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# ON THE STABILITY OF A FIXED POINT ALGEBRA $C^*(E)^\gamma$ OF A GAUGE ACTION ON A GRAPH $C^*$-ALGEBRA

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Abstract. The fixed point algebra $C^*(E)^\gamma$ of a gauge action $\gamma$ on a graph $C^*$-algebra $C^*(E)$ and its AF subalgebras $C^*(E)^\gamma_v$ associated to each vertex $v$ do play an important role for the study of dynamical properties of $C^*(E)$. In this paper, we consider the stability of $C^*(E)^\gamma$ (an AF algebra is either stable or equipped with a (nonzero bounded) trace). It is known that $C^*(E)^\gamma$ is stably isomorphic to a graph $C^*$-algebra $C^*(E_\mathbb{Z}\times E)$ which we observe being stable. We first give an explicit isomorphism from $C^*(E)^\gamma$ to a full hereditary $C^*$-subalgebra of $C^*(E_\mathbb{N}\times E)\,(\subset C^*(E_\mathbb{Z}\times E))$ and then show that $C^*(E_\mathbb{N}\times E)$ is stable whenever $C^*(E)^\gamma$ is so. Thus $C^*(E)^\gamma$ cannot be stable if $C^*(E_\mathbb{N}\times E)$ admits a trace. It is shown that this is the case if the vertex matrix of $E$ has an eigenvector with an eigenvalue $\lambda > 1$. The AF algebras $C^*(E)^\gamma_v$ are shown to be nonstable whenever $E$ is irreducible. Several examples are discussed.

## 1. Introduction

Let $E$ be a row finite directed graph and $C^*(E)$ be the graph $C^*$-algebra of $E$ generated by a universal Cuntz-Krieger $E$ family $\{p_v, s_e\}$ (for example, see [1, 3, 18, 19, 22]). Then by the universal property, the gauge action $\gamma$ of $\mathbb{T}$, $\gamma_z(p_v) = p_v$, $\gamma_z(s_e) = zs_e$, is well defined and the fixed point algebra $C^*(E)^\gamma$ turns out to be an AF algebra. In fact, it is known in [17] using results of [25] and [19] on groupoid $C^*$-algebras that $C^*(E)^\gamma$ is strong Morita equivalent (hence stably isomorphic by [7]) to the graph $C^*$-algebra $C^*(E_\mathbb{Z}\times E)$ of the Cartesian product graph $E_\mathbb{Z}\times E$ ($E_\mathbb{Z}\times E$ is the graph $Z\times E$ in [17]). Since $E_\mathbb{Z}\times E$ has no loops, its graph $C^*$-algebra $C^*(E_\mathbb{Z}\times E)$ is an AF algebra ([18]). In this paper we are concerned with the question whether $C^*(E)^\gamma$ is in fact isomorphic to $C^*(E_\mathbb{Z}\times E)$. For this, we observe that $C^*(E_\mathbb{Z}\times E)$ is always stable, that is, $C^*(E_\mathbb{Z}\times E)\cong C^*(E_\mathbb{Z}\times E)\otimes\mathcal{K}$, where $\mathcal{K}$ is the $C^*$-algebra of compact operators on a separable infinite dimensional Hilbert space. Thus

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the question is equivalent to asking if $C^*(E)^\gamma$ is stable. But we will see that $C^*(E)^\gamma$ may not be stable (then it should admit a nonzero bounded trace since every AF algebra is either stable or equipped with such a nonzero bounded trace by [4, 24]).

The fixed point algebra $C^*(E)^\gamma$ and its AF subalgebras $C^*(E)^\gamma_v$ associated to each vertex $v$ of $E$ do play an important role for the study of the dynamical properties of $C^*(E)$. For example, if $E$ is locally finite, $C^*(E)^\gamma$ contains a $C^*$-subalgebra isomorphic to the commutative $C^*$-algebra $C_0(X_E)$ of continuous functions (vanishing at infinity) on the locally compact shift space $X_E$ of one-sided infinite paths, and it is shown in [13] that if $X_E$ and $X_F$ are topologically conjugate, the graph $C^*$-algebras $C^*(E)$ and $C^*(F)$ are isomorphic. Moreover, for $E$ irreducible, the topological entropy $ht(\Phi_E)$ (in the sense of [8, 28]) of the canonical completely positive map $\Phi_E$ on $C^*(E)$ is equal to that of the restriction $\Phi_E|_{C^*(E)^\gamma}$ ([15]). $C^*(E)^\gamma_v$ is a $\Phi_E$-invariant subalgebra of $C^*(E)^\gamma$ such that the topological entropy $ht(\Phi_E|_{C^*(E)^\gamma_v})$ is equal to the loop entropy of the graph $E$ if $E$ is a locally finite irreducible infinite graph [13]. The restriction of $\Phi_E$ onto the commutative subalgebra isomorphic to $C_0(X_E)$ corresponds to the $*$-homomorphism on $C_0(X_E)$ induced by the continuous shift map on $X_E$.

It is known in [7] that every full hereditary $C^*$-subalgebra $B$ of a $C^*$-algebra $A$ is stably isomorphic to $A$. We will define an isomorphism from $C^*(E)^\gamma$ onto a full hereditary $C^*$-subalgebra $A_\gamma$ of a graph $C^*$-algebra $C^*(E_\mathbb{N} \times E)$ which itself can be viewed as a full hereditary $C^*$-subalgebra of $C^*(E_\mathbb{Z} \times E)$ (thus $C^*(E)^\gamma$ is stably isomorphic to $C^*(E_\mathbb{Z} \times E)$ as proved in [17]). The isomorphism is obtained by using the fact that $C^*(E)^\gamma$ can be identified with a full hereditary $C^*$-subalgebra of the crossed product $C^*(E) \times_\gamma \mathbb{T}$ because $\mathbb{T}$ is compact ([16, 26]) and the concrete isomorphism between $C^*(E) \times_\gamma \mathbb{T}$ and $C^*(E_\mathbb{Z} \times E)$ constructed in [15]. (It was already known in [17] that these two algebras $C^*(E) \times_\gamma \mathbb{T}$ and $C^*(E_\mathbb{Z} \times E)$ are isomorphic, but with no explicit isomorphism.) The ideal structure of $C^*(E)^\gamma$ has been studied in [20].

We show in Theorem 4.2 that if $C^*(E)^\gamma$ is stable, so is $C^*(E_\mathbb{N} \times E)$, which implies that the $C^*$-algebras $A_\gamma (\cong C^*(E)^\gamma) \subset C^*(E_\mathbb{N} \times E) \subset C^*(E_\mathbb{Z} \times E)$ are all isomorphic if and only if $C^*(E)^\gamma$ is stable. (In particular, $C^*(E)^\gamma$ can be realized as a graph $C^*$-algebra.) By an example we also show that the converse of the theorem may not be true. Theorem 4.2 is useful especially when we want to prove nonstability of $C^*(E)^\gamma$. Of course, $C^*(E)^\gamma$ is possibly stable. Actually a locally finite irreducible (infinite) graph $E$ is given for which $C^*(E)^\gamma$ is stable (we prove that $C^*(E_\mathbb{N} \times E)$ cannot admit a nonzero bounded trace). In Theorem 5.1, we give a condition in terms of the vertex matrix of $E$ under which $C^*(E_\mathbb{N} \times E)$ admits a bounded trace, hence $C^*(E)^\gamma$ is not stable by Theorem 4.2. Examples of $E$ with nonstable $C^*(E)^\gamma$ are discussed. Finally we prove that the AF subalgebras $C^*(E)^\gamma_v$ of $C^*(E)^\gamma$ are not stable if $E$ is irreducible.

## 2. Preliminaries

**Crossed products by compact groups and fixed point algebras**. Let $A$ be a $C^*$-algebra and $\alpha$ be an action of a compact group $G$ on $A$. Then the $*$-algebra $C(G, A)$ of continuous functions from $G$ to $A$ with the following convolution (as multiplication) and involution

$$f * g(t) = \int_G f(s)\alpha_s(g(s^{-1}t))ds,$$
$$f^*(t) = \alpha_t(f(t^{-1})^*)$$

is dense in the crossed product $A\times_\alpha G$, where $ds$ is the normalized Haar measure on $G$ (see [21, 7.7] or [10, 8.3.1]). If $\tilde{A}$ denotes the smallest unitization of $A$ (so $\tilde{A} = A$ if $A$ is unital), every continuous function $h : G \to \tilde{A}$ belongs to the multiplier algebra of $A \times_\alpha G$. In particular, the constant function $1_G : G \to \tilde{A}$ given by $1_G(s) = 1$, $s \in G$, is a projection of the multiplier algebra of $A \times_\alpha G$ ([26]). Thus $1_G(A \times_\alpha G)1_G$ is a hereditary $C^*$-subalgebra of $A \times_\alpha G$.

*Remark* 2.1. Let $\alpha$ be an action of a compact group $G$ on a $C^*$-algebra $A$.

(i) For a function $f \in C(G)(\subset C(G, \tilde{A}))$ and an element $x \in A$, define $f \cdot x \in C(G, A)$ by

$$(f \cdot x)(s) = f(s)x,\ \ s \in G.$$

Then $\text{span}\{f \cdot x \mid f \in C(G),\ x \in A\}$ is dense in $A \times_\alpha G$.

(ii) If $A^\alpha := \{a \in A \mid \alpha_g(a) = a \ \text{ for all } g \in G\}$ is the fixed point algebra of $\alpha$, identifying $x \in A^\alpha$ and the constant function $1_G \cdot x$ in $C(G, A)$ with the value $x$ everywhere we see that

$$x \mapsto 1_G \cdot x : A^\alpha \to\ 1_G(A \times_\alpha G)1_G \tag{1}$$

is an isomorphism of $A^\alpha$ onto the hereditary subalgebra $1_G(A \times_\alpha G)1_G$ of $A \times_\alpha G$ ([26]).

**Graph $C^*$-algebras**. A *directed graph* $E = (E^0, E^1, r, s)$ consists of the vertex set $E^0$, the edge set $E^1$, and the range, source maps $r, s : E^1 \to E^0$. $E$ is called *row finite* if each vertex of $E$ emits only finitely many edges and *locally finite* if it is row finite and each vertex receives only finitely many edges. By $E^n$ we denote the set of all finite paths $\alpha = e_1 \cdots e_n$ $(r(e_i) = s(e_{i+1}), 1 \le i \le n-1)$ of *length* $n$ $(|\alpha| = n)$ (Vertices are finite paths of length 0). Then $E^* = \cup_{n\ge 0}E^n$ denotes the set of all finite paths. Infinite paths $e_1e_2e_3\cdots$ or $\cdots e_3e_2e_1$ can be considered and the maps $r$ or $s$ naturally extend to $E^*$ and the infinite paths. A vertex $v$ is called a *sink* if $s^{-1}(v) = \emptyset$ and a *source* if $r^{-1}(v) = \emptyset$. In this paper, we consider only row finite graphs. For $v, w \in E^0$, we write $v \gg w$ if there is a path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$.

Now we collect some definitions from [18] and [19] that we will be using below:

(i) $E$ is *irreducible* if $v \gg w$ for any $v, w \in E^0$.

(ii) A finite path $\beta$ is a *loop* if $s(\beta) = r(\beta)$ and $|\beta| > 0$.
(iii) An *exit* of a subgraph $F$ of $E$ is an edge $e \in E^1$ with $s(e) \in F^0$ and $r(e) \notin F^0$. $E$ has *property* (L) if every loop has an exit. A graph with no loops has the property vacuously.
(iv) $E$ has *property* (K) if for any vertex $v$ and a loop $\beta = \beta_1\beta_2 \cdots \beta_{|\beta|}$ with $s(\beta) = v$ there is another loop $\alpha = \alpha_1\alpha_2 \cdots \alpha_{|\alpha|}$ with $s(\alpha) = v$ such that $\alpha_i \neq \beta_i$ for some $i \leq \min\{|\alpha|, |\beta|\}$. (K) implies (L).

It is now well known ([3, 18, 19, 22]) that there exists a universal $C^*$-algebra $C^*(E)$, called the *graph $C^*$-algebra*, associated with a row finite graph $E$ generated by a *Cuntz-Krieger $E$-family* which consists of operators $\{s_e, p_v \mid e \in E^1, v \in E^0\}$ such that $\{s_e\}_{e\in E^1}$ are partial isometries and $\{p_v\}_{v\in E^0}$ are mutually orthogonal projections satisfying the relations

$$s_e^* s_e = p_{r(e)} \quad \text{and} \quad p_v = \sum_{s(e)=v} s_e s_e^* \quad \text{if } s^{-1}(v) \neq \emptyset.$$

(We simply write $C^*(E) = C^*(s_e, p_v)$ if $C^*(E)$ is generated by $\{s_e, p_v \mid e \in E^1, v \in E^0\}$.) For each $\alpha = \alpha_1\alpha_2 \cdots \alpha_{|\alpha|} \in E^*$, $\alpha_i \in E^1$, $s_\alpha$ denotes the partial isometry $s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{|\alpha|}}$ ($s_v = s_v^* = p_v$ for $v \in E^0$). Note that for every $\alpha \in E^*$,

$$s_\alpha s_\alpha^* \leq p_{s(\alpha)} \quad \text{and} \quad s_\alpha^* s_\alpha = p_{r(\alpha)}.$$

*Remark* 2.2. Let $C^*(E) = C^*(s_e, p_v)$ be the graph $C^*$-algebra associated with a row finite graph $E$. We will need the following basic facts which can be easily found in [1], [3], [18], [19], [22], etc.

(i) $C^*(E) = \overline{\operatorname{span}}\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*\}$ since

$$s_\alpha^* s_\beta = \begin{cases} s_\mu^*, & \text{if } \alpha = \beta\mu, \\ s_\nu, & \text{if } \beta = \alpha\nu, \\ 0, & \text{otherwise.} \end{cases}$$

Also $s_\alpha s_\beta^* = 0$ if $r(\alpha) \neq r(\beta)$.
(ii) For each $p_v \in C^*(E)$ and $n \in \mathbb{N}$, if $E$ is row finite,

$$p_v = \sum_{\substack{s(\alpha)=v \\ |\alpha|=n}} s_\alpha s_\alpha^*.$$

(iii) Let $E^0 := \{v_1, v_2, v_3, \dots\}$. Then the set of projections $\{\sum_{i=1}^n p_{v_i} \mid n \geq 1\}$ forms an approximate identity for $C^*(E)$. $C^*(E)$ is unital if and only if $E^0$ is finite.
(iv) If $E$ has property (L), in particular if $E$ has no loops, every Cuntz-Krieger $E$-family of nonzero operators generates a $C^*$-algebra isomorphic to $C^*(E)$.
(v) If $V \subset E^0$ is a hereditary subset ($v \in V$, $v \gg w$ implies $w \in V$), $\mathcal{I}(V) := \overline{\operatorname{span}}\{s_\alpha s_\beta^* \mid r(\alpha) = r(\beta) \in V\}$ is an ideal of $C^*(E)$. Furthermore for $E$ with property $(K)$, $V \to \mathcal{I}(V)$ constitutes a bijection

between the set of saturated hereditary vertex subsets of $E^0$ and the ideals of $C^*(E)$ ($V \subset E^0$ is saturated if $r(s^{-1}(v)) \subset V$ implies $v \in V$).

Let $G$ be a countable group. Recall ([17]) that for a graph $E$ and a function $c : E^1 \to G$, the *skew product* graph $E(c)$ is defined to be $(G \times E^0, G \times E^1, r, s)$, where

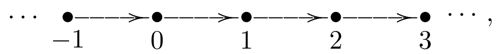
$$s(g,e) = (g, s(e)) \text{ and } r(g,e) = (gc(e), r(e)).$$

For two graphs $E$ and $F$, the *Cartesian product* is the graph

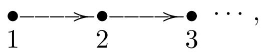
$$E \times F = (E^0 \times F^0, E^1 \times F^1, r, s),$$

where $r(e,f) = (r(e), r(f))$ and $s(e,f) = (s(e), s(f))$. For example, if

$E_{\mathbb{Z}}$ :

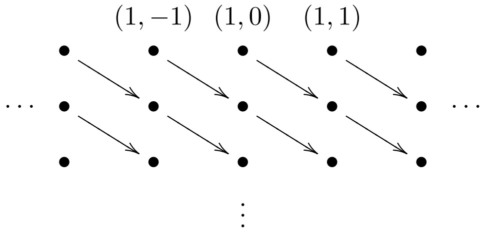



$E_{\mathbb{N}}$ :




then $E_{\mathbb{N}} \times E_{\mathbb{Z}}$ is as follows;

$E_{\mathbb{N}} \times E_{\mathbb{Z}}$ :




Note that $E_{\mathbb{Z}} \times E$ or $E_{\mathbb{N}} \times E$ have no loops for every $E$. Moreover, $E_{\mathbb{Z}} \times E = E(c)$ if $c : E^1 \to \mathbb{Z}$ is given $c(e) = 1$. For ease of notation, we denote an edge $x$ of $E_{\mathbb{Z}} \times E$ by $(n, e)$ ($n \in \mathbb{Z}$, $e \in E^1$) if $s(x) = (n, s(e))$ and $r(x) = (n+1, r(e))$. For paths of $E_{\mathbb{Z}} \times E$ (or $E_{\mathbb{N}} \times E$), we use similar notations, namely we write $(n, \alpha)$ for a path $(n, \alpha_1)(n+1, \alpha_2) \cdots (n + |\alpha| - 1, \alpha_{|\alpha|})$.

## 3. $C^*(E)^\gamma$, $C^*(E_{\mathbb{N}} \times E)$, and $C^*(E_{\mathbb{Z}} \times E)$

By the universal property of $C^*(E) = C^*(s_e, p_v)$, there exists an action $\gamma$ (called the *gauge action*) of $\mathbb{T}$ on $C^*(E)$ given by

$$\gamma_z(s_e) = z s_e, \ \gamma_z(p_v) = p_v, \ z \in \mathbb{T}.$$

The fixed point algebra of $\gamma$ is

$$C^*(E)^\gamma = \overline{\text{span}}\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*, \ |\alpha| = |\beta|\}.$$

Applying some results of [25] on groupoid $C^*$-algebras it is proved in [17] that $C^*(E_{\mathbb{Z}} \times E) \cong C^*(E) \times_\gamma \mathbb{T}$. But one can also give an explicit isomorphism:

**Proposition 3.1** ([15]). *Let $E$ be a row-finite graph with no sinks. If $C^*(E) = C^*(p_v, s_e)$ and $C^*(E_{\mathbb{Z}} \times E) = C^*(p_{(n,v)}, s_{(n,e)})$, then there is an isomorphism $\eta : C^*(E_{\mathbb{Z}} \times E) \to C^*(E) \times_\gamma \mathbb{T}$ such that*

$$\eta(p_{(m,v)}) = z^m \cdot p_v, \quad \eta(s_{(m,e)}) = z^m \cdot s_e,$$

*where $m \in \mathbb{Z}$, $v \in E^0$, and $e \in E^1$.*

Since the graph $E_{\mathbb{N}} \times E$ has property (L) for every $E$, $C^*(E_{\mathbb{N}} \times E)$ can be identified with the $C^*$-subalgebra

$$C^*(E_{\mathbb{N}} \times E) = C^*\{p_{(n,v)}, s_{(n,e)} \mid n \in \mathbb{N},\ v \in E^0,\ e \in E^1\}$$

of $C^*(E_{\mathbb{Z}} \times E)$ (Remark 2.2.(iv)).

**Proposition 3.2.** *Let $E$ be a row-finite graph with no sinks.*

(i) *If $F$ is a subgraph of $E$ with no exits, then $B_F := \overline{\text{span}}\{s_\alpha s_\beta^* \mid \alpha, \beta \in F^*\}$ is a hereditary $C^*$-subalgebra of $C^*(E)$ that generates the ideal*

$$\mathcal{I}(F^0) = \overline{\text{span}}\{s_\alpha s_\beta^* \in C^*(E) \mid r(\alpha) = r(\beta) \in F^0\}.$$

(ii) *$C^*(E_{\mathbb{N}} \times E)$ is a full hereditary $C^*$-subalgebra of $C^*(E_{\mathbb{Z}} \times E)$.*

*Proof.* (i) From $B_F \cdot C^*(E) \subset B_F$ we see that $B_F$ is a hereditary $C^*$-subalgebra of $C^*(E)$. Since $F^0$ is a hereditary vertex subset, by Remark 2.2.(v), $\mathcal{I}(F^0)$ is an ideal of $C^*(E)$. $B_F \subset \mathcal{I}(F^0)$ is obvious and $\mathcal{I}(F^0)$ is generated by $B_F$ because $s_\alpha s_\beta^* = s_\alpha p_{r(\alpha)} s_\beta^*$ and $p_{r(\alpha)} \in B_F$ if $s_\alpha s_\beta^* \in \mathcal{I}(F^0)$.

(ii) Let $C^*(E_{\mathbb{Z}} \times E) = C^*(p_{(n,v)}, s_{(n,e)})$, $n \in \mathbb{Z}$, $v \in E^0$, and $e \in E^1$. Since $E_{\mathbb{N}} \times E$ has no exits, by (i), $C^*(E_{\mathbb{N}} \times E)$ is a hereditary subalgebra generating the ideal

$$\mathcal{I} = \overline{\text{span}}\{s_{(n,\alpha)} s_{(m,\beta)}^* \mid r(n,\alpha) = r(m,\beta) \in (E_{\mathbb{N}} \times E)^0\}.$$

Since $E$ has no sinks, every element of the form $s_\mu s_\nu^* \in C^*(E_{\mathbb{Z}} \times E)$ can be written as the finite sum of elements $s_\alpha s_\beta^*$ with $r(\alpha) = r(\beta) \in (E_{\mathbb{N}} \times E)^0$ by Remark 2.2.(ii), so that $s_\mu s_\nu^* \in \mathcal{I}$. □

**Proposition 3.3.** *Let $E$ be a locally finite graph with no sinks and sources. Then $C^*(E)^\gamma$ is isomorphic to the full hereditary $C^*$-subalgebra*

$$A_\gamma := \overline{\text{span}}\{s_{(1,\alpha)} s_{(1,\beta)}^* \mid \alpha, \beta \in E^* \text{ and } |\alpha| = |\beta|\}$$

*of $C^*(E_{\mathbb{N}} \times E)$.*

*Proof.* Let $\eta : C^*(E_{\mathbb{Z}} \times E) \to C^*(E) \times_\gamma \mathbb{T}$ be the isomorphism of Proposition 3.1. We show that $\eta(A_\gamma) = B_\gamma$, where

$$B_\gamma := \overline{\text{span}}\{1_G \cdot s_\alpha s_\beta^* \mid \alpha, \beta \in E^* \text{ and } |\alpha| = |\beta|\}$$

is isomorphic to $C^*(E)^\gamma$ by (1) of Remarks 2.1. Then, since the hereditary $C^*$-subalgebra $B_\gamma$ is full in $C^*(E) \times_\gamma \mathbb{T}$ by [16, Proposition 5.4 and Theorem 6.3], so is $A_\gamma = \eta^{-1}(B_\gamma)$ in $C^*(E_{\mathbb{Z}} \times E)$.

First note that if $x = s_\alpha s_\beta^*$, $y = s_\mu s_\nu^*$, $f(z) = z^n$, and $g(z) = z^k$, then

$$(f \cdot x) * (g \cdot y)(z) = z^k \big(xy \int_{\mathbb{T}} w^{n-k+|\mu|-|\nu|} dw\big),$$
$$(f \cdot x)^*(z) = f(z)\gamma_z(x)^*$$

from which we have for $\alpha = \alpha_1\alpha_2 \cdots \alpha_n \in E^n$,

$$\begin{aligned}\eta(s_{(1,\alpha)}) &= \eta(s_{(1,\alpha_1)}) * \eta(s_{(2,\alpha_2)}) * \cdots * \eta(s_{(n,\alpha_n)})\\ &= (z^1 \cdot s_{\alpha_1}) * (z^2 \cdot s_{\alpha_2}) * \cdots * (z^n \cdot s_{\alpha_n})\\ &= z^n \cdot s_\alpha.\end{aligned}$$

Thus if $\alpha$, $\beta \in E^n$, then

$$\begin{aligned}&\eta(s_{(1,\alpha)} s_{(1,\beta)}^*) = \eta(s_{(1,\alpha)}) * \eta(s_{(1,\beta)})^*\\ &= (z^n \cdot s_\alpha) * (z^n \cdot s_\beta)^* = (z^n \cdot s_\alpha) * (1_G \cdot s_\beta^*) = 1_G \cdot s_\alpha s_\beta^*.\end{aligned}$$

□

*Remark* 3.4. It is known in [17] that $C^*(E)^\gamma$ and $C^*(E_{\mathbb{Z}} \times E)$ are stably isomorphic (or strong Morita equivalent) if $E$ is a row finite graph with no sinks, which also immediately follows from Proposition 3.2 and Proposition 3.3 above since every $C^*$-algebra is stably isomorphic to its full hereditary $C^*$-subalgebras.

## 4. Stable case

Note that the algebras $C^*(E)^\gamma$, $C^*(E_{\mathbb{Z}} \times E)$, and $C^*(E_{\mathbb{N}} \times E)$ are all AF. An AF algebra is known to be stable ($A \cong A \otimes \mathcal{K}$) unless it admits a nonzero bounded trace [4, 24].

The following lemma is immediate from [11, Lemma 2.1] and [12, Theorem 3.3]. For two projections $p$, $q$, we write $p \precsim q$ if $p$ is equivalent to a subprojection of $q$.

**Lemma 4.1.** *Let $A$ be a $C^*$-algebra with an approximate identity $(p_n)_{n\ge 1}$ consisting of projections with $p_1 \le p_2 \le \cdots$. Then we have the following*:

- (i) *$A$ is stable if and only if for every $n$, there is an $m > n$ such that $p_n \precsim p_m - p_n$.*
- (ii) *For a row-finite graph $E$, $C^*(E) = C^*(p_v, s_e)$ is stable if and only if for each finite subset $V \subset E^0$, there is a finite set $W \subset E^0$ with $V \cap W = \emptyset$ such that $\sum_{v\in V} p_v \precsim \sum_{w\in W} p_w$.*

For a finite subset $V \subset E^0$, let $p_V := \sum_{v\in V} p_v$ and let

$$E^n_{s^{-1}(V)} := \{\alpha \in E^n \mid s(\alpha) \in V\}.$$

Theorem 4.2(ii) below shows that $C^*(E)^\gamma$ and $C^*(E_{\mathbb{Z}} \times E)$ (and $C^*(E_{\mathbb{N}} \times E)$) are all isomorphic if and only if $C^*(E)^\gamma$ is stable. A vertex $v \in E^0$ is *left-infinite* if there is an infinite path $\alpha$ ending at $v$ such that all edges of $\alpha$ are distinct (see [11, Lemma 2.11]) and $E$ is *left-infinite* if every vertex of $E$ is left-infinite. It is known in [11, Lemma 2.13] that if $E$ is a locally finite left-infinite graph, $C^*(E)$ is stable. But the converse need not be true, in fact, $E_{\mathbb{N}} \times E$ is not left invertible while $C^*(E_{\mathbb{N}} \times E)$ is possibly stable (see Theorem 4.2(iii) and examples of this section).

**Theorem 4.2.** *Let $E$ be a locally finite infinite graph with no sinks and sources. Then we have the following*:

(i) *Let $c : E^1 \to G$ be a function. If $E$ is left-infinite, so is $E(c)$.*
(ii) *$C^*(E_{\mathbb{Z}} \times E)$ is stable.*
(iii) *If $C^*(E)^\gamma$ is stable, then both $C^*(E)$ and $C^*(E_{\mathbb{N}} \times E)$ are stable.*

*Proof.* (i) Let $E$ be left-infinite and $(g_0, v_0) \in E(c)^0$. Since $v_0 \in E^0$ is left-infinite, there is an infinite path $\alpha$ consisting of distinct edges, $\alpha = \cdots \alpha_3\alpha_2\alpha_1$ with $r(\alpha_1) = v_0$. Then the infinite path

$$\cdots (g_0c(\alpha_1)^{-1}c(\alpha_2)^{-1}c(\alpha_3)^{-1}, \alpha_3)(g_0c(\alpha_1)^{-1}c(\alpha_2)^{-1}, \alpha_2)(g_0c(\alpha_1)^{-1}, \alpha_1)$$

ending at $(g_0, v_0)$ has distinct edges. Hence $E(c)$ is left-infinite.

(ii) Note that $E_{\mathbb{Z}} \times E$ is left-infinite.

(iii) Suppose $C^*(E)^\gamma$ is stable. Since $C^*(E)^\gamma$ contains an approximate identity $\{\sum_{i=1}^n p_{v_i} \mid n = 1, 2, \dots\}$ of $C^*(E)$ (Remark 2.2.(iii)), applying Lemma 4.1 we see that $C^*(E)$ is stable. For stability of $C^*(E_{\mathbb{N}} \times E)$, let $E_{\mathbb{N}_n} \times E$ $(n \geq 1)$ be the subgraph of $E_{\mathbb{N}} \times E$ with $(E_{\mathbb{N}_n} \times E)^0 = \{(k, v) \mid k \geq n,\ v \in E^0\}$ and $(E_{\mathbb{N}_n} \times E)^1 = \{(k, e) \mid k \geq n,\ e \in E^1\}$. Clearly, $\varphi_n : C^*(E_{\mathbb{N}} \times E) \to C^*(E_{\mathbb{N}_n} \times E)$, $\varphi_n(p_{(i,v)}) = p_{(i+n,v)}$, $\varphi_n(s_{(j,e)}) = s_{(j+n,e)}$ $(i,\ j \geq 1)$, is an isomorphism. For each $k \geq 1$ and a finite subset $V \subset E^0$, set

$$[1, k] \times V := \{(i, v) \in (E_{\mathbb{N}} \times E)^0 \mid 1 \leq i \leq k,\ v \in V\}.$$

Then the corresponding projection $p_{[1,k]\times V}$ can be written as

$$p_{[1,k]\times V} = \sum_{n=1}^{k} p_{\{n\}\times V} = \sum_{n=1}^{k} \big( \sum_{v \in V} p_{(n,v)} \big).$$

For each $n$, consider the projection $\varphi_n^{-1}(p_{\{n\}\times V}) = p_{\{1\}\times V}$ in $C^*(E_{\mathbb{N}} \times E)$. Since $p_{\{1\}\times V}$ belongs to $A_\gamma (\cong C^*(E)^\gamma)$ and we assume that $C^*(E)^\gamma$ is stable, by Lemma 4.1.(i) there exists a finite vertex set $W \subset E^0$ with $V \cap W = \emptyset$ and a partial isometry $x \in C^*(E)^\gamma$ such that $x^*x = p_{\{1\}\times V}$ and $xx^* \leq p_{\{1\}\times W}$. Then $x_n := \varphi_n(x)$ is a partial isometry in $C^*(E_{\mathbb{N}_n} \times E)$ satisfying $x_n^*x_n = p_{\{n\}\times V}$ and $x_nx_n^* \leq p_{\{n\}\times W}$. Now $X := \sum_{n=1}^k x_n \in C^*(E_{\mathbb{N}} \times E)$ is a partial isometry such that $X^*X = p_{[1,k]\times V}$ and $XX^* \leq p_{[1,k]\times W}$. This completes the proof since every finite vertex subset of $(E_{\mathbb{N}} \times E)^0$ is contained in $[1, k] \times V$ for some $k$ and $V$ and $([1, k] \times V) \cap ([1, k] \times W) = \emptyset$. □

**Proposition 4.3.** *Let $E$ be a locally finite infinite graph without sinks or sources. Then we have the following*:

(i) *$C^*(E)^\gamma$ is stable if for every finite subset $V \subset E^0$, there is an $l \in \mathbb{N}$ and a finite vertex subset $W \subset E^0$ with $V \cap W = \emptyset$ such that for each $\alpha \in E^l_{s^{-1}(V)}$, there is $\alpha' \in E^l_{s^{-1}(W)}$ with $r(\alpha) = r(\alpha')$ such that $\alpha \mapsto \alpha'$ is injective.*

(ii) *If every vertex of $E$ receives at most one edge, then $C^*(E_{\mathbb{N}} \times E)$ is stable.*

*Proof.* (i) is obvious by Proposition 3.3 and Lemma 4.1.

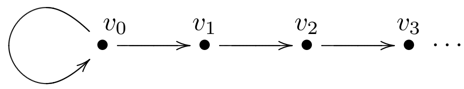
(ii) Let $\{p_{(n,v)}, s_{(n,e)}\}$ be the Cuntz-Krieger $(E_{\mathbb{N}} \times E)$-family and $V$ be a finite subset of $(E_{\mathbb{N}} \times E)^0$. Then there is $k_0 \in \mathbb{N}$ such that $(n, v) \in V$ implies $n < k_0$. For each $(n, v) \in V$, consider the following set of paths

$$S^{k_0}_{(n,v)} := (E_{\mathbb{N}} \times E)^{k_0}_{s^{-1}(n,v)} = \{(n, \alpha) \in (E_{\mathbb{N}} \times E)^{k_0} \mid s(n, \alpha) = (n, v)\}.$$

Then $p_{(n,v)} = \sum_{(n,\alpha)\in S^{k_0}_{(n,v)}} s_{(n,\alpha)} s^*_{(n,\alpha)}$ is equivalent to $\sum_{(n,\alpha)\in S^{k_0}_{(n,v)}} s^*_{(n,\alpha)} s_{(n,\alpha)}$, note here that the projections $s^*_{(n,\alpha)} s_{(n,\alpha)} = p_{r(n,\alpha)}$ are mutually orthogonal since there is only one path with range $r(n, \alpha)$ and with length $k_0$. Moreover, if $(m, w) \in W := \{r(n, \alpha) \mid (n, \alpha) \in S^{k_0}_{(n,v)}\}$, then $m \geq k_0$ and so $(m, v) \notin V$. Therefore we have $V \cap W = \emptyset$ and $\sum_{(n,v)\in V} p_{(n,v)} \sim \sum_{(m,w)\in W} p_{(m,w)}$. Thus by Lemma 4.1 the assertion follows. □

**Example 4.4.** For the following graph $E$, $C^*(E_{\mathbb{N}} \times E)$ is stable. But $C^*(E)^\gamma$ is not.

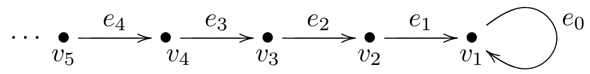
$E$ :




By Proposition 4.3(ii), $C^*(E_{\mathbb{N}} \times E)$ is stable. But $C^*(E)$ is not stable by [11, Lemma 2.16] since it has a quotient $C^*$-algebra isomorphic to the nonstable algebra $C(\mathbb{T}) \cong C^*(E)/\mathcal{I}$, where $\mathcal{I}$ is the ideal corresponding to the saturated hereditary vertex subset $\{v_1, v_2, \dots\}$. Thus $C^*(E)^\gamma$ is not stable by Theorem 4.2.(iii).

**Example 4.5.** $C^*(F)^\gamma$ is stable if $F$ is as follows:

$F$ :



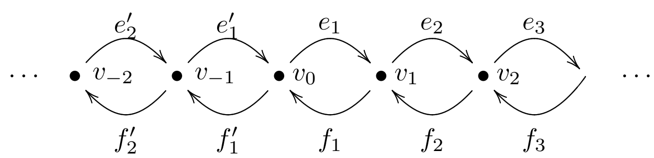

In fact, the increasing sequence of projections $p_n := \sum_{i=1}^n p_{v_i}$, $n \geq 1$, is an approximate identity for $C^*(E)^\gamma$ such that each $p_n$ is equivalent to $p_m - p_n$ for some $m > n$ in $C^*(E)^\gamma$: The partial isometry

$$s := s_{e_{2n-1}\cdots e_1} s^*_{e_{n-1}\cdots e_1 e_0^n} + \cdots + s_{e_n \cdots e_1} s^*_{e_0^n}$$

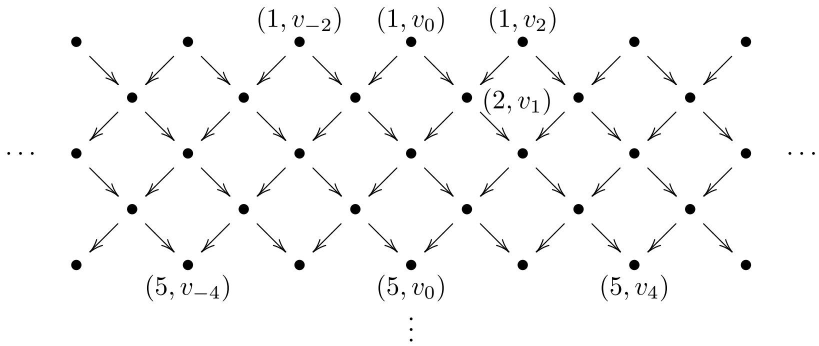
of $C^*(E)^\gamma$ satisfies $s^*s = p_n$ and $ss^* = p_{2n} - p_n$. Thus the stability of $C^*(E)^\gamma$ follows from Lemma 4.1.

**Example 4.6.** $C^*(E)^\gamma$ is stable for the following irreducible graph $E$:

$E$ :



We show that $A_\gamma$ (of Proposition 3.3) is stable. Since $E_{\mathbb{N}}\times E$ has property (K) and there are only two nontrivial saturated hereditary vertex subsets $V_0$, $V_1$ in $(E_{\mathbb{N}}\times E)^0$ ($V_0\supset\{(1,k)\mid k \text{ is even}\}$ and $V_1\supset\{(1,k)\mid k\text{ is odd}\}$) such that $V_0\cap V_1=\emptyset$, we see that $C^*(E_{\mathbb{N}}\times E)$ has only two nontrivial (proper) ideals $\mathcal{I}(V_0)$ and $\mathcal{I}(V_1)$. Moreover $\mathcal{I}(V_0)\cap\mathcal{I}(V_1)=\{0\}$ because $V_0\cap V_1=\emptyset$. Since $A_\gamma$ is a full hereditary $C^*$-subalgebra of $C^*(E_{\mathbb{N}}\times E)$, $\mathcal{I}\mapsto A_\gamma\cap\mathcal{I}$ establishes a bijection between the sets of ideals of $C^*(E_{\mathbb{N}}\times E)$ and $A_\gamma$. Thus $A_\gamma$ has two nontrivial ideals $A_\gamma\cap\mathcal{I}(V_0)$ and $A_\gamma\cap\mathcal{I}(V_1)$. But actually these are isomorphic and $A_\gamma=(A_\gamma\cap\mathcal{I}(V_0))\oplus(A_\gamma\cap\mathcal{I}(V_1))$. If $A_\gamma$ is not stable, there exists a nonzero bounded trace $\tau$. Then $\tau|_{A_\gamma\cap\mathcal{I}(V_i)}$ is nonzero for some $i=0,1$. Assume that $\tau|_{A_\gamma\cap\mathcal{I}(V_0)}$ is nonzero. Note that the projections $\{p_n:=\sum_{k=-n}^{n}p_{(1,v_k)}\}_n$ forms an approximate identity for $A_\gamma\cap\mathcal{I}(V_0)$. Then $\tau(p_{(1,v_{2k})})\neq 0$ for some $k$. We may assume that $\tau(p_{(1,v_0)})=1$. Consider the following subgraph of $E_{\mathbb{N}}\times E$.




If $(1,\alpha)$, $(1,\beta)\in(E_{\mathbb{N}}\times E)^{2k}$ are paths from $(1,v_0)$ to $(2k+1,v_{2i})$, $-k\le i\le k$, then $x:=s_{(1,\alpha)}s^*_{(1,\beta)}\in A_\gamma$ satisfies

$$xx^*=s_{(1,\alpha)}s^*_{(1,\alpha)},\quad x^*x=s_{(1,\beta)}s^*_{(1,\beta)}.$$

Thus $\tau(s_{(1,\alpha)}s^*_{(1,\alpha)})=\tau(s_{(1,\beta)}s^*_{(1,\beta)})$, hence for each $k\ge 1$,

$$1=\tau(p_{(1,v_0)})=\sum_{\substack{\alpha\in E^{2k}\\ s(\alpha)=v_0}}\tau(s_{(1,\alpha)}s^*_{(1,\alpha)})=\sum_{v_{2i}}\sum_{\substack{\alpha\in E^{2k}\\ s(\alpha)=v_0\\ r(\alpha)=v_{2i}}}\tau(s_{(1,\alpha)}s^*_{(1,\alpha)}),$$

where $-k\le i\le k$. If $K_{2i}$ is the number of paths $\alpha\in E^{2k}$ with $s(\alpha)=v_0$ and $r(\alpha)=v_{2i}$, then

$$K_{2i}=\binom{2k}{k-i}=\frac{(2k)!}{(k+i)!(k-i)!}\le\binom{2k}{k}=K_0.$$

Let $t_{2i}:=\tau(s_{(1,\alpha)}s^*_{(1,\alpha)})$ for $\alpha\in E^{2k}$ with $s(\alpha)=v_0$, $r(\alpha)=v_{2i}$. Then

$$1=\tau(p_{(1,v_0)})=\sum_{i=-k}^{k}K_{2i}t_{2i}=\sum_{i=-k}^{k}\binom{2k}{k-i}t_{2i}\le\sum_{i=-k}^{k}\binom{2k}{k}t_{2i}.$$

On the other hand, for each $i$, there are $2^{2k}$ paths $\mu \in E^{2k}$ with $r(\mu) = v_{2i}$. Thus we have that

$$\|\tau\| \geq \sum_{i=-k}^{k} 2^{2k} t_{2i}.$$

Now we show by induction on $k$ that $2^{2k} > k^{1/3} \cdot \binom{2k}{k}$ for all $k \geq 1$. In fact, the inequality holds for $k = 1$. Suppose $2^{2k} > k^{1/3} \cdot \binom{2k}{k}$, then

$$\begin{aligned} 2^{2k} > k^{1/3} \cdot \binom{2k}{k} &= k^{1/3} \cdot \frac{(2k)!}{(k!)^2} \\ &= k^{1/3} \cdot \frac{(2k)!(2k+1)(2k+2)}{(k!)^2(k+1)(k+1)} \cdot \frac{(k+1)(k+1)}{(2k+1)(2(k+1))} \\ &= (k+1)^{1/3} \cdot \frac{(2(k+1))!}{((k+1)!)^2} \cdot \frac{k^{1/3}}{(k+1)^{1/3}} \cdot \frac{k+1}{2(2k+1)}, \end{aligned}$$

from which we have

$$\begin{aligned} 2^{2k+2} &> (k+1)^{1/3} \cdot \binom{2(k+1)}{k+1} \cdot \frac{k^{1/3}}{(k+1)^{1/3}} \cdot \frac{4k+4}{4k+2} \\ &> (k+1)^{1/3} \cdot \binom{2(k+1)}{k+1} \end{aligned}$$

since $\frac{k^{1/3}}{(k+1)^{1/3}} \cdot \frac{4k+4}{4k+2} > 1$ for all $k \geq 1$. Then

$$\|\tau\| \geq \sum_{i=-k}^{k} 2^{2k} t_{2i} > \sum_{i=-k}^{k} k^{1/3} \binom{2k}{k} t_{2i} \geq k^{1/3}$$

which goes to $\infty$ as $k \to \infty$, a contradiction to the boundedness of $\tau$.

## 5. Nonstable case

In this section, we consider locally finite infinite graphs $E$ for which $C^*(E_{\mathbb{N}} \times E)$ have bounded traces (hence, not stable). Of course, then $C^*(E)^\gamma$ is not stable by Theorem 4.2.

Recall ([11, Definition 2.7]) that $\tau : E^0 \to [0, \infty)$ is a *bounded graph-trace* if

$$\tau(v) = \sum_{\{e \mid s(e) = v\}} \tau(r(e)) \quad \text{and} \quad \sum_{v \in E^0} \tau(v) < \infty.$$

If $E$ has no loops, every bounded graph-trace on $E$ extends to a bounded trace $E$ ([11, Lemma 2.8]).

**Theorem 5.1.** *Let $E$ be a locally finite infinite graph with no sinks and sources. Let $E^0 := \{1, 2, \dots\}$ and $A = (a_{ij})$ be the vertex matrix of $E$, that is, $A$ is an $E^0 \times E^0$ matrix with $a_{ij}$ edges from vertex $i$ to vertex $j$. If there is an eigenvector $\xi = (\xi_1, \xi_2, \dots)$ of $A$ with an eigenvalue $\lambda$ ($A\xi = \lambda\xi$) such that*

(i) $\lambda > 1$,

(ii) $\xi_i \geq 0$ *for each* $i \in E^0$ *and* $0 < \sum_{i \geq 1} \xi_i < \infty$, *then $C^*(E_{\mathbb{N}} \times E)$ admits a bounded trace. In particular, $C^*(E_{\mathbb{N}} \times E)$ is not stable.*

*Proof.* Since $E_{\mathbb{N}} \times E$ has no loops, it is enough to claim that there is a bounded graph-trace $\tau$ on $E_{\mathbb{N}} \times E$. Define $\tau : (E_{\mathbb{N}} \times E)^0 \to [0, \infty)$ by

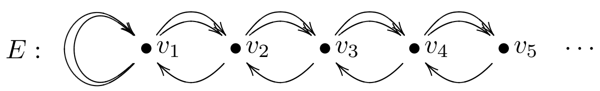
$$\tau(p_{(n,i)}) = \frac{1}{\lambda^{n-1}}\, \xi_i, \quad n \geq 1,\ i \geq 1.$$

Then the sum $\sum_{(n,i)} \tau(p_{(n,i)}) = \sum_n (\sum_i \frac{1}{\lambda^{n-1}}\, \xi_i) = \sum_{n\geq 1} \frac{1}{\lambda^{n-1}} \cdot \sum_{i\geq 1} \xi_i$ converges. Also $A\xi = \lambda\xi$ (hence, $\xi_i = \frac{1}{\lambda} \sum_j a_{ij}\xi_j$) implies that for each $i \in E^0$,

$$\begin{aligned} \tau(p_{(n,i)}) &= \frac{1}{\lambda^{n-1}}\, \xi_i = \frac{1}{\lambda^n} \sum_j a_{ij}\xi_j = \sum_j a_{ij} \frac{1}{\lambda^n}\, \xi_j \\ &= \sum_j a_{ij} \tau(p_{(n+1,j)}) = \sum_{\{(n,e) \mid s(n,e)=(n,i)\}} \tau(p_{r(n,e)}). \end{aligned}$$

□

**Example 5.2.** $C^*(E)^\gamma$ is not stable if $E$ is an irreducible infinite graph as below:




The vertex matrix

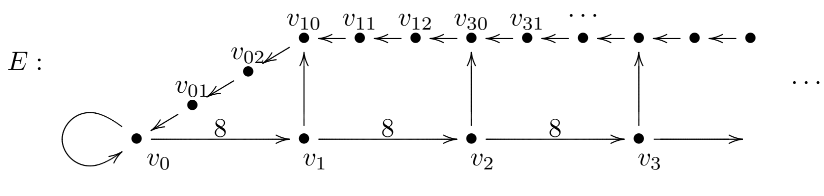
$$\begin{pmatrix} 2 & 2 & 0 & 0 & \cdots \\ 1 & 0 & 2 & 0 & \cdots \\ 0 & 1 & 0 & 2 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has an eigenvector $\xi = (\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \dots)$ with $\lambda = 3$. Thus $C^*(E_{\mathbb{N}} \times E)$ is not stable by Theorem 5.1 and so $C^*(E)^\gamma$ is not stable by Theorem 4.2.

**Example 5.3.** It is known in [27] that for a pair of positive real numbers $1 < p \leq q$, there exists an irreducible infinite graph $E_{p,q}$ with

$$h_l(E_{p,q}) = \log p \ \text{ and } \ h_b(E_{p,q}) = \log q.$$

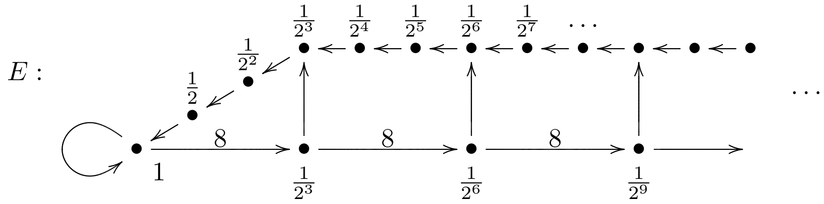
The following graph $E := E_{2,8}$ satisfies $h_l(E) = \log 2$ and $h_b(E) = \log 8$. There are 8 edges from the vertex $v_n$ to the vertex $v_{n+1}$ for each $n \geq 0$ (Example 3.3 of [14]).



Note that if a vector $\xi = (\xi_v)_{v\in E^0}$ satisfies

$$\sum_{w\in E^0} a_{vw}\xi_w = 2\xi_v, \quad v \in E^0, \tag{2}$$

$\xi$ is an eigenvector of the vertex matrix $A$ such that $A\xi = 2\xi$. Let $\xi = (\xi_v)_{v\in E^0}$ be the vector with $\xi_v > 0$ as follows:



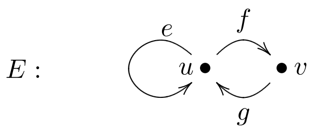

Then (2) can be shown at every vertex $v$, and by Theorem 5.1 with $\lambda = 2$, $C^*(E_{\mathbb{N}} \times E)$ admits a bounded trace, hence is not stable. Hence $C^*(E)^\gamma$ is not stable by Theorem 4.2 again.

Now we consider an AF subalgebra $C^*(E)^\gamma_v$ of $C^*(E)^\gamma$ for each $v \in E^0$,

$$C^*(E)^\gamma_v = \overline{\operatorname{span}}\{s_\alpha s_\beta^* \in C^*(E)^\gamma \mid r(\alpha) = r(\beta) = v\}.$$

The following example shows that $C^*(E)^\gamma_u$ and $C^*(E)^\gamma_v$ may not be isomorphic if $u \neq v$. The AF algebras $C^*(E)^\gamma$ and $C^*(E)^\gamma_v$ were denoted $\mathcal{A}_E$ and $\mathcal{A}_E(v)$, respectively, in [13, 15].

**Example 5.4.** Consider the following irreducible finite graph $E$:




Let $C^*(E)$ be generated by a Cuntz-Krieger $E$-family $\{p_u, p_v, s_e, s_f, s_g\}$. Then

$$C^*(E)^\gamma_u = C^*(E)^\gamma \quad \text{and} \quad C^*(E)^\gamma_u \ncong C^*(E)^\gamma_v.$$

In fact, if $s_\alpha e_\beta^* \in C^*(E)^\gamma_v$, namely $r(\alpha) = r(\beta) = v$ ($|\alpha| = |\beta|$), then $s_\alpha s_\beta^* = s_\alpha p_v s_\beta^* = s_\alpha(s_g s_g^*)s_\beta^* \in C^*(E)^\gamma_u$. Thus $C^*(E)^\gamma_v \subset C^*(E)^\gamma_u$ and hence $C^*(E)^\gamma_u = C^*(E)^\gamma$. On the other hand, $C^*(E)^\gamma_v$ has an approximate identity consisting of projections $q_n$, where $q_n := p_v + \sum_{k=0}^n s_{e^k f} s_{e^k f}^*$. Since

$$\|1 - q_n\| = \|(p_u + p_v) - q_n\| = \|s_{e^{n+1}} s_{e^{n+1}}^*\| = 1,$$

it follows that $C^*(E)^\gamma_v$ is nonunital while $C^*(E)^\gamma_u$ is unital with unit $p_u + p_v$. Thus $C^*(E)^\gamma_u$ is not isomorphic to $C^*(E)^\gamma_v$.

**Theorem 5.5.** *Let $E$ be a locally finite irreducible infinite graph and $v \in E^0$. Then $C^*(E)^\gamma_v$ admits a nonzero bounded trace. In particular, $C^*(E)^\gamma_v$ is not stable.*

*Proof.* For each $n \geq 0$, put

$$C^*(E)^\gamma_{v,n} := \text{span}\{s_\alpha s_\beta^* \in C^*(E)^\gamma_v \mid |\alpha| = |\beta| \leq n\}.$$

Then $\{C^*(E)^\gamma_{v,n}\}_{n\geq 0}$ is an increasing sequence of finite dimensional $C^*$-subalgebras of $C^*(E)^\gamma$ such that

$$C^*(E)^\gamma_v = \overline{\cup_{n=0}^\infty C^*(E)^\gamma_{v,n}}.$$

Since $E$ is irreducible, the elements in the set

$$\omega(v,n) := \{s_\alpha s_\beta^* \in C^*(E)^\gamma_v \mid |\alpha| = |\beta| \leq n\}$$

are linearly independent by [14, Lemma 3.7], a linear map on $C^*(E)^\gamma_{v,n}$ is determined by its values on $s_\alpha s_\beta^* \in \omega(n,v)$. We define linear functionals

$$\tau_n : C^*(E)^\gamma_{v,n} \to \mathbb{C},\ n \geq 0$$

as follows. Let $\tau_0(p_v) = \frac{1}{2}$. For $n \geq 1$, define $\tau_n : C^*(E)^\gamma_{v,n} \to \mathbb{C}$ by

$$\tau_n(s_\alpha s_\beta^*) = \begin{cases} 1/2, & \text{if } \alpha = \beta = v, \\ \frac{1}{N_k 2^{k+1}}, & \text{if } \alpha = \beta \in E^k,\ 1 \leq k \leq n, \\ 0, & \text{otherwise,} \end{cases}$$

where $N_k := |\{\alpha \in E^k \mid r(\alpha) = v\}|$. Extend $\tau_n$ to a linear map on $C^*(E)^\gamma_{v,n}$. Then $\tau_n|_{C^*(E)^\gamma_{v,n-1}} = \tau_{n-1}$ is obvious. Now let

$$\tau : \cup_{n=0}^\infty C^*(E)^\gamma_{v,n} \to \mathbb{C}$$

be the linear map given by $\tau(x) = \tau_n(x)$ if $x \in C^*(E)^\gamma_{v,n}$. To see that $\tau$ is a trace, it suffices to show $\tau((s_\mu s_\nu^*)(s_\alpha s_\beta^*)) = \tau((s_\alpha s_\beta^*)(s_\mu s_\nu^*))$ for $s_\alpha s_\beta^*$, $s_\mu s_\nu^* \in \omega(n,v)$. But

$$\tau((s_\alpha s_\beta^*)(s_\mu s_\nu^*)) = \begin{cases} \frac{1}{N_k 2^{k+1}}, & \text{if } \beta = \mu\delta \text{ and } \alpha = \nu\delta \in E^k, \\ \frac{1}{N_k 2^{k+1}}, & \text{if } \mu = \beta\delta \text{ and } \nu = \alpha\delta \in E^k, \\ 0, & \text{otherwise,} \end{cases}$$

which implies that

$$\tau((s_\mu s_\nu^*)(s_\alpha s_\beta^*)) = \tau((s_\alpha s_\beta^*)(s_\mu s_\nu^*)).$$

Now we show that $\tau(X^*X) \geq 0$ for any $X \in \text{span}(\omega(v,n)) = C^*(E)^\gamma_{v,n}$. For this, choose $s_\alpha s_\beta^*$ with the smallest length $|\alpha|$ among the terms appearing in the expression of $X$. Then decompose $X = Y + Z$ in a way that the terms in $Y$ are of the form $\lambda\, s_{\alpha\mu} s_{\beta\nu}^*$ ($\lambda \in \mathbb{C}$ and $r(\alpha\mu) = r(\beta\nu) = v$, hence $\mu, \nu$ must be loops at $v$ whenever $|\mu| = |\nu| \geq 1$) and $Z$ is the sum of the remainders. Then $\tau(X^*X) = \tau(Y^*Y) + \tau(Z^*Z)$. If we show $\tau(Y^*Y) \geq 0$, the same argument can be applied (to $Z^*Z$) repeatedly to prove $\tau(X^*X) \geq 0$ since $X$ has only finite terms. Moreover, by decomposing $Y$ if needed, it is enough to consider $Y$ of the form

$$Y = \lambda_0\, s_\alpha s_\beta^* + \lambda_1 s_{\alpha\mu_1} s_{\beta\nu_1}^* + \cdots + \lambda_k s_{\alpha\mu_1\cdots\mu_k} s_{\beta\nu_1\cdots\nu_k}^* \tag{3}$$

for some loops $\mu_j, \nu_j$ at $v$ with $|\mu_j| = |\nu_j|$, $j = 1, \dots, k$. Clearly

$$\tau(Y^*Y) = |\lambda_0|^2 \tau(s_\beta s_\beta^*) \geq 0 \ \text{ if } \ Y = \lambda_0 s_\alpha s_\beta^*.$$

If $Y = \lambda_0 s_\alpha s_\beta^* + \lambda_1 s_{\alpha\mu_1} s_{\beta\nu_1}^*$, we have

$$\begin{aligned}\tau(Y^*Y) &= \tau\big((\overline{\lambda}_0 s_\beta s_\alpha^* + \overline{\lambda}_1 s_{\beta\nu_1} s_{\alpha\mu_1}^*)(\lambda_0 s_\alpha s_\beta^* + \lambda_1 s_{\alpha\mu_1} s_{\beta\nu_1}^*)\big) \\ &= |\lambda_0|^2 \tau(s_\beta s_\beta^*) + |\lambda_1|^2 \tau(s_{\beta\nu_1} s_{\beta\nu_1}^*) + \overline{\lambda}_0 \lambda_1 \tau(s_{\beta\mu_1} s_{\beta\nu_1}^*) + \overline{\lambda}_1 \lambda_0 \tau(s_{\beta\nu_1} s_{\beta\mu_1}^*).\end{aligned}$$

Hence $\tau(Y^*Y) = |\lambda_0|^2 \tau(s_\beta s_\beta^*) + |\lambda_1|^2 \tau(s_{\beta\nu_1} s_{\beta\nu_1}^*) \geq 0$ if $\mu_1 \neq \nu_1$. In case $\mu_1 = \nu_1$, from $\tau(s_\beta s_\beta^*) \geq \tau(s_{\beta\mu_1} s_{\beta\mu_1}^*))$, we have

$$\begin{aligned}\tau(Y^*Y) &= |\lambda_0|^2 \tau(s_\beta s_\beta^*) + |\lambda_1|^2 \tau(s_{\beta\mu_1} s_{\beta\mu_1}^*) + \overline{\lambda}_0 \lambda_1 \tau(s_{\beta\nu_1} s_{\beta\nu_1}^*) + \overline{\lambda}_1 \lambda_0 \tau(s_{\beta\nu_1} s_{\beta\nu_1}^*) \\ &\geq (|\lambda_0|^2 + |\lambda_1|^2 + \overline{\lambda}_0 \lambda_1 + \overline{\lambda}_1 \lambda_0) \tau(s_{\beta\nu_1} s_{\beta\nu_1}^*) \\ &= |\lambda_0 + \lambda_1|^2 \tau(s_{\beta\nu_1} s_{\beta\nu_1}^*) \\ &\geq 0.\end{aligned}$$

Now for $Y$ in (3), let $l$ be the smallest number such that $\mu_{l+1} \neq \nu_{l+1}$. Then with $\mu_0 := \alpha$ and $\nu_0 := \beta$, a computation shows that

$$\begin{aligned}\tau(Y^*Y) &= \sum_{i=0}^{k} |\lambda_i|^2 \tau(s_{\nu_0\nu_1\cdots\nu_i} s_{\nu_0\nu_1\cdots\nu_i}^*) + \sum_{\substack{i \neq j \\ 0 \leq i,j \leq l}} (\lambda_i \overline{\lambda}_j + \lambda_i \overline{\lambda}_i) \tau(s_{\nu_0\nu_1\cdots\nu_l} s_{\nu_0\nu_1\cdots\nu_l}^*) \\ &\geq |\lambda_0 + \cdots + \lambda_l|^2 \tau(s_{\nu_0\nu_1\cdots\nu_l} s_{\nu_0\nu_1\cdots\nu_l}^*) \\ &\geq 0.\end{aligned}$$

Thus $\tau_n : C^*(E)^\gamma_{v,n} \to \mathbb{C}$ is a positive trace for each $n$. Hence $\|\tau_n\| = \tau_n(1_n)$, where $1_n$ is the unit of $C^*(E)^\gamma_{v,n}$. But

$$\tau_n(1_n) \leq \sum_{\substack{r(\alpha)=v \\ |\alpha| \leq n}} \tau_n(s_\alpha s_\alpha^*) \leq \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}} \leq 1,$$

which means that $\tau$ is a bounded trace on the dense subalgebra $\cup_{n=0}^\infty C^*(E)^\gamma_{v,n}$ of $C^*(E)^\gamma_v$. Thus $\tau$ extends to a bounded trace on $C^*(E)^\gamma_v$. □

*Remark* 5.6. The assertion in Theorem 5.5 may not be true if $E$ is not irreducible (see Example 4.5). It would be very interesting to find a necessary and sufficient condition, especially in graph theoretical terms, under which $C^*(E)^\gamma$ becomes stable.

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