}, \bar{t}), \ldots, \bar{u}_{n}^{k-1}(\bar{t}, \bar{t}), \bar{u}_{1}^{k}(\bar{t}, \bar{t}), \ldots, \bar{u}_{k}^{k}(\bar{t}, \bar{t})\right), \\
& \quad(r=1, \ldots, k-1)
\end{align*}
$$

$$
\begin{align*}
\bar{u}_{s}^{k}= & G_{s}^{k}\left(\bar{t} \mid \bar{u}_{k}^{k-1}, \ldots, \bar{u}_{n}^{k-1}, \bar{u}_{k}^{k}, \ldots, \bar{u}_{k}^{k}\right)  \tag{3.26}\\
= & g_{s}^{k}\left(\bar{t}, x_{k}(\bar{t}), \bar{u}_{k}^{k-1}(\bar{t}, \bar{t}), \ldots, \bar{u}_{n}^{k-1}(\bar{t}, \bar{t}), \bar{u}_{1}^{k}(\bar{t}, \bar{t}), \ldots, \bar{u}_{k}^{k}(\bar{t}, \bar{t})\right) \\
& \quad(r=k+1, \ldots, n)
\end{align*}
$$

Differentiating both sides of (3.25) with respect to $\bar{t}$ yields

$$
\begin{aligned}
u_{r}^{k-1,1}= & \sum_{q=k}^{n} \frac{\partial g_{r}^{k-1}}{\partial \bar{u}_{q}^{k-1}} v_{q}^{k-1,1}+\sum_{p=1}^{k} \frac{\partial g_{r}^{k-1}}{\partial \bar{u}_{p}^{k}} v_{p}^{k, 1}+\frac{\partial g_{r}^{k-1}}{\partial t}+\frac{\partial g_{r}^{k-1}}{\partial x} F_{k} \\
= & \sum_{q=k}^{n} \frac{\partial g_{r}^{k-1}}{\partial \bar{u}_{q}^{k-1}}\left(\sum_{i=1}^{n} a_{q i}^{k-1,1} u_{i}^{k-1,1}+b_{q}^{k-1,1}\right) \\
& +\sum_{p=1}^{k} \frac{\partial g_{r}^{k-1}}{\partial \bar{u}_{p}^{k}}\left(\sum_{i=1}^{n} a_{p i}^{k, 1} u_{i}^{k, 1}+b_{p}^{k, 1}\right)+\frac{\partial g_{r}^{k-1}}{\partial t}+\frac{\partial g_{r}^{k-1}}{\partial x} F_{k}, \\
& (r=1, \ldots, k-1) .
\end{aligned}
$$

Repeating $m$ times we get that for $j=1, \ldots, m$

$$
\begin{align*}
u_{r}^{k-1, j}= & \sum_{q=k}^{n} \frac{\partial g_{r}^{k-1}}{\partial \bar{u}_{q}^{k-1}}\left(\sum_{i_{j}=1}^{n}\left(\sum_{i_{1}, \ldots, i_{j-1}=1}^{n} a_{q i_{1}}^{k-1,1}, a_{i_{1} i_{2}}^{k-1,2}, \ldots, a_{i_{j-1} i_{j}}^{k-1, j}\right) u_{i_{j}}^{k-1, j}\right)  \tag{3.27}\\
& +\sum_{p=1}^{k} \frac{\partial g_{r}^{k-1}}{\partial \bar{u}_{p}^{k}}\left(\sum_{i_{j}=1}^{n}\left(\sum_{i_{1}, \ldots, i_{j-1}=1}^{n} a_{p i_{1}}^{k, 1}, a_{i_{1} i_{2}}^{k, 2}, \ldots, a_{i_{j-1} i_{j}}^{k, j}\right) u_{i_{j}}^{k, j}\right)+F_{r}^{k-1, j} \\
\triangleq & \sum_{q=k}^{n} \frac{\partial g_{r}^{k-1}}{\partial \bar{u}_{q}^{k-1}}\left(\sum_{i=1}^{n} \bar{a}_{q i}^{k-1, j} u_{i}^{k-1, j}\right)+\sum_{p=1}^{k} \frac{\partial g_{r}^{k-1}}{\partial \bar{u}_{p}^{k}}\left(\sum_{i=1}^{n} \bar{a}_{p i}^{k, j} u_{i}^{k, j}\right) \\
& +F_{r}^{k-1, j}, \quad(r=1, \ldots, k-1)
\end{align*}
$$

here $a^{k-1, j}$ and $a^{k, j}$ are functions of $(t, x, \bar{u}), F_{r}^{k-1, j}$ are functions of $\left(t, x, u^{p, q}\right)$ $(p=1, \ldots, n-1 ; q=0, \ldots, j-1)$, which are at least $C^{1}$, and

$$
\begin{equation*}
a_{l i}^{k-1, j}(0,0, \widehat{u})=\tau_{l}^{k-1} \delta_{l i}, \quad(l, i=1, \ldots, n ; j=1, \ldots, m) . \tag{3.28}
\end{equation*}
$$

Therefore we obtain

$$
\begin{equation*}
\bar{a}_{l i}^{k-1, j}(0,0, \widehat{u})=\left(\tau_{l}^{k-1}\right)^{j} \delta_{l i}, \quad(l, i=1, \ldots, n ; j=1, \ldots, m), \tag{3.29}
\end{equation*}
$$

and $a^{k, j}$ also have expressions similar to (3.28). Likewise, for (3.26) and odd $k$, similar results can be obtained, and (3.28), (3.29) hold.

Lemma 3.1. In the absence of the centered wave, by equations (3.9)-(3.20) the derivatives of the solution $\bar{u}(\bar{t}, \bar{x})$ of orders $\leq m-1$ at the origin can be determined uniquely if and only if

$$
\operatorname{det}\left|I-\Theta_{j}\right| \neq 0(j=1, \ldots, m-1)
$$

where matrices $\Theta_{j}$ are defined by (2.15), (2.16).

Proof. Letting $(\bar{t}, \bar{x})=(0,0)$ in (3.27) and noting (3.29), it follows

$$
\begin{aligned}
u_{r}^{k-1, j}= & \sum_{q=k}^{n} \frac{\partial g_{r}^{k-1}}{\partial \bar{u}_{q}^{k-1}}\left(\tau_{q}^{k-1}\right)^{j} u_{q}^{k-1, j}(0,0) \\
& +\sum_{p=1}^{k} \frac{\partial g_{r}^{k-1}}{\partial \bar{u}_{p}^{k}}\left(\tau_{p}^{k}\right)^{j} u_{p}^{k, j}(0,0)+F_{r}^{k-1, j}(0,0)
\end{aligned}
$$

In view of $(2.11)$ and (2.13) we get an $n(n-1)(m-1)$ system in terms of $u_{i}^{k, j}(0,0)(k=1, \ldots, n-1 ; j=1, \ldots, m-1 ; i=1, \ldots, n)$, whose Jacobi matrix is of the following form

$$
\left(\begin{array}{cccc}
I-\Theta_{1} & & & \\
& I-\Theta_{2} & & 0 \\
& & \ddots & \\
& * & & I-\Theta_{m-1}
\end{array}\right)
$$

Hence

$$
\prod_{j=1}^{m-1} \operatorname{det}\left|I-\Theta_{j}\right| \neq 0
$$

if and only if the system has a unique solution, the proof of Lemma 3.1 is complete.

By Lemma 3.1, we can give the following boundary conditions for the derivatives of $\bar{u}$ of orders $<m$. As $k=1, \ldots, n-1$,

$$
\left\{\begin{array}{c}
u_{r}^{k, j}=u_{r}^{k, j}(0,0)+\int_{0}^{\bar{t}} u_{r}^{k, j+1}(\bar{t}, \bar{t}) d \bar{t} \quad \text { on } \bar{x}=\bar{t}  \tag{3.30}\\
(r=1, \ldots, k ; j=0, \ldots, m-1) \\
u_{s}^{k, j}=u_{s}^{k, j}(0,0)+\int_{0}^{\bar{t}} u_{s}^{k, j+1}(\bar{t}, 0) d \bar{t} \quad \text { on } \bar{x}=0 \\
\quad(s=k+1, \ldots, n ; j=0, \ldots, m-1)
\end{array}\right.
$$

for odd $k$, and

$$
\left\{\begin{array}{c}
u_{r}^{k, j}=u_{r}^{k, j}(0,0)+\int_{0}^{\bar{t}} u_{r}^{k, j+1}(\bar{t}, 0) d \bar{t} \quad \text { on } \bar{x}=0  \tag{3.31}\\
(r=1, \ldots, k ; j=0, \ldots, m-1) \\
u_{s}^{k, j}=u_{s}^{k, j}(0,0)+\int_{0}^{\bar{t}} u_{s}^{k, j+1}(\bar{t}, \bar{t}) d \bar{t} \quad \text { on } \bar{x}=\bar{t} \\
\quad(s=k+1, \ldots, n ; j=0, \ldots, m-1)
\end{array}\right.
$$

for even $k$. For the $m$-th order derivatives of $\bar{u}$, letting $j=m$ in (3.27), it follows
(3.32)
$u_{1}^{1, m}=\sum_{q=2}^{n} \frac{\partial g_{1}^{1}}{\partial \bar{u}_{q}^{1}}\left(\sum_{i=1}^{n} \bar{a}_{q i}^{1, m} u_{i}^{1, m}\right)+\sum_{p=1}^{2} \frac{\partial g_{1}^{1}}{\partial \bar{u}_{p}^{2}}\left(\sum_{i=1}^{n} \bar{a}_{p i}^{2, m} u_{i}^{2, m}\right)+F_{1}^{1, m} \quad$ on $\quad \bar{x}=\bar{t}$,

$$
\begin{equation*}
u_{s}^{1, m}=\frac{\partial g_{s}^{1}}{\partial \bar{u}_{1}^{1}}\left(\sum_{i=1}^{n} \bar{a}_{1 i}^{1, m} u_{i}^{1, m}\right)+F_{s}^{1, m} \quad \text { on } \quad \bar{x}=0, \quad(s=2, \ldots, n) \tag{3.33}
\end{equation*}
$$

As $k=2, \ldots, n-2$, we have

$$
\begin{array}{r}
u_{r}^{k, m}=\sum_{q=k+1}^{n} \frac{\partial g_{r}^{k}}{\partial \bar{u}_{q}^{k}}\left(\sum_{i=1}^{n} \bar{a}_{q i}^{k, m} u_{i}^{k, m}\right)+\sum_{p=1}^{k+1} \frac{\partial g_{r}^{k}}{\partial \bar{u}_{p}^{k+1}}\left(\sum_{i=1}^{n} \bar{a}_{p i}^{k+1, m} u_{i}^{k+1, m}\right)+F_{r}^{k, m}  \tag{3.34}\\
\text { on } \bar{x}=\bar{t}, \quad(r=1, \ldots, k)
\end{array}
$$

$$
\begin{array}{r}
u_{s}^{k, m}=\sum_{p=1}^{k} \frac{\partial g_{s}^{k}}{\partial \bar{u}_{p}^{k}}\left(\sum_{i=1}^{n} \bar{a}_{p i}^{k, m} u_{i}^{k, m}\right)+\sum_{q=k}^{n} \frac{\partial g_{s}^{k}}{\partial \bar{u}_{q}^{k-1}}\left(\sum_{i=1}^{n} \bar{a}_{q i}^{k-1, m} u_{i}^{k-1, m}\right)+F_{s}^{k, m}  \tag{3.35}\\
\text { on } \quad \bar{x}=0,(s=k+1, \ldots, n)
\end{array}
$$

for odd $k$.
For even $k$, we only need to take values of (3.34) on $\bar{x}=0$, and to take values of (3.35) on $\bar{x}=\bar{t}$. As $n$ is even, we have

$$
\begin{gather*}
u_{r}^{n-1, m}=\frac{\partial g_{r}^{n-1}}{\partial \bar{u}_{n}^{n-1}}\left(\sum_{i=1}^{n} \bar{a}_{n i}^{n-1, m} u_{i}^{n-1, m}\right)+F_{r}^{n-1, m} \\
\text { on } \quad \bar{x}=\bar{t},(r=1, \ldots, n-1),  \tag{3.36}\\
u_{n}^{n-1, m}=\sum_{p=1}^{n-1} \frac{\partial g_{n}^{n-1}}{\partial \bar{u}_{p}^{n-1}}\left(\sum_{i=1}^{n} \bar{a}_{p i}^{n-1, m} u_{i}^{n-1, m}\right) \\
\\
+\sum_{q=n-1}^{n} \frac{\partial g_{n}^{n-1}}{\partial \bar{u}_{q}^{n-2}}\left(\sum_{i=1}^{n} \bar{a}_{q i}^{n-2, m} u_{i}^{n-2, m}\right)  \tag{3.37}\\
+F_{n}^{n-1, m} \quad \text { on } \quad \bar{x}=0 .
\end{gather*}
$$

Likewise, for odd $n$, we can obtain the result for odd $n$ by taking values of (3.36) on $\bar{x}=0$, and taking values of (3.37) on $\bar{x}=\bar{t}$.

Thus, we obtain an $n(n-1)(m+1)$ system (3.23) of the functional form on $\bar{D}(\delta)$ in terms of $u_{i}^{k, j}(k=1, \ldots, n-1 ; i=1, \ldots, n ; j=0, \ldots, m)$ and boundary conditions (3.30)-(3.37) and (3.20). Using Theorem 6.1 of Chapter 2 in [18] yields the following lemma.

Lemma 3.2. The generalized Riemann problem (3.1), (3.2) admits a unique piecewise $C^{m+1}$ solution if and only if the functional boundary value problem, (3.23), (3.20), (3.30)-(3.37), admits a unique $C^{1}$ solution on $\bar{D}(\delta)$.

In what follows we shall prove Theorem 2.1, that is to prove if

$$
\left\|\Theta_{m}\right\|=\left\|\bar{\Theta}_{m}\right\|<1
$$

then the problem (3.23), (3.20), (3.30)-(3.37) admits a unique $C^{1}$ solution on the angular domain $\bar{D}(\delta)$. To this end, we need to use Theorem 6.1 of Chapter 2 in [18](see the Appendix)

Proof of Theorem 2.1. According to Lemma 3.2, we know that finding the piecewise $C^{m+1}$ solution of the generalized Riemann problem (3.1), (3.2) is equivalent to finding $C^{1}$ solution of the functional boundary value problem, (3.23), (3.20), (3.30)-(3.37) on the angular domain $\bar{D}(\delta)$. We first check the conditions (i)-(xi) of Theorem 6.1 in Chapter 2 [18].

Here $u=u_{i}^{k, j}(k=1, \ldots, n-1 ; j=0, \ldots, m ; i=1, \ldots, n), \alpha=0, \beta=$ $0, N=n(n-1)(m+1) ; \zeta_{l i}, \quad \lambda_{l}, \mu_{l}(l, i=1, \ldots, N)$ are given by (3.23), $G_{l}(l=1, \ldots, N)$ are given by $(3.30)-(3.37), u^{k, 0}(0,0)(k=1, \ldots, n-1)$ are defined by the solution of the Riemann problem (3.3), (3.4), and $u^{k, j}(0,0)(k=$ $1, \ldots, n-1 ; j=1, \ldots, m)$ are obtained by means of Lemma 3.1. Moreover, in this case, from (2.15) and (2.16) it easily follows

$$
\Theta_{m}=\left.\frac{\partial\left(g_{1}^{1}, \ldots, g_{n}^{1}, \ldots, g_{1}^{n-1}, \ldots, g_{n}^{n-1}\right)}{\partial\left(\bar{u}_{1}^{1}, \ldots, \bar{u}_{n}^{1}, \ldots, \bar{u}_{1}^{n-1}, \ldots, \bar{u}_{n}^{n-1}\right)}\right|_{\bar{t}=\bar{x}=0} \cdot \tau^{m}
$$

where $\tau$ is defined by (2.17). Noting (3.22) and (3.24), we have

$$
\zeta_{l i}^{0}=\delta_{l i}, \quad(l, i=1, \ldots, N)
$$

We first verify conditions (i)-(vii) for system (3.23).
By the expressions of $\zeta_{l i}, \lambda_{l}$ and $\mu_{l}(l, i=1, \ldots, N)$, we know they are $C^{1}$ functions, hence (i) is trivial.

For (ii), since $v \in \sum\left(\delta \mid \Omega_{1}\right)$, obviously we have

$$
\begin{equation*}
\|v(\bar{t}, \bar{x})-v(0,0)\| \leq \varepsilon\left(\delta, \Omega_{1}\right) \tag{3.38}
\end{equation*}
$$

Applying (3.7), (3.8), (3.19), (3.20) and the mean value theorem it follows that in $A_{k} O A_{k+1}$ (taking odd $k$ for an example, for even $k$ the result is similar).

$$
\begin{align*}
x_{k}(\bar{t}, \bar{x})= & \frac{\bar{x}}{\bar{t}} x_{k+1}(\bar{t})+\left(1-\frac{\bar{x}}{\bar{t}}\right) x_{k}(\bar{t}) \\
= & \frac{\bar{x}}{\bar{t}}\left(\bar{t} F_{k+1}\left(\widetilde{t}, x_{k+1}(\widetilde{t}), v\left(\widetilde{t}, x_{k+1}(\widetilde{t})\right)\right)\right) \\
& \left.\left.\left.+\left(1-\frac{\bar{x}}{\bar{t}}\right)\left(\bar{t} F_{k} \widetilde{\widetilde{t}}, x_{k} \widetilde{\widetilde{t}}\right), v \widetilde{\widetilde{t}}, x_{k} \widetilde{\widetilde{t}}\right)\right)\right), \quad(0 \leq \widetilde{t}, \widetilde{\widetilde{t}} \leq \bar{t}) . \tag{3.39}
\end{align*}
$$

Since $F_{k}$ and $F_{k+1}$ are at least $C^{1}$ functions, in view of (3.38), we conclude

$$
\left\{\begin{array}{l}
\left|F_{k+1}\left(\widetilde{t}, x_{k+1}(\widetilde{t}), v\left(\widetilde{t}, x_{k+1}(\widetilde{t})\right)\right)\right| \leq\left|F_{k+1}(0,0, v(0,0))\right|+\varepsilon\left(\delta, \Omega_{1}\right)  \tag{3.40}\\
\left.\left.\left.\mid F_{k} \widetilde{\widetilde{t}}, x_{k} \widetilde{\widetilde{t}}\right), v \widetilde{\widetilde{t}}, x_{k}(\widetilde{t})\right)\right)\left|\leq\left|F_{k}(0,0, v(0,0))\right|+\varepsilon\left(\delta, \Omega_{1}\right)\right.
\end{array}\right.
$$

Substituting (3.40) into (3.39), one yields

$$
\begin{equation*}
\left|x_{k}(\bar{t}, \bar{x})\right| \leq \varepsilon\left(\delta, \Omega_{1}\right) . \tag{3.41}
\end{equation*}
$$

As a result, since $\mu \in C^{1}$,

$$
\begin{aligned}
& \left|\mu\left(\bar{t}, x_{k}(\bar{t}, \bar{x}), v(\bar{t}, \bar{x})\right)\right| \\
\leq & |\mu(0,0, v(0,0))|+\left\lvert\, \bar{t} \frac{\partial \mu}{\partial t}\left(\eta_{1} \bar{t}, 0, v(0,0)\right)+x_{k}(\bar{t}, \bar{x}) \frac{\partial \mu}{\partial x}\left(0, \eta_{2} x_{k}(\bar{t}, \bar{x}), v(0,0)\right)\right. \\
& \left.+\sum_{i=1}^{N}\left(v_{i}(\bar{t}, \bar{x})-v_{i}(0,0)\right) \frac{\partial \mu}{\partial v_{i}}\left(0,0, \eta_{3} v(\bar{t}, \bar{x})+\left(1-\eta_{3}\right) v(0,0)\right) \right\rvert\, \\
\leq & |\mu(0,0, v(0,0))|+\varepsilon\left(\delta, \Omega_{1}\right)
\end{aligned}
$$

where $0 \leq \eta_{1}, \eta_{2}, \eta_{3} \leq 1$. Therefore the verification of (ii) is complete.
For (iii), because the functions in $\Gamma[v]$ are continuous, and the continuous function in a closed interval can assume the maximum, hence there exists a constant $K_{1}$ depending only on $\Omega_{1}$ such that

$$
\|\Gamma[v]\| \leq K_{1} .
$$

For (iv) and (v), by means of checking (ii) it follows that

$$
\begin{align*}
& \omega(\eta \mid v) \leq \widetilde{\omega}_{0}(\eta), \\
& \omega(\eta \mid x) \leq \widetilde{\omega}_{0}(\eta), \tag{3.42}
\end{align*}
$$

where $\widetilde{\omega}_{0}(\eta)$ is a function depending only on $\Omega_{1}$, and $\widetilde{\omega}_{0}(\eta) \rightarrow 0$ as $\eta \rightarrow 0$. By the expressions of $\zeta_{l i}, \lambda_{l}, \mu_{l}(l, i=1, \ldots, N)$ we know (iv) and (v) hold.

For (vi), $k=1, \ldots, N,(3.19),(3.20)$ imply that there exist constants $K_{2}, K_{3}$ such that (in (3.43), take even $k$ for an example, for odd $k$ the result is similar)

$$
\begin{align*}
& \left|x_{k}\left(\bar{t} \mid v^{\prime}\right)-x_{k}\left(\bar{t} \mid v^{\prime \prime}\right)\right|  \tag{3.43}\\
= & \left|\int_{0}^{\bar{t}}\left[F_{k}\left(\bar{t}, x_{k}\left(\bar{t} \mid v^{\prime}\right), v^{\prime}(\bar{t}, \bar{t})\right)-F_{k}\left(\bar{t}, x_{k}\left(\bar{t} \mid v^{\prime \prime}\right), v^{\prime \prime}(\bar{t}, \bar{t})\right)\right] d \bar{t}\right| \\
\leq & K_{2}\left\|v^{\prime}-v^{\prime \prime}\right\|+K_{3} \int_{0}^{\bar{t}}\left|x_{k}\left(\bar{t} \mid v^{\prime}\right)-x_{k}\left(\bar{t} \mid v^{\prime \prime}\right)\right| d \bar{t} .
\end{align*}
$$

By (3.43) and Gronwall's inequality, it yields that there exists a constant $K_{4}$ depending only on $\delta$ and $\Omega_{1}$ such that

$$
\begin{equation*}
\left|x_{k}\left(\bar{t} \mid v^{\prime}\right)-x_{k}\left(\bar{t} \mid v^{\prime \prime}\right)\right| \leq K_{4}\left\|v^{\prime}-v^{\prime \prime}\right\| . \tag{3.44}
\end{equation*}
$$

Hence for $k=1, \ldots, n-1$, there exists a constant $K_{5}$ depending only on $\delta$ and $\Omega_{1}$ such that

$$
\begin{align*}
& \left|x_{k}\left(\bar{t}, \bar{x} \mid v^{\prime}\right)-x_{k}\left(\bar{t}, \bar{x} \mid v^{\prime \prime}\right)\right|  \tag{3.45}\\
= & \mid \overline{\bar{x}} \int_{0}^{\bar{t}}\left(F_{k+1}\left(\bar{t}, x_{k+1}\left(\bar{t} \mid v^{\prime}\right), v^{\prime}\right)-F_{k+1}\left(\bar{t}, x_{k+1}\left(\bar{t} \mid v^{\prime \prime}\right), v^{\prime \prime}\right)\right) d \bar{t} \\
& \left.+\left(1-\frac{\bar{x}}{\bar{t}}\right) \int_{0}^{\bar{t}}\left(F_{k}\left(\bar{t}, x_{k}\left(\bar{t} \mid v^{\prime}\right), v^{\prime}\right)-F_{k}\left(\bar{t}, x_{k}\left(\bar{t} \mid v^{\prime \prime}\right), v^{\prime \prime}\right)\right) d \bar{t} \right\rvert\, \\
\leq & K_{5}\left\|v^{\prime}-v^{\prime \prime}\right\|
\end{align*}
$$

for odd $k$. Similarly for even $k$ (3.45) holds. Therefore by the expressions of $\zeta_{l i}, \lambda_{l}, \mu_{l}(l, i=1, \ldots, N)$ we can get (vi).

By the expressions (3.15) of $\lambda$ we easily obtain (vii) also holds.
So far we have proved the system (3.23) satisfies the conditions (i)-(vii). In the sequel, we shall show the boundary conditions (3.30)-(3.37) satisfy conditions (viii)-(xi). Taking (3.34) for an example, others can be tackled similarly.
(viii) is still trivial.

For (ix), let

$$
G_{r}^{k, m}(\bar{t})=u_{r}^{k, m}(\bar{t}, \bar{t} \mid v(\bar{t}, \bar{x})),
$$

differentiating (3.61) with respect to $\bar{t}$ yields

$$
\begin{aligned}
\left(G_{r}^{k, m}(\bar{t})\right)^{\prime}= & \left\{\sum_{q=k+1}^{n} \frac{\partial g_{r}^{k}}{\partial \bar{u}_{q}^{k}}\left(\sum_{i=1}^{n} \bar{a}_{q i}^{k, m}\left(\frac{\partial v_{i}^{k, m}}{\partial \bar{t}}+\frac{\partial v_{i}^{k, m}}{\partial \bar{x}}\right)\right)\right. \\
& +\sum_{p=1}^{k+1} \frac{\partial g_{r}^{k}}{\partial \bar{u}_{p}^{k+1}}\left(\sum_{i=1}^{n} \bar{a}_{p i}^{k+1, m}\left(\frac{\partial v_{i}^{k+1, m}}{\partial \bar{t}}+\frac{\partial v_{i}^{k+1, m}}{\partial \bar{x}}\right)\right) \\
& +\sum_{j=0}^{m-1} \widetilde{F}_{r}^{k, m, j}\left(\frac{\partial v^{k, j}}{\partial \bar{t}}+\frac{\partial v^{k, j}}{\partial \bar{x}}\right) \\
& +\sum_{j=0}^{m-1} \widetilde{\widetilde{F}}_{r}^{k, m, j}\left(\frac{\partial v^{k+1, j}}{\partial \bar{t}}+\frac{\partial v^{k+1, j}}{\partial \bar{x}}\right) \\
& \left.+\frac{\partial F_{r}^{k, m}}{\partial t}+\frac{\partial F_{r}^{k, m}}{\partial x} F_{k+1}\right\}\left.\right|_{\bar{x}=\bar{t}}
\end{aligned}
$$

where $\frac{\partial g_{r}^{k}}{\partial \bar{u}_{q}^{k}}, \frac{\partial g_{r}^{k}}{\partial \bar{u}_{p}^{k+1}}$ are $C^{m}$ functions, $\bar{a}_{q i}^{k, m}, \bar{a}_{p i}^{k+1, m}$ given by (3.27) are $C^{1}$ functions, $\widetilde{F}_{r}^{k, m, j}, \widetilde{\widetilde{F}}_{r}^{k, m, j}, \frac{\partial F_{r}^{k, m}}{\partial t}, \frac{\partial F_{r}^{k, m}}{\partial x}$ are continuous functions of $\left(\bar{t}, x_{k+1}(\bar{t}), v^{p, q}\right)$ ( $p=1, \ldots, n-1 ; q=0, \ldots, m-1$ ), and $F_{k+1}$ given by (3.19) are $C^{m+1}$ functions.

Since $v(\bar{t}, \bar{x}) \in \Sigma\left(\delta \mid \Omega_{1}\right)$, obviously it holds

$$
\|v(\bar{t}, \bar{x})-v(0,0)\| \leq \Omega_{1} \delta
$$

Noticing (3.41), we obtain

$$
\begin{gathered}
\left\{\begin{array}{l}
\left.\left|\frac{\partial g_{r}^{k}}{\partial \bar{u}_{q}^{k}}-\frac{\partial g_{r}^{k}}{\partial \bar{u}_{q}^{k}}\right|_{\bar{t}=\bar{x}=0} \right\rvert\, \leq \varepsilon\left(\delta, \Omega_{1}\right), \quad(q=k+1, \ldots, n), \\
\left.\left|\frac{\partial g_{r}^{k}}{\partial \bar{u}_{p}^{k+1}}-\frac{\partial g_{r}^{k}}{\partial \bar{u}_{p}^{k+1}}\right|_{\bar{t}=\bar{x}=0} \right\rvert\, \leq \varepsilon\left(\delta, \Omega_{1}\right), \quad(p=1, \ldots, k+1),
\end{array}\right. \\
\left\{\begin{array}{l}
\left|\bar{a}_{q i}^{k, m}-\bar{a}_{q i}^{k, m}\right|_{\bar{t}=\bar{x}=0} \mid \leq \varepsilon\left(\delta, \Omega_{1}\right), \quad(q=k+1, \ldots, n ; i=1, \ldots, n), \\
\left|\bar{a}_{p i}^{k+1, m}-\bar{a}_{p i}^{k+1, m}\right|_{\bar{t}=\bar{x}=0} \mid \leq \varepsilon\left(\delta, \Omega_{1}\right), \quad(p=1, \ldots, k+1 ; i=1, \ldots, n), \\
\left\{\begin{array}{l}
\left|\widetilde{F}_{r}^{k, m, j}\right| \leq\left|\widetilde{F}_{r}^{k, m, j}\right| \bar{t}=\bar{x}=0 \\
\left|\widetilde{\widetilde{F}}_{r}^{k, m, j}\right| \leq \varepsilon\left(\delta, \Omega_{1}\right), \\
\end{array}\right. \\
\left|\frac{\partial \widetilde{\widetilde{F}}_{r}^{k, m, j}| |_{\bar{t}=\bar{x}=0}^{k, m} \mid+\varepsilon\left(\delta, \Omega_{1}\right),}{\partial t}+\frac{\partial F_{r}^{k, m}}{\partial x} F_{k+1}\right| \leq\left|\left(\frac{\partial F_{r}^{k, m}}{\partial t}+\frac{\partial F_{r}^{k, m}}{\partial x} F_{k+1}\right)_{\bar{t}=\bar{x}=0}\right|+\varepsilon\left(\delta, \Omega_{1}\right) .
\end{array}\right.
\end{gathered}
$$

Thus, letting

$$
\begin{gathered}
\bar{M}_{r}=\max _{j=0, \ldots, m-1}\left(\left|\widetilde{F}_{r}^{k, m, j}\right|_{\bar{t}=\bar{x}=0}\left|,\left|\widetilde{\widetilde{F}}_{r}^{k, m, j}\right|_{\bar{t}=\bar{x}=0}\right|\right), \\
\left.R_{2}=\left|\left(\frac{\partial F_{r}^{k, m}}{\partial t}+\frac{\partial F_{r}^{k, m}}{\partial x} F_{k+1}\right)\right|_{\bar{t}=\bar{x}=0} \right\rvert\,
\end{gathered}
$$

and noting (3.29), we obtain

$$
\begin{aligned}
& \left\|G_{r}^{k, m}(\bar{t})^{\prime}\right\| \\
\leq & \sum_{q=k+1}^{n} \sum_{i=1}^{n}\left(\left.\frac{\partial g_{r}^{k}}{\partial \bar{u}_{q}^{k}}\right|_{\bar{t}=\bar{x}=0} \cdot\left(\tau_{q}^{k}\right)^{m} \delta_{q i}+\varepsilon\left(\delta, \Omega_{1}\right)\right)\left\|\frac{\partial v_{i}^{k, m}}{\partial \bar{t}}+\frac{\partial v_{i}^{k, m}}{\partial \bar{x}}\right\| \\
& +\sum_{p=1}^{k+1} \sum_{i=1}^{n}\left(\left.\frac{\partial g_{r}^{k}}{\partial \bar{u}_{p}^{k+1}}\right|_{\bar{t}=\bar{x}=0} \cdot\left(\tau_{p}^{k+1}\right)^{m} \delta_{p i}+\varepsilon\left(\delta, \Omega_{1}\right)\right)\left\|\frac{\partial v_{i}^{k+1, m}}{\partial \bar{t}}+\frac{\partial v_{i}^{k+1, m}}{\partial \bar{x}}\right\| \\
& +\sum_{j=0}^{m-1}\left(\bar{M}_{r}+\varepsilon\left(\delta, \Omega_{1}\right)\right)\left(\left\|\frac{\partial v^{k, j}}{\partial \bar{t}}+\frac{\partial v^{k, j}}{\partial \bar{x}}\right\|+\left\|\frac{\partial v^{k+1, j}}{\partial \bar{t}}+\frac{\partial v^{k+1, j}}{\partial \bar{x}}\right\|\right) \\
& +R_{2}+\varepsilon\left(\delta, \Omega_{1}\right) .
\end{aligned}
$$

As for condition (x), when $v \in \Sigma\left(\delta, \Omega_{1}\right)$, it holds

$$
\Omega(\eta \mid v) \leq \Omega_{1} \eta
$$

For continuous functions $f, g$, we have

$$
\begin{aligned}
& \Omega(\eta \mid f \cdot g) \leq\|f\| \Omega(\eta \mid g)+\|g\| \Omega(\eta \mid f) \\
& \Omega(\eta \mid f(g)) \leq \omega(\Omega(\eta \mid g) \mid f)
\end{aligned}
$$

Recalling (3.42), we can directly obtain (x) from (3.46).
As for condition (xi), since $F_{r}^{k, m}$ in (3.34) are $C^{1}$ functions, in view of (3.44), we can get (xi).

Since (3.30) and (3.31) are of integral form, it is easy to verify

$$
\left\{\begin{array}{l}
\left\|G_{r}^{k, j}(\bar{t})^{\prime}\right\| \leq \varepsilon\left(\delta, \Omega_{1}\right)\left\|\frac{\partial v}{\partial \bar{t}}+\frac{\partial v}{\partial \bar{x}}\right\|+R_{2}+\varepsilon\left(\delta, \Omega_{1}\right), \\
\omega\left(\eta \mid G_{r}^{k, j}(\bar{t})^{\prime}\right) \leq \varepsilon\left(\delta, \Omega_{1}\right) \Omega\left(\eta \left\lvert\,\left(\frac{\partial v}{\partial \bar{t}}+\frac{\partial v}{\partial \bar{x}}\right)\right.\right)+\omega_{2}(\eta), \quad(j=0, \ldots, m-1) \\
\left\|G_{r}^{k, j}\left(\bar{t} \mid v^{\prime}\right)-G_{r}^{k, j}\left(\bar{t} \mid v^{\prime \prime}\right)\right\| \leq \varepsilon\left(\delta, \Omega_{1}\right)\left\|v^{\prime}-v^{\prime \prime}\right\| .
\end{array}\right.
$$

Thus, we obtain the characterizing matrix of the functional boundary value problem as follows

$$
A=\left(\begin{array}{cc}
0 & \\
* \ldots \ldots * & \bar{\Theta}_{m}
\end{array}\right)
$$

It is easy to see

$$
\|A\|_{\min }=\left\|\bar{\Theta}_{m}\right\|_{\min }
$$

this completes the proof of Theorem 2.1.
To prove Theorem 2.2, we need the following regularity lemma.
Lemma 3.3. Assume that the functional boundary value problem (3.9)-(3.20) admits a unique $C^{m+1}$ solution $\bar{u}(\bar{t}, \bar{x})$ on $\bar{D}\left(\delta_{0}\right)$, and $\left\|\bar{\Theta}_{m}\right\|<1$. If the coefficients of (3.6) and initial conditions (3.2) are $C^{M+m+1}$ functions ( $M \geq 0$ ), and $(M+m+1)$-th order derivatives of $\mu(t, x, u)$ are Lipschitz continuous with respect to $u$, then there exists a positive constant $\delta^{*} \leq \delta_{0}$ independent of $M$, such that $\bar{u}$ is a $C^{M+m+1}$ solution of (3.9)-(3.20) on $\bar{D}\left(\delta^{*}\right)$.

In [21], the authors showed the following regularity lemma of typical boundary value problem.

Lemma 3.4. Suppose that the typical boundary value problem

$$
\left\{\begin{array}{l}
\sum_{i=1}^{N} \zeta_{l i}(t, x, u)\left(\frac{\partial u_{i}}{\partial t}+\lambda_{l}(t, x, u) \frac{\partial u_{i}}{\partial x}\right)=\mu_{l}(t, x, u), \quad l=1, \ldots, N  \tag{3.47}\\
x=t: u_{r}=G_{r}\left(t, u_{k+1}, \ldots, u_{N}\right), \quad r=1, \ldots, K \\
x=0: u_{s}=G_{s}\left(t, u_{1}, \ldots, u_{k}\right), \quad s=K+1, \ldots, N
\end{array}\right.
$$

on the angular domain $\bar{D}\left(\delta_{0}\right)$ admits a unique $C^{1}$ solution, whose corresponding $\left\|\widetilde{\Theta}_{1}\right\|<1$. If $\zeta, \lambda, \mu, G_{r}, G_{s}$ are $C^{M+1}(M \geq 0)$ functions, and $(M+1)$-th order derivatives of $\mu$ are Lipschitz continuous with respect to $u$, then there exists a positive constant $\delta^{*} \leq \delta_{0}$ independent of $M$, such that $u$ is a $C^{M+1}$ solution of (3.47) on $\bar{D}\left(\delta^{*}\right)$.

Proof of Lemma 3.3. If $\bar{u}=\bar{u}(\bar{t}, \bar{x})$ is a $C^{m+1}$ solution of (3.9)-(3.20) on $\bar{D}\left(\delta_{0}\right)$, then by Lemma 3.2, $u_{i}^{k, j}(k=1, \ldots, n-1 ; j=0, \ldots, m ; i=1, \ldots, n)$ is a $C^{1}$ solution of the functional boundary value problem (3.23), (3.20), (3.30)(3.37). Regarding $x_{k}(\bar{t})$ obtained $(k=1, \ldots, n)$ as known functions, then we easily know $u_{i}^{k, j}$ is a $C^{1}$ solution of typical boundary value problem (3.47), $\zeta, \lambda, \mu, G_{r}, G_{s}$ are at least $C^{2}$ functions, second order derivatives of $\mu$ with respect to $u$ are Lipschitz continuous, and $\widetilde{\bar{\Theta}}_{1}=\bar{\Theta}_{m}$. By Lemma 3.4, there exists a positive constant $\delta^{*} \leq \delta_{0}$ such that $u_{i}^{k, j}$ is a $C^{2}$ solution of (3.47) on $\bar{D}\left(\delta^{*}\right)$. Then it follows from (3.20) that $x_{k}(\bar{t})(k=1, \ldots, n)$ are at least $C^{3}$ functions, so $\zeta, \lambda, \mu, G_{r}, G_{s}$ are at least $C^{3}$ functions. Repeated application of Lemma 3.4 implies that $u_{i}^{k, j}$ is a $C^{m+1}$ of (3.47) on $\delta^{*}$. In view of Lemma 3.2 , we obtain $u$ is a $C^{M+m+1}$ solution of (3.9)-(3.20) on $\bar{D}\left(\delta^{*}\right)$. The proof of Lemma 3.3 is complete.

Proof of Theorem 2.2. Since any element of $\bar{\Theta}_{j}$ tends to 0 as $j \rightarrow+\infty$, there exist a positive integer $m \geq m_{0}$ and a positive constant $\delta_{0}$ such that the functional boundary value problem (3.9)-(3.20) admits a unique $C^{m+1}$ solution $\bar{u}=\bar{u}(\bar{t}, \bar{x})$ on $\bar{D}\left(\delta_{0}\right)$. Owing to Lemma 3.3, we obtain that $\bar{u}$ is a $C^{\infty}$ solution of (3.9)-(3.20) on $\bar{D}\left(\delta^{*}\right)$. By the equivalence of the generalized Riemann problem and the functional boundary value problem (3.9)-(3.20), one yields that $u=$ $u(t, x)$ is a $C^{\infty}$ solution of the generalized Riemann problem in a neighborhood of the origin.

## 4. Appendix

Let

$$
R(\delta)=\{(t, x) \mid 0 \leq t \leq \delta, \quad \beta t \leq x \leq \alpha t\}, \quad(\alpha>\beta)
$$

be an angular domain. Consider on this domain the following boundary value problem in functional form:

$$
\begin{align*}
& \sum_{j=1}^{n} \zeta_{l j}(t, x, \mid u)\left(\frac{\partial u_{j}}{\partial t}+\lambda_{l}(t, x \mid u) \frac{\partial u_{j}}{\partial x}\right)=\mu_{l}(t, x \mid u), \quad(l=1, \ldots, n)  \tag{4.1}\\
& \sum_{j=1}^{n} \zeta_{r j}^{0}=G_{r}(t, u) \quad \text { on } \quad x=\alpha t, \quad(r=1, \ldots, m)  \tag{4.2}\\
& \sum_{j=1}^{n} \zeta_{s j}^{0}=G_{s}(t, u) \quad \text { on } \quad x=\beta t, \quad(s=1, \ldots, n) \tag{4.3}
\end{align*}
$$

where the coefficients $\zeta_{l j}, \lambda_{l}, \mu_{l}$ and the boundary conditions $G_{l}(l, j=1, \ldots, n)$ are assumed to be functionals of the unknown function $u=u(t, x)$, and

$$
\zeta_{l j}^{0} \triangleq \zeta_{l j}(0,0 \mid 0)=\left.\zeta_{l j}(t, x \mid v)\right|_{v \equiv 0, t=x=0}
$$

Let

$$
\Sigma(\delta)=\left\{v(t, x) \mid v \in C_{1}[R(\delta)], v(0,0)=0\right\}
$$

and

$$
\Sigma\left(\delta \mid \Omega_{1}\right)=\left\{v(t, x) \mid v \in \Sigma(\delta),\|q\| \leq \Omega_{1}\right\}
$$

where

$$
\begin{array}{ll}
q=\left\{q_{i}\right\}: q_{l}=\frac{\partial v_{l}}{\partial t}+\beta \frac{v_{l}}{\partial x}, \quad q_{n+l}=\frac{\partial v_{l}}{\partial t}+\alpha \frac{v_{l}}{\partial x}, \quad(l=1, \ldots, n), \\
q^{*}=\left\{q_{i}^{*}\right\}: q_{l}^{*}=\sum_{j=1}^{n} \zeta_{l j}^{0} q_{j}, \quad q_{n+l}^{*}=\sum_{j=1}^{n} \zeta_{l j}^{0} q_{n+j}, \quad(l=1, \ldots, n) .
\end{array}
$$

For $v \in C^{1}[R(\delta)]$, define

$$
\left\{\begin{array}{l}
\tilde{\zeta}_{l j}=\zeta_{l j}(t, x \mid v(x, t)) \\
\tilde{\lambda}_{l}(t, x)=\lambda_{l}(t, x \mid v(t, x)), \quad(l, j=1, \ldots, n) \\
\tilde{\mu}_{l}=\mu_{l}(t, x \mid v(t, x))
\end{array}\right.
$$

and

$$
\begin{aligned}
\Gamma_{2}[v]=\{ & \left\{\tilde{\zeta}_{l j}, \frac{\partial \tilde{\zeta}_{l j}}{\partial t}, \frac{\partial \tilde{\zeta}_{l j}}{\partial x}, \tilde{\lambda}_{l}, \frac{\partial \tilde{\lambda}_{l}}{\partial x}, \tilde{\mu}_{l}, \frac{\partial \tilde{\mu}_{l}}{\partial x}, \frac{1}{\operatorname{det}\left|\tilde{\zeta}_{l j}\right|}, \frac{1}{\alpha-\tilde{\lambda}_{r}(t, \alpha t)}, \frac{1}{\tilde{\lambda}_{s}(t, \beta t)-\beta}\right\} \\
& (l, j=1, \ldots, n ; r=1, \ldots, m ; s=m+1, \ldots, n)
\end{aligned}
$$

Assume that the functional coefficients of system (4.1) satisfy the following conditions:
(i) For any $v \in C^{1}[R(\delta)]$, the values of the functions $\tilde{\zeta}_{l j}(t, x), \tilde{\lambda}_{l}(t, x)$, $\tilde{\mu}_{l}(t, x)$ $(l, j=1, \ldots, n)$ on any domain $R\left(\delta^{\prime}\right)\left(0 \leq \delta^{\prime} \leq \delta\right)$ depend only on the values of the function $v(t, x)$ on $R\left(\delta^{\prime}\right)$, and all functions in $\Gamma_{2}[v]$ are continuous on $R(\delta)$;
(ii) On $R(\delta)$, for any $v \in \Sigma\left(\delta \mid \Omega_{1}\right)$,

$$
\|\tilde{\mu}\| \leq R_{1}+\varepsilon\left(\delta, \Omega_{1}\right)
$$

where $R_{1}$ is independent of $\delta$ and $\Omega_{1}$, and for any fixed $\Omega_{1}$,

$$
\begin{equation*}
\varepsilon\left(\delta, \Omega_{1}\right) \rightarrow 0 \quad \text { as } \quad \delta \rightarrow 0 \tag{4.4}
\end{equation*}
$$

(iii) On $R(\delta)$, for any $v \in \Sigma\left(\delta \mid \Omega_{1}\right)$,

$$
\left\|\Gamma_{2}[v]\right\| \leq K_{1}
$$

where $K_{1}$ depends only on $\Omega_{1}$;
(iv) On $R(\delta)$, for any $v \in \Sigma\left(\delta \mid \Omega_{1}\right)$,

$$
\omega\left(\eta \mid \tilde{\lambda}_{l}\right)+\omega\left(\eta \mid \tilde{\mu}_{l}\right) \leq \omega_{0}(\eta)
$$

where $\omega(\eta \mid \tilde{\lambda}), \omega(\eta \mid \tilde{\mu})$ are defined by

$$
\omega(\eta \mid w)=\sup _{\substack{i=1, \ldots, n \\\left(t^{\prime}, x^{\prime}\right),\left(t^{\prime}, x^{\prime}\right) \in R(\delta) \\\left|t^{\prime}-t^{\prime \prime}\right| \leq \eta,\left|x^{\prime}-x^{\prime \prime}\right| \leq \eta}}\left|w_{i}\left(t^{\prime}, x^{\prime}\right)-w_{i}\left(t^{\prime \prime}, x^{\prime \prime}\right)\right|
$$

$w$ is an $n$ dimensional vector valued function, and $\omega_{0}(\eta)$ is a nonnegative function depending only on $\Omega_{1}$ and $\omega_{0}(\eta) \rightarrow 0$ as $\eta \rightarrow 0$;
(v) On $R(\delta)$, for any $v \in \Sigma\left(\delta \mid \Omega_{1}\right)$,

$$
\omega\left(\eta \mid \Gamma_{2}[v]\right) \leq K_{2} \omega(\eta \mid q)+\omega_{1}(\eta)
$$

where $\omega_{1}(\eta)$ has the same property as $\omega_{0}(\eta)$, and $K_{2}$ depends only on $\Omega_{1}$;
(vi) On $R(\delta)$, for any $v^{\prime}, v^{\prime \prime} \in \Sigma\left(\delta \mid \Omega_{1}\right)$,

$$
\begin{aligned}
& \left\|\zeta_{l j}\left(t, x \mid v^{\prime}\right)-\zeta_{l j}\left(t, x \mid v^{\prime \prime}\right)\right\| \leq K_{3}\left\|v^{\prime}-v^{\prime \prime}\right\|, \\
& \left\|\lambda_{l}\left(t, x \mid v^{\prime}\right)-\lambda_{l}\left(t, x \mid v^{\prime \prime}\right)\right\| \leq K_{3}\left\|v^{\prime}-v^{\prime \prime}\right\|, \\
& \left\|\mu_{l}\left(t, x \mid v^{\prime}\right)-\mu_{l}\left(t, x \mid v^{\prime \prime}\right)\right\| \leq K_{3}\left\|v^{\prime}-v^{\prime \prime}\right\|,
\end{aligned}
$$

where $K_{3}$ also depends only on $\Omega_{1}$;
(vii) Let

$$
\lambda_{l}^{0}=\lambda_{l}(0,0 \mid 0)=\left.\lambda(t, x \mid v)\right|_{t=0, x=0, v \equiv 0} \quad(l=1, \ldots, n)
$$

Then for $r=1, \ldots, m$,

$$
\lambda_{r}^{0}<\beta
$$

or for any $v \in \Sigma\left(\delta \mid \Omega_{1}\right)$,

$$
\left.\lambda_{r}(t, x \mid v)\right|_{x=\beta t} \leq \beta
$$

Similarly for $s=m+1, \ldots, n$,

$$
\lambda_{s}^{0}>\alpha
$$

or for any $v \in \Sigma\left(\delta \mid \Omega_{1}\right)$,

$$
\left.\lambda_{s}(t, x \mid v)\right|_{x=\alpha t} \leq \alpha
$$

For $v \in C^{1}[R(\delta)]$, define

$$
\begin{cases}\tilde{G}_{r}(t)=\left.G_{r}(t \mid v)\right|_{x=\alpha t}, & (r=1, \ldots, m) \\ \tilde{G}_{s}(t)=\left.G_{s}(t \mid v)\right|_{x=\beta t}, & (s=m+1, \ldots, n)\end{cases}
$$

We suppose that the functional boundary functions in (4.2), (4.3) satisfy the following conditions;
(viii) For any $v \in C^{1}[R(\delta)], \tilde{G}_{l}(t)(l=1, \ldots, n)$ are $C^{1}$ functions on the interval $0<t \leq \delta$. Moreover, the values of the functions $\tilde{G}_{l}(t)$ on $0 \leq t \leq$ $\delta^{\prime}\left(0 \leq \delta^{\prime} \leq \delta\right)$ depend only on the values of the functions $v(t, x)$ on $R\left(\delta^{\prime}\right)$. In particular, $\tilde{G}_{l}(0)(l=1, \ldots, n)$ depend only on $v(0,0)$;
(ix) On $0 \leq t \leq \delta$, for any $v \in \Sigma\left(\delta \mid \Omega_{1}\right)$,
$\left\|\tilde{G}_{l}^{\prime}(t)\right\| \leq \sum_{k=1}^{n}\left(\theta_{l k}+\varepsilon\left(\delta, \Omega_{1}\right)\right) \operatorname{Max}\left(\left\|q_{k}^{*}\right\|,\left\|q_{n+k}^{*}\right\|\right)+R_{2}+\varepsilon\left(\delta, \Omega_{1}\right),(l=1, \ldots, n)$, where $\theta_{l k}$ and $R_{2}$ are nonnegative constants independent of $\delta$ and $\Omega_{1}, \varepsilon\left(\delta, \Omega_{1}\right)$ satisfies (4.4);
(x) On $0 \leq t \leq \delta$, for any $v \in \Sigma\left(\delta \mid \Omega_{1}\right)$,
$\omega\left(\eta \mid \tilde{G}^{\prime}\right)_{l}(t) \leq \sum_{k=1}^{n}\left(\theta_{l k}+\varepsilon\left(\delta, \Omega_{1}\right)\right) \operatorname{Max}\left(\Omega\left(\eta \mid q_{k}^{*}\right), \Omega\left(\eta \mid q_{n+k}^{*}\right)\right)+\omega_{2}(\eta),(l=1, \ldots, n)$,
where $\Omega\left(\eta \mid q_{i}\right)$ denotes the modulus of the continuity of $q_{i}$ on $R(\delta)(i=1, \ldots, 2 n)$, and $\omega_{2}(\eta)$ is a nonnegative function depending only on $\Omega_{1}$ with $\omega_{2}(\eta) \rightarrow 0$ as $\eta \rightarrow 0$;
(xi) On $R(\delta)$, for any $v^{\prime}, v^{\prime \prime} \in \Sigma\left(\delta \mid \Omega_{1}\right)$,

$$
\left\|G_{l}\left(t, x \mid v^{\prime}\right)-G_{l}\left(t, x \mid v^{\prime \prime}\right)\right\| \leq \sum_{k=1}^{n}\left(\theta_{l k}+\varepsilon\left(\delta, \Omega_{1}\right)\right)\left\|v_{k}^{\prime *}-v_{k}^{\prime \prime *}\right\|, \quad(l=1, \ldots, n)
$$

where

$$
v_{k}^{\prime *}=\sum_{j=1}^{n} \zeta_{k j}^{0} v_{j}^{\prime}, \quad v_{k}^{\prime \prime *}=\sum_{j=1}^{n} \zeta_{k j}^{0} v_{j}^{\prime \prime}, \quad(k=1, \ldots, n)
$$

Under the preceding assumptions, problem (4.1)-(4.3) is called a typical boundary value problem in functional form and the matrix

$$
H=\left(\theta_{l k}\right)
$$

is called the characterizing matrix of this problem. Then the following theorem holds.

Theorem 4.1. If the minimal characterizing number of $H$ is less than 1, i.e.,

$$
\theta_{\min }=|H|_{\min }<1,
$$

then for sufficiently small $\delta>0$, the typical boundary value problem in functional form, (4.1)-(4.3), admits a unique solution $u=u(t, x)$ on $R(\delta)$.

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