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SOME BILINEAR ESTIMATES

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Abstract. We establish some estimates on the hyper bilinear Hilbert transform on both Euclidean space and torus. We also use a transference method to obtain a Kenig-Stein's estimate on bilinear fractional integrals on the n-torus.

1. Introduction

Let $B : S(\mathbb{R}^n) \times S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ be a continuous bilinear operator, from the product of Schwartz spaces into the space of tempered distributions, which commutes with simultaneous translations. Then there exists $m \in S'(\mathbb{R}^n \times \mathbb{R}^n)$, the symbol of B, such that

$$B(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m(\xi,\eta)\,\widehat{f}(\xi)\,\widehat{g}(\eta)\,e^{2\pi i\langle x,(\xi+\eta)\rangle}d\xi d\eta.$$

Such an operator and its variants have been extensively studied in recent years. When $m(\xi,\eta)$ is smooth, the L^p boundedness problem was well studied by Coifman and Meyer. The problem becomes tougher when the symbol of a bilinear operator is nonsmooth. To illustrate the later case, one famous example is the bilinear Hilbert transform

$$H(f,g)(x) = p.v.\pi^{-1}\int_{\mathbb{R}} f(x-t)\,g(x+t)\,t^{-1}dt$$

whose $L^2 \times L^\infty \to L^2$ boundedness was a question related to a long time conjecture of Calderón about the uniform boundedness with respect to $\alpha \in [-1,0]$ of the family of Hilbert transforms h_α defined by

$$h_\alpha(f,g)(x) = p.v.\pi^{-1}\int_{\mathbb{R}} f(x-t)\,g(x+\alpha t)\,t^{-1}dt.$$

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Historically, the interest in the bilinear Hilbert transform arose from the study of the Cauchy integral and the Hilbert transform on Lipschitz curves and, as a first step in study of these, the first commutator of Calderón. The reader can find more details in [17].

One can easily check that the symbol of H is $i\ \mathrm{sgn}\,(\xi-\eta)$, which is not continuous along the line $\xi=\eta$.

The operator B has its periodic version on the n-torus T^n :

$$\widetilde{B}\left(\widetilde{f},\widetilde{g}\right)(x)=\sum_{k\in\mathbb{Z}^n}\sum_{\nu\in\mathbb{Z}^n}m\,(k,\upsilon)\,a_k b_\upsilon e^{2\pi i\langle x,(k+\upsilon)\rangle},$$

where $\{a_k\}$ and $\{b_\upsilon\}$ are the Fourier coefficients of $\widetilde{f},\widetilde{g}\in C^\infty(T^n)$, respectively. Thus the periodic version of H is

$$\widetilde{H}\left(\widetilde{f},\widetilde{g}\right)(x)=i\sum_{k\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}}\mathrm{sgn}\,(k-\upsilon)\,a_k b_\upsilon e^{2\pi i x(k+\upsilon)}.$$

The following celebrated theorem of Lacey and Thiele solves the conjecture of Calderón:

Theorem A ([13]). *Let* $1<q,r\le\infty$ *and* $\frac{2}{3}<p<\infty$. *Then*

$$\|H\,(f,g)\|_{L^p(\mathbb{R})}\le C\,\|f\|_{L^q(\mathbb{R})}\,\|g\|_{L^r(\mathbb{R})}\,,$$

provided $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$.

Using a transference method, we can obtain an analog for the periodic version:

Theorem B ([5]). *Let* $1<q,r\le\infty$ *and* $\frac{2}{3}<p<\infty$. *Then*

$$\left\|\widetilde{H}\left(\widetilde{f},\widetilde{g}\right)\right\|_{L^p(T)}\le C\left\|\widetilde{f}\right\|_{L^q(T)}\|\widetilde{g}\|_{L^r(T)}\,,$$

provided $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$.

To study the operator B in the one dimension case, Gilbert and Nahmod established some more general theorems by considering some related cone operators. For $n=1$, fix an angle θ, let

$$C_{P_\theta}:(f,g)\to\int_{P_\theta}m\,(\xi,\eta)\,\widehat{f}\,(\xi)\,\widehat{g}\,(\eta)\,e^{2\pi i x(\xi+\eta)}d\xi d\eta$$

be the cone operator associated with the half plane

$$P_\theta=\left\{(\xi,\eta)\in\mathbb{R}^2:\xi\tan\theta-\eta>0\right\}.$$

In [6], Gilbert and Nahmod proved the following interesting result.

Theorem C ([6])**.** *Let $m = m(\xi, \eta)$ be a function having derivatives of all orders in the half plane P_θ such that for any $\beta \in \mathbb{Z}^+ \times \mathbb{Z}^+$,*

$$\left|D^\beta m(\xi,\eta)\right| \le C\left(\mathrm{dist}\left((\xi,\eta), \partial P_\theta\right)\right)^{-|\beta|},\ |\beta| > 0.$$

Then, if ∂P_θ is not one of the coordinate axes and $\theta \ne -\frac{\pi}{4}$, C_{P_θ} is bounded from $L^r(\mathbb{R}) \times L^q(\mathbb{R})$ into $L^p(\mathbb{R})$ with $\frac{1}{q} + \frac{1}{r} = \frac{1}{p} < \frac{3}{2}$.

The importance of the above result has to do, in particular, with its possible extensions to the multilinear setting, as done in the work of Muscalu, Tao, and Thiele [15] and to the x-variable setting (that is the multiplier depends on x, ξ, η) initiated in the work of Bényi, Nahmod and Torres [1]; see also [2].

In this paper, we will use Theorem C to study the hyper bilinear Hilbert transform

$$H_\alpha(f,g)(x) = p.v.\pi^{-1}\int_{\mathbb{R}} f(x-t)\,g(x+t)\,t^{-1}|t|^{-\alpha}\,dt,\ 0 \le \alpha < 1.$$

One easily checks that, up to a constant multiple, the symbol of H_α is $m(\xi,\eta) = i\ \mathrm{sgn}(\xi-\eta)|\xi-\eta|^\alpha$, which is continuous, but not smooth. Also, the periodic version of H_α is

$$\widetilde{H}_\alpha\left(\widetilde{f},\widetilde{g}\right)(x) = i\sum_k\sum_\nu \mathrm{sgn}(k-\upsilon)|k-\upsilon|^\alpha a_k b_\upsilon e^{2\pi i x(k+\upsilon)}.$$

One of the main purposes of this paper is to study the L^p boundedness of H_α. Then, using a transference method, we obtain an analog for the periodic version $\widetilde{H_\alpha}$. Our first result is stated in the following theorem:

Theorem 1. *If $0 \le \alpha < 1$, $1 < q, r < \infty$, and $\frac{1}{q} + \frac{1}{r} = \frac{1}{p} < \frac{3}{2}$, then we have*

$$\|H_\alpha(f,g)\|_{L^p(\mathbb{R})} \le C\|f\|_{L^q_\alpha(\mathbb{R})}\|g\|_{L^r(\mathbb{R})} + \|f\|_{L^q(\mathbb{R})}\|g\|_{L^r_\alpha(\mathbb{R})};$$
$$\left\|\widetilde{H}_\alpha\left(\widetilde{f},\widetilde{g}\right)\right\|_{L^p(T)} \le C\left\|\widetilde{f}\right\|_{L^q_\alpha(T)}\|\widetilde{g}\|_{L^r(T)} + \left\|\widetilde{f}\right\|_{L^q(T)}\|\widetilde{g}\|_{L^r_\alpha(T)},$$

where L^p_α is the homogeneous Sobolev L^p space of order α.

Remark. In the theorem, if we take $g(x) = 1$ and $r = \infty$, then we obtain results for the classical hyper Hilbert transform (see [8]).

Other interesting operators are bilinear operators with non-singular kernels. For simplicity, we introduce the bilinear fractional integral F_α on $\mathbb{R}^n$ studied by Kenig and Stein:

$$F_\alpha(f,g)(x) = \int_{\mathbb{R}^n} f(x+t)\,g(x-t)\,|t|^{\alpha-n}\,dt,\ 0 < \alpha < n.$$

In [12], Kenig and Stein established the following theorem (see also [4] for the study of a rough bilinear fractional integral).

Theorem D. *Assume that* $0 < \alpha < n$, $\frac{1}{r} + \frac{1}{q} > \frac{\alpha}{n}$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - \frac{\alpha}{n}$, *and* $1 \le q, r \le \infty$. *Then*

(a) *if* $1 < q$, r, *then*

$$\|F_\alpha(f,g)\|_{L^p(\mathbb{R}^n)} \le C \|f\|_{L^q(\mathbb{R}^n)} \|g\|_{L^r(\mathbb{R}^n)};$$

(b) *if* $1 \le q$, r *and either* q *or* r *is one, then*

$$\|F_\alpha(f,g)\|_{L^{p,\infty}(\mathbb{R}^n)} \le C \|f\|_{L^q(\mathbb{R}^n)} \|g\|_{L^r(\mathbb{R}^n)}.$$

After a simple computation, one sees that, up to a constant, the symbol of F_α is $|\xi - \eta|^{-\alpha}$, $\xi, \eta \in \mathbb{R}^n$. Thus the periodic version $\widetilde{F}_\alpha$ of F_α on the n-torus T^n is defined by

$$\widetilde{F}_\alpha\left(\widetilde{f}, \widetilde{g}\right)(x) = \sum_{(k,j)\in\mathbb{Z}^n\times\mathbb{Z}^n : k\neq j} |k-j|^{-\alpha} a_k b_j e^{2\pi i\langle x,(k+j)\rangle}. \tag{1}$$

Recall that on a compact Lie group G, following Stein [16, p. 58], the Riesz potential on G is defined by (see [3])

$$I_\alpha(f)(x) = \int_G f\left(xy^{-1}\right) K_\alpha(y)\, dy,$$

where $K_\alpha(y)$ is the kernel function defined by

$$K_\alpha(y) = -\Gamma\left(\frac{\alpha}{2}\right)^{-1} \int_0^\infty t^{\frac{\alpha}{2}} \nabla^2 W_t(y)\, dt \tag{2}$$

and $W_t(y)$ is the heat kernel on G. We can use $K_\alpha(y)$ to define the bilinear Riesz potential

$$I_\alpha(f,g)(x) = \int_G f(xy)\, g\left(xy^{-1}\right) K_\alpha(y)\, dy. \tag{3}$$

Taking Fourier series, one easily checks that, in the distribution sense, the definitions of (1) and (3) are equivalent if one takes G to be the n-torus T^n. Thus, naturally, we expect that we may use a transference method (for instance see [5] for theorems of DeLeeuw type) to transfer the boundedness result of F_α to those of $\widetilde{F}_\alpha$. However, it is known that, in general, the classical transference method fails even in the linear case if $p \neq q$ (see [10], for example). To overcome this obstacle, we will use a combination of the classical transference method and methods used in [12]. We establish the following periodic analog of Theorem D.

Theorem 2. *Assume that* $0 < \alpha < n$, $\frac{1}{r} + \frac{1}{q} > \frac{\alpha}{n}$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - \frac{\alpha}{n}$, *and* $1 \le q, r \le \infty$. *Then*

(a) *if* $1 < q$, r, *then*

$$\left\|\widetilde{F}_\alpha\left(\widetilde{f}, \widetilde{g}\right)\right\|_{L^p(T^n)} \le C \left\|\widetilde{f}\right\|_{L^q(T^n)} \|\widetilde{g}\|_{L^r(T^n)};$$

(b) *if* $1 \le q, r$ *and either* q *or* r *is one, then*

$$\left\|\widetilde{F}_\alpha\left(\widetilde{f}, \widetilde{g}\right)\right\|_{L^{p,\infty}(T^n)} \le C \left\|\widetilde{f}\right\|_{L^q(T^n)} \|\widetilde{g}\|_{L^r(T^n)}.$$

The plan of the paper is as follows: in Section 2, we recall the definition of homogeneous Sobolev spaces and state an easy lemma that will be needed later. Two main theorems, Theorem 1 and Theorem 2, are proved In Sections 3 and 4, respectively. We use the letter "C" to denote (possibly different) constants that are independent of the essential variables in the argument.

2. Sobolev spaces and Lemma 1

Let Φ be a fixed function in $S(\mathbb{R}^n)$ such that the Fourier transform $\widehat{\Phi}$ of Φ has support in $\left\{\xi \in \mathbb{R} : \frac{1}{2} \leq |\xi| \leq 2\right\}$ and satisfies $\left|\widehat{\Phi}(\xi)\right| \geq C > 0$ if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$. The homogeneous Sobolev space $L^p_\alpha(\mathbb{R}^n)$ of order α is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f\|_{L^p_\alpha(\mathbb{R}^n)} = \left\| \left\{ \int_0^\infty \left(t^{-\alpha} |\Phi_t * f(x)|\right)^2 t^{-1} dt \right\}^{\frac{1}{2}} \right\|_{L^p_x(\mathbb{R}^n)} < \infty,\ 1 < p < \infty.$$

The homogeneous Sobolev space $L^p_\alpha(T^n)$ of order α is the collection of all $\widetilde{f} \in S'(T^n)$ such that

$$\left\|\widetilde{f}\right\|_{L^p_\alpha(T^n)} = \left\| \left\{ \int_0^\infty \left(t^{-\alpha} \left|\widetilde{\Phi}_t * \widetilde{f}(x)\right|\right)^2 t^{-1} dt \right\}^{\frac{1}{2}} \right\|_{L^p_x(T^n)} < \infty,\ 1 < p < \infty.$$

In the above definition,

$$\Phi_t(x) = t^{-n}\Phi\left(\frac{x}{t}\right) \text{ and } \widetilde{\Phi}_t(x) = \sum_{k \in \mathbb{Z}^n} t^{-n}\Phi\left(\frac{x+k}{t}\right).$$

Since the definition is independent of the choice of Φ, it is easy to check that

$$\|R_\alpha f\|_{L^p(\mathbb{R}^n)} \cong \|f\|_{L^p_\alpha(\mathbb{R}^n)},\ \text{with } (R_\alpha f)\hat{}\,(\xi) = |\xi|^\alpha \widehat{f}(\xi).$$

Also, if α is a positive integer, then $\|f\|_{L^p_\alpha(\mathbb{R}^n)} \cong \|D^\alpha f\|_{L^p(\mathbb{R}^n)}$.

Let δ be a positive number less than $1/2$, and define the n-cells Q and Ω_δ by

$$Q = \left[-\frac{1}{2}, \frac{1}{2}\right]^n, \quad \Omega_\delta = \left[-\frac{1}{2} - \delta, \frac{1}{2} + \delta\right]^n.$$

Q is the fundamental cube on which

$$\int_{T^n} \widetilde{f}(x)\, dx = \int_Q \widetilde{f}(x)\, dx, \quad \forall \widetilde{f} \in L^1(T^n).$$

Let Ψ be a function in $S(R^n)$ satisfying $\mathrm{supp}(\Psi) \subseteq \Omega_\delta,\ 0 \leq \Psi(x) \leq 1$ and $\Psi(x) \equiv 1$ on Q. We denote $\Psi^{\frac{1}{N}}(x) = \Psi\left(\frac{x}{N}\right)$ for an integer N. The following lemma can be found in [5].

Lemma 1. *Let B be a bilinear operator with symbol $m(\xi,\eta)$. For any $C^{\infty}(T^n)$ functions $\widetilde{f}(x)=\sum_{k\in\mathbb{Z}^n}a_k e^{2\pi i\langle k,x\rangle}$ and $\widetilde{g}(x)=\sum_{\upsilon\in\mathbb{Z}^n}b_\upsilon e^{2\pi i\langle \upsilon,x\rangle}$, one has*

$$\Psi\left(\frac{x}{N}\right)^2\widetilde{B}\left(\widetilde{f},\widetilde{g}\right)(x)-B\left(\Psi^{\frac{1}{N}}\widetilde{f},\Psi^{\frac{1}{N}}\widetilde{g}\right)(x)=-E_N\left(\widetilde{f},\widetilde{g}\right)(x),$$

where the error term $E_N(\widetilde{f},\widetilde{g})(x)$ is equal to

$$\sum_k\sum_\nu a_k b_\upsilon e^{2\pi i\langle k+\upsilon,x\rangle}\int_{\mathbb{R}^{2n}}\widehat{\Psi}(u)\,\widehat{\Psi}(v)\,\Delta_N m_{k,\upsilon}(u,v)\,e^{2\pi i\left\langle u+v,\frac{x}{N}\right\rangle}dudv,$$

and

$$\Delta_N m_{k,\upsilon}(u,v)=m\left(k+\frac{u}{N},\upsilon+\frac{v}{N}\right)-m(k,\upsilon).$$

Proof. Check the Fourier transforms on both sides of the equality involving B and the error E_N. □

3. Proof of Theorem 1

The ideas used in the proof of part (a) of our theorem are reminiscent of the well-established techniques involving commutator type estimates that go back to the work of Kato and Ponce [11] and the extension of the Leibniz rule for fractional derivatives to the general setting of bilinear pseudodifferential operators in [1]. By symmetry, we can assume

$$H_\alpha(f,g)(x)=\int_{\xi>\eta}(\xi-\eta)^\alpha\,\widehat{f}(\xi)\,\widehat{g}(\eta)\,e^{2\pi i x(\xi+\eta)}d\xi d\eta.$$

Thus H_α is a cone operator C_{P_θ} with $\theta=\frac{\pi}{4}$. We will use Theorem C to obtain the boundedness of H_α. We also note dist $\left((\xi,\eta),\partial P_{\frac{\pi}{4}}\right)=|\xi-\eta|$ for any point (ξ,η). We partition the unity as

$$1=\chi_1(\xi,\eta)+\chi_2(\xi,\eta)+\chi_3(\xi,\eta)+\chi_4(\xi,\eta)+\chi_5(\xi,\eta),$$

where $\chi_1(\xi,\eta)$ is supported on the region $\{\xi>\eta:|\xi|\geq 4|\eta|\}$ and equal to 1 on the region $\{\xi>\eta:|\xi|\geq 8|\eta|\}$, $\chi_2(\xi,\eta)$ is supported on the region $\{\xi>\eta:|\eta|\geq 4|\xi|\}$ and equal to 1 on the region $\{\xi>\eta:|\eta|\geq 8|\xi|\}$ and χ_3, χ_4, χ_5 are supported on regions Ω_3, Ω_4, Ω_5 in which $|\xi|\cong|\eta|$, and such that we have bounds

$$\left|\nabla^j\chi_i(\xi,\eta)\right|\leq C\left(\xi^2+\eta^2\right)^{-\frac{j}{2}}$$

for all $j\geq 0$ and $i=1,2,3,4,5$. We can then partition

$$H_\alpha(f,g)(x)=\sum_{j=1}^{5}T_j(f,g)(x),$$

where

$$T_j(f,g)(x)=\int_{\xi>\eta}(\xi-\eta)^\alpha\,\chi_j(\xi,\eta)\,\widehat{f}(\xi)\,\widehat{g}(\eta)\,e^{2\pi i x(\xi+\eta)}d\xi d\eta.$$

We will estimate each T_j. For $j = 1$, we write

$$T_1(f,g)(x) = \int_{\xi>\eta} m_1(\xi,\eta)(R_\alpha f)\,\hat{}\,(\xi)\,\widehat{g}(\eta)\,e^{2\pi i x(\xi+\eta)}d\xi d\eta$$

with

$$m_1(\xi,\eta) = (\xi-\eta)^\alpha \chi_1(\xi,\eta)\,|\xi|^{-\alpha}.$$

Note that in the support of m_1, one has

$$|\xi| \cong |\xi-\eta| \cong \left(\xi^2+\eta^2\right)^{\frac{1}{2}}.$$

Thus m_1 satisfies the condition in Theorem C, so that we have

$$\|T_1(f,g)\|_{L^p(\mathbb{R})} \le C\,\|R_\alpha f\|_{L^q(\mathbb{R})}\,\|g\|_{L^r(\mathbb{R})} \cong C\,\|f\|_{L^q_\alpha(\mathbb{R})}\,\|g\|_{L^r(\mathbb{R})}.$$

A similar argument shows that

$$\|T_2(f,g)\|_{L^p(\mathbb{R})} \le C\,\|f\|_{L^q(\mathbb{R})}\,\|g\|_{L^r_\alpha(\mathbb{R})}.$$

In the supports of m_3, m_4, m_5, we have $|\xi| \cong |\eta| \cong \left(\xi^2+\eta^2\right)^{\frac{1}{2}}$, and $|\xi| \ge c\,|\xi-\eta|$ for some $c>0$. Thus for $j = 3,4,5$, and any nonnegative α_1, α_2 with $\alpha_1+\alpha_2 = \alpha$, we have the better estimates

$$\|T_j(f,g)\|_{L^p(\mathbb{R})} \le C\,\|f\|_{L^q_{\alpha_1}(\mathbb{R})}\,\|g\|_{L^r_{\alpha_2}(\mathbb{R})}.$$

Part (a) of Theorem 1 is proved. We now prove part (b) of the theorem. Taking $\delta = \frac{1}{4}$ and applying Lemma 1 on the operators H_α and $\widetilde{H}_\alpha$, one easily sees that $E_N(\widetilde{f},\widetilde{g})(x)$ converges to zero uniformly as $N \to \infty$. Thus, following an idea in the proof of Theorem 3 of [5] and noting that $\widetilde{H}_\alpha(\widetilde{f},\widetilde{g})$ is a periodic function, one has

$$\left\|\widetilde{H}_\alpha\left(\widetilde{f},\widetilde{g}\right)\right\|_{L^p(T)} \cong \left\{N^{-1}\int_{-N/2}^{N/2} \mid \widetilde{H}_\alpha\left(\widetilde{f},\widetilde{g}\right)(x)\mid^p dx\right\}^{1/p}$$

for any integer N. By the choice of Ψ, we have

$$\left\|\widetilde{H}_\alpha\left(\widetilde{f},\widetilde{g}\right)\right\|_{L^p(T)} \cong \left\{N^{-1}\int_{-N/2}^{N/2} \mid \Psi(x/N)^2\widetilde{H}_\alpha\left(\widetilde{f},\widetilde{g}\right)(x)\mid^p dx\right\}^{1/p}.$$

By Lemma 1, we now have

$$\left\|\widetilde{H}_\alpha\left(\widetilde{f},\widetilde{g}\right)\right\|_{L^p(T)} \preceq \lim_{N\to\infty} N^{-\frac{1}{p}}\left\|H_\alpha\left(\Psi^{\frac{1}{N}}\widetilde{f},\Psi^{\frac{1}{N}}\widetilde{g}\right)\right\|_{L^p(\mathbb{R})}.$$

Thus, by (a) of the theorem, we know that $\left\|\widetilde{H}_\alpha(\widetilde{f},\widetilde{g})\right\|_{L^p(T)}$ is bounded by

$$C\lim_{N\to\infty} N^{-\frac{1}{p}}\left\{\left\|\Psi^{\frac{1}{N}}\widetilde{f}\right\|_{L^q_\alpha(\mathbb{R})}\left\|\Psi^{\frac{1}{N}}\widetilde{g}\right\|_{L^r(\mathbb{R})} + \left\|\Psi^{\frac{1}{N}}\widetilde{f}\right\|_{L^q(\mathbb{R})}\left\|\Psi^{\frac{1}{N}}\widetilde{g}\right\|_{L^r_\alpha(\mathbb{R})}\right\}.$$

By the choice of $\Psi^{1/N}(x)$ and noting that $\widetilde{f}$ is a periodic function, it is easy to see that for any $q > 0$

$$N^{-\frac{1}{q}} \left\| \Psi^{\frac{1}{N}} \widetilde{f} \right\|_{L^q(\mathbb{R})} \preceq \left(\frac{1}{N} \int_{-N}^{N} | \widetilde{f}(x) |^q \, dx \right)^{1/q} \simeq \| \widetilde{f} \|_{L^q(T)} \tag{4}$$

uniformly for all positive integers N. Thus, to complete the proof, it remains to show

$$\lim_{N\to\infty} N^{-1} \left\| \Psi^{\frac{1}{N}} \widetilde{f} \right\|^q_{L^q_\alpha(\mathbb{R})} \le C \left\| \widetilde{f} \right\|^q_{L^q_\alpha(T)}.$$

By the definition, we write

$$\begin{aligned} N^{-1} \left\| \Psi^{\frac{1}{N}} \widetilde{f} \right\|^q_{L^q_\alpha(\mathbb{R})} &= N^{-1} \int\limits_{|y|\ge 4N} \left\{ \int_0^\infty \left(t^{-\alpha} \left| \Phi_t * \left(\Psi^{\frac{1}{N}} \widetilde{f} \right)(y) \right| \right)^2 t^{-1} dt \right\}^{\frac{q}{2}} dy \\ &\quad + N^{-1} \int\limits_{|y|<4N} \left\{ \int_0^\infty \left(t^{-\alpha} \left| \Phi_t * \left(\Psi^{\frac{1}{N}} \widetilde{f} \right)(y) \right| \right)^2 t^{-1} dt \right\}^{\frac{q}{2}} dy \\ &= L(N) + R(N). \end{aligned}$$

To estimate $L(N)$, note that $\Psi^{\frac{1}{N}}(x)$ has compact support on $\left[-\frac{3N}{4}, \frac{3N}{4}\right]$. Thus if $|y| \ge 4N$, then $|y - x| \ge 2N$. Let $M < |y|$ be a positive number. We further write

$$\begin{aligned} L(N) &= N^{-1} \int\limits_{|y|\ge 4N} \left\{ \int_M^{|y|} \left(t^{-\alpha} \left| \Phi_t * \left(\Psi^{\frac{1}{N}} \widetilde{f} \right)(y) \right| \right)^2 t^{-1} dt \right\}^{\frac{q}{2}} dy \\ &\quad + N^{-1} \int\limits_{|y|\ge 4N} \left\{ \int_{|y|}^\infty \left(t^{-\alpha} \left| \Phi_t * \left(\Psi^{\frac{1}{N}} \widetilde{f} \right)(y) \right| \right)^2 t^{-1} dt \right\}^{\frac{q}{2}} dy \\ &\quad + N^{-1} \int\limits_{|y|\ge 4N} \left\{ \int_0^M \left(t^{-\alpha} \left| \Phi_t * \left(\Psi^{\frac{1}{N}} \widetilde{f} \right)(y) \right| \right)^2 t^{-1} dt \right\}^{\frac{q}{2}} dy \\ &= L_1(N) + L_2(N) + L_3(N). \end{aligned}$$

Note that when $|y| \ge 4N$ and $x \in \operatorname{supp}\left(\Psi^{\frac{1}{N}}\right)$, one has

$$|\Phi_t(x-y)| \le C\,(t + |y-x|)^{-1} \le C\,(t+|y|)^{-1}.$$

Thus, by a simple computation and (4),

$$\begin{aligned} L_2(N) &\le N^{-1} \int\limits_{|y|\ge 4N} \left\{ \int_{|y|}^\infty t^{-2\alpha-3} dt \right\}^{\frac{q}{2}} dy \left\| \Psi^{\frac{1}{N}} \widetilde{f} \right\|^q_{L^1(\mathbb{R})} \\ &\preceq N^{-q\alpha} \left\| \widetilde{f} \right\|^q_{L^1(T)} = o(1), \text{ as } N \to \infty. \end{aligned}$$

To estimate $L_3(N)$, we note that $\Phi_t(x) = O(t|x|^{-2})$. Then

$$L_3(N) \cong N^{-1} \int\limits_{|y|\geq 4N} |y|^{-2q} \left(\int_0^M t^{-\alpha+1} dt \right)^{\frac{q}{2}} dy \left\| \Psi^{\frac{1}{N}} \widetilde{f} \right\|_{L(\mathbb{R})}^q$$
$$\preceq M^{-q\alpha/2} \left\| \widetilde{f} \right\|_{L^1(T)}^q .$$

Also, it is easy to check that

$$L_1(N) \leq CN^{-1} \int\limits_{|y|\geq 4N} \left(\int_M^{|y|} (t+|y|)^{-2} t^{-1-2\alpha} dt \right)^{\frac{q}{2}} dy \left\| \Psi^{\frac{1}{N}} \widetilde{f} \right\|_{L(\mathbb{R})}^q$$
$$\preceq M^{-q\alpha} \left\| \widetilde{f} \right\|_{L^1(T)}^q .$$

Choosing $M = N^{1/2}$, we obtain that $\lim_{N\to\infty} L(N) = 0$. Thus, to finish the proof of the theorem, it remains to show

$$\lim_{N\to\infty} R(N) \leq C \left\| \widetilde{f} \right\|_{L^q_\alpha(T)}^q ,$$

where

$$R(N) = CN^{-1} \int\limits_{|y|<4N} \left\{ \int_0^\infty \left(t^{-\alpha} \left| \Phi_t * \left(\Psi^{\frac{1}{N}} \widetilde{f} \right)(y) \right| \right)^2 t^{-1} dt \right\}^{\frac{q}{2}} dy.$$

Choose both Ψ and Φ to be radial. For $\widetilde{f}(x) = \sum_k a_k e^{2\pi i k x}$, by the Plancherel theorem and an easy computation, we see that

$$\Phi_t * \left(\Psi^{\frac{1}{N}} \widetilde{f} \right)(y) \cong \sum a_k e^{2\pi i k y} \int_{\mathbb{R}} N \widehat{\Psi}(Nx) \left(\widehat{\Phi}(t(x+k)) - \widehat{\Phi}(tx) \right) e^{2\pi i x y} dx$$
$$+ C\Psi\left(\frac{y}{N}\right) \widetilde{\Phi}_t * \widetilde{f}(y) .$$

Thus, $R(N)$ is dominated by

$$CN^{-1} \int\limits_{|y|<4N} \left\{ \sum |a_k| \int_0^\infty \left(t^{-\alpha} \int_{\mathbb{R}} \left| N\widehat{\Psi}(Nx) \left(\widehat{\Phi}(t(x+k)) - \widehat{\Phi}(tx) \right) \right| dx \right)^2 t^{-1} dt \right\}^{\frac{q}{2}} dy$$
$$+ CN^{-1} \int\limits_{|y|<4N} \left\{ \int_0^\infty \left(t^{-\alpha} \left| \Psi\left(\frac{y}{N}\right) \widetilde{\Phi}_t * \widetilde{f}(y) \right| \right)^2 t^{-1} dt \right\}^{\frac{q}{2}} dy.$$

Since $\{a_k\}$ tends to zero rapidly as $k \to \infty$, one easily checks that the first term above converges to zero as $N \to \infty$. The second term is bounded by

$$C \int\limits_Q \left\{ \int_0^\infty \left(t^{-\alpha} \left| \widetilde{\Phi}_t * \widetilde{f}(y) \right| \right)^2 t^{-1} dt \right\}^{\frac{q}{2}} dy \preceq C \left\| \widetilde{f} \right\|_{L^q_\alpha(T)}^q .$$

The proof is completed.

4. Proof of Theorem 2

Let Φ be the function as in the definition of L^p_α and let $\phi_k(\xi) = \widehat{\Phi}\left(2^k\xi\right)$ and $\phi(\xi) = \widehat{\Phi}\left(\xi\right)$. Then

$$\text{supp}\phi_k \subseteq \left\{\xi : 2^{-k} \leq |\xi| \leq 2^{-k+1}\right\}.$$

We may assume that ϕ is radial and $\sum_{k\in\mathbb{Z}} \phi_k\left(x\right) \equiv 1$ for all $x \in \mathbb{R}^n$. Thus for any $f, g \in S\left(\mathbb{R}^n\right)$, we have

$$F_\alpha\left(f,g\right)\left(x\right) = \sum_k \int_{\mathbb{R}^n} f\left(x+t\right) g\left(x-t\right) |t|^{\alpha-n} \phi_k\left(t\right) dt = \sum_{k\in\mathbb{Z}} L_k\left(f,g\right)\left(x\right).$$

Similarly, for any $\widetilde{f}, \widetilde{g} \in S\left(T^n\right)$, we have

$$\begin{aligned}\widetilde{F}_\alpha\left(\widetilde{f},\widetilde{g}\right)\left(x\right) &= \sum_k \int_Q \widetilde{f}\left(x+t\right)\widetilde{g}\left(x-t\right) |t|^{\alpha-n} K_\alpha\left(t\right)\phi_k\left(t\right) dt \\ &= \sum_{k=0}^{\infty} \widetilde{L}_k\left(\widetilde{f},\widetilde{g}\right)\left(x\right).\end{aligned}$$

The following is Lemma 5 in [12]:

Lemma 2 (Kenig-Stein)**.**

(i) $\left\|L_k\left(f,g\right)\right\|_{L^{\frac{1}{2}}\left(\mathbb{R}^n\right)} \leq C2^{-k\alpha} \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}$;

(ii) $\left\|L_k\left(f,g\right)\right\|_{L^1\left(\mathbb{R}^n\right)} \leq C2^{k(n-\alpha)} \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}$.

Using the above lemma, Kenig and Stein proved that F_α is bounded from $L^1\left(\mathbb{R}^n\right) \times L^1\left(\mathbb{R}^n\right)$ to $L^{q,\infty}\left(\mathbb{R}^n\right)$ with $\frac{1}{q} = 2 - \frac{\alpha}{n}$. Also, it is trivial to see that

$$F_\alpha\left(f,g\right)\left(x\right) \leq \|f\|_\infty I_\alpha\left(f\right)\left(x\right), \quad F_\alpha\left(f,g\right)\left(x\right) \leq \|g\|_\infty I_\alpha\left(g\right)\left(x\right),$$

where

$$I_\alpha\left(f\right)\left(x\right) = \int_{\mathbb{R}^n} f\left(x-y\right)|y|^{-n+\alpha} dy$$

is the ordinary fractional integral. Thus one easily obtains the boundedness of $F_\alpha : L^\infty \times L^r \to L^q$ and $L^r \times L^\infty \to L^q$ with $\frac{1}{q} = \frac{1}{r} - \frac{\alpha}{n}$. Then Theorem D follows by complex bilinear interpolations as in the work of [9], see also the work of [7].

Now return to the case $\widetilde{F}_\alpha(\widetilde{f},\widetilde{g})\left(x\right)$. We may assume that both $\widetilde{f}$ and $\widetilde{g}$ are nonnegative. An easy computation shows that the kernel defined in (2) satisfies $|K_\alpha\left(y\right)| \leq C\,|y|^{-n+\alpha}$. Thus we also have

$$\widetilde{F}_\alpha\left(\widetilde{f},\widetilde{g}\right)\left(x\right) \leq \left\|\widetilde{f}\right\|_{L^\infty(T^n)} \int_Q \widetilde{g}\left(x-t\right)|K_\alpha\left(t\right)| dt,$$

$$\widetilde{F}_\alpha\left(\widetilde{f},\widetilde{g}\right)(x) \le \|\widetilde{g}\|_{L^\infty(T^n)} \int_Q \widetilde{f}(x-t)\,|K_\alpha(t)|\,dt,$$

where

$$J_\alpha\left(\widetilde{f}\right)(x) = \int_Q \widetilde{f}(x-t)\,|K_\alpha(t)|\,dt$$

is the ordinary Riesz potential on the n-torus whose boundedness of $L^\infty \times L^r \to L^q$ and $L^r \times L^\infty \to L^q$ with $\frac{1}{q} = \frac{1}{r} - \frac{\alpha}{n}$ is well known. Thus to prove Theorem 2, by interpolation, it suffices to establish the boundedness of $L^1(T^n) \times L^1(T^n) \to L^{q,\infty}(T^n)$ with $\frac{1}{q} = 2 - \frac{\alpha}{n}$. To this end, by checking the proof in [12], one only needs to establish the following lemma which is an analogous version of Lemma 2:

Lemma 3.

(i) $\left\|\widetilde{L}_k\left(\widetilde{f},\widetilde{g}\right)\right\|_{L^{\frac{1}{2}}(T^n)} \le C2^{-k\alpha}\left\|\widetilde{f}\right\|_{L^1(T^n)}\|\widetilde{g}\|_{L^1(T^n)}$;

(ii) $\left\|\widetilde{L}_k\left(\widetilde{f},\widetilde{g}\right)\right\|_{L^1(\mathbb{R}^n)} \le C2^{k(n-\alpha)}\left\|\widetilde{f}\right\|_{L^1(T^n)}\|\widetilde{g}\|_{L^1(T^n)}$.

Proof. The inequality in (ii) is an easy consequence of the Fubini theorem. To prove (i) we will use Lemma 1 to transfer the result of Lemma 2. We may assume that both $\widetilde{f}$ and $\widetilde{g}$ are nonnegative. Thus we have

$$\left|\widetilde{L}_k\left(\widetilde{f},\widetilde{g}\right)(x)\right| \le C\int_Q \widetilde{f}(x+t)\,\widetilde{g}(x-t)\,|t|^{-n+\alpha}\,\phi_k(t)\,dt.$$

Without loss of generality, we may again write

$$\left|\widetilde{L}_k\left(\widetilde{f},\widetilde{g}\right)(x)\right| = \int_Q \widetilde{f}(x+t)\,\widetilde{g}(x-t)\,|t|^{-n+\alpha}\,\phi_k(t)\,dt.$$

By Lemma 1, it is easy to see that

$$\begin{aligned}
&\left\|\widetilde{L}_k\left(\widetilde{f},\widetilde{g}\right)\right\|_{L^{\frac{1}{2}}(T^n)}^{\frac{1}{2}} \\
&\cong N^{-n}\int_{NQ}\Psi\left(\frac{x}{N}\right)^2\left|\widetilde{L}_k\left(\widetilde{f},\widetilde{g}\right)(x)\right|^{\frac{1}{2}}dx \\
&\le N^{-n}\int_{\mathbb{R}^n}\left|L_k\left(\Psi^{\frac{1}{N}}\widetilde{f},\Psi^{\frac{1}{N}}\widetilde{g}\right)(x)\right|^{\frac{1}{2}}dx + N^{-n}\int_{NQ}\left|E_N\left(\widetilde{f},\widetilde{g}\right)(x)\right|^{\frac{1}{2}}dx.
\end{aligned}$$

We may check that E_N converges to zero uniformly on$x$ as $N \to \infty$. Letting $N \to \infty$ and by (i) of Lemma 2, we obtain that

$$\begin{aligned}
\|\,\widetilde{L}_k\left(\widetilde{f},\widetilde{g}\right)\|_{L^{\frac{1}{2}}(T^n)} &\preceq \lim_{N\to\infty} N^{-2n}\left\{\int_{\mathbb{R}^n}\left|L_k\left(\Psi^{\frac{1}{N}}\widetilde{f},\Psi^{\frac{1}{N}}\widetilde{g}\right)(x)\right|^{\frac{1}{2}}dx\right\}^2 \\
&\preceq \lim_{N\to\infty} 2^{-k\alpha}\left\|N^{-n}\Psi^{\frac{1}{N}}\widetilde{f}\right\|_{L^1(\mathbb{R}^n)}\left\|N^{-n}\Psi^{\frac{1}{N}}\widetilde{g}\right\|_{L^1(\mathbb{R}^n)} \\
&\cong 2^{-k\alpha}\left\|\widetilde{f}\right\|_{L^1(T^n)}\|\widetilde{g}\|_{L^1(T^n)}.
\end{aligned}$$

The proof is finished. □

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