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L^p-BOUNDEDNESS FOR THE COMMUTATORS OF ROUGH OSCILLATORY SINGULAR INTEGRALS WITH NON-CONVOLUTION PHASES

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Abstract. In this paper, the author studies the k-th commutators of oscillatory singular integral operators with a BMO function and phases more general than polynomials. For $1 < p < \infty$, the L^p-boundedness of such operators are obtained provided their kernels belong to the spaces $L(\log^+ L)^{k+1}(S^{n-1})$. The results of the corresponding maximal operators are also established.

1. Introduction

Let us consider the following oscillatory singular integral operator

$$T_\Phi f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{i\Phi(x,y)} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy, \tag{1.1}$$

where $\mathbb{R}^n$ denotes the n-dimensional Euclidean space ($n \geq 2$), $\Phi(x,y)$ is a suitable mapping on $\mathbb{R}^n \times \mathbb{R}^n$, $\Omega(x)$ is homogeneous of degree zero and has mean value zero on the unit sphere S^{n-1} of $\mathbb{R}^n$. As well-known, operators of the type (1.1) have arisen in the study of singular integrals on lower dimensional varieties and the singular Radon transform. For the background information about these operators, we refer the readers to consult [12, 13, 14]. When $\Phi(x,y) = P(x,y)$ is a real-valued polynomial mapping on $\mathbb{R}^n \times \mathbb{R}^n$, we denote T_Φ by T_P. The class of operators T_P was first studied by Ricci and Stein [12]. They proved that the operator T_P is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ provided that $\Omega \in C^1(S^{n-1})$. Later on, the condition $\Omega \in C^1(S^{n-1})$ was relaxed to $\Omega \in L^q(S^{n-1})$ for some $q > 1$ by Lu and Zhang [10]. Subsequently, this result was improved by many authors (see [1, 2, 7, 9] et al.). It is worth pointing out that Al-Salman et al [1, 2] studied a more general class of oscillatory singular

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integral operators T_Φ for phase functions Φ of the form

$$\Phi(x,y)=\sum_{j=0}^{l}P_j(x)\phi_j(y-x), \tag{1.2}$$

where $\phi_j:\mathbb{R}^n\longrightarrow\mathbb{R}$ is a homogeneous function which is real analytic on S^{n-1}, and P_j is a real-valued polynomial on $\mathbb{R}^n$. It is clear that the class of such functions Φ contains properly the class of all real-valued polynomial mapping P on $\mathbb{R}^n\times\mathbb{R}^n$. Al-Salman et al [1, 2] proved that T_Φ is bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$, provided $\Omega\in L\log^+L(S^{n-1})$ or $B_q^{0,0}(S^{n-1})$ for some $q>1$, where $B_q^{0,0}$ denotes the block space introduced by Jiang and Lu [8].

The purpose of this paper is to study the higher order commutators related the oscillatory singular operators defined by (1.1). Let k be a positive integer, $b\in\mathrm{BMO}(\mathbb{R}^n)$. Define the k-th order commutator $T^\Phi_{b,k}$ generated by T_Φ and b as follows:

$$T^\Phi_{b,k}f(x)=\text{p.v.}\int_{\mathbb{R}^n}e^{i\Phi(x,y)}[b(x)-b(y)]^k\frac{\Omega(x-y)}{|x-y|^n}f(y)dy, \tag{1.3}$$

where $\Phi(x,y)$ satisfies (1.2). When $\Phi(x,y)=P(x,y)$ is a real-valued polynomial mapping on $\mathbb{R}^n\times\mathbb{R}^n$, we denote $T^\Phi_{b,k}$ by $T^P_{b,k}$.

For $b\in\mathrm{BMO}(\mathbb{R}_+)$ (the radial BMO function class), Ding and Lu [4] (resp., Lu and Wu [9]) gave the weighted L^p-boundedness of $T^P_{b,k}$ $(1<p<\infty)$, if $\Omega\in L\log^+L(S^{n-1})$ (resp., $\Omega\in B_q^{0,0}(S^{n-1})$). For the general $b\in\mathrm{BMO}(\mathbb{R}^n)$, Ding [3] showed that $T^P_{b,k}$ is bounded on $L^p(\mathbb{R}^n)\,(1<p<\infty)$ with bound $C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$ independent of the coefficients of $P(x,y)$ (also see [5]), if $\Omega\in\bigcup_{r>1}L^r(S^{n-1})$. Subsequently, Ma and Hu [11] extended the result in [3] to the case of $\Omega\in L(\log^+L)^{k+1}(S^{n-1})$ for $p=2$ (also see [15] for the other improvement). A natural question is whether $\Omega\in L(\log^+L)^{k+1}(S^{n-1})$ is also sufficient for implying the L^p-boundedness of $T^P_{b,k}$ for $p\neq2$, $1<p<\infty$. In this paper, we will give a affirmative answer for this question. In fact, we shall establish the more general result as follows.

Theorem 1.1. *Let $k\geq1$, $\mathbb{N}^0$ denote the set of all nonnegative integers, Ω be homogeneous of degree zero with mean value zero on S^{n-1}, $b\in\mathrm{BMO}(\mathbb{R}^n)$. Suppose that $\Omega\in L(\log^+L)^{k+1}(S^{n-1})$, $\{d_j,\,m_j:\ 0\leq j\leq l\}\subset\mathbb{N}^0$ and that $\Phi(x,y)=\sum_{j=0}^{l}P_j(x)\phi_j(y-x)$, where $\phi_j:\mathbb{R}^n\longrightarrow\mathbb{R}$ is a homogeneous of degree m_j which is real analytic on S^{n-1}, and $P_j(x)$ is a real-valued polynomial on $\mathbb{R}^n$ with degree d_j. If ϕ_j is constant function whenever $m_j=0$, then for $1<p<\infty$,*

$$\|T^\Phi_{b,k}f\|_p\leq C_p\left(1+\|\Omega\|_{L(\log^+L)^{k+1}(S^{n-1})}\right)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p,$$

where C_p is independent of the coefficients of the polynomials $\{P_j:\,0\leq j\leq l\}$.

Remark 1.1. From [1, 2], the class of such functions Φ in Theorem 1.1 contains properly the class of all real-valued polynomial mappings on $\mathbb{R}^n\times\mathbb{R}^n$. For example, for $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$,

$$\Phi(x,y) = \left\{ (y_n - x_n)\sin\left((y_n - x_n)^3 \left(\sum_{j=1}^{n} (y_j - x_j)^2 \right)^{-3/2} \right) \right\} \prod_{j=1}^{n} x_j^2$$

satisfies the assumptions in Theorem 1.1, but it is not a polynomial. Therefore, Theorem 1.1 extends the result of [11] by both expanding the range of the phase function "Φ" and the range of "p".

In addition, for the corresponding maximal operator defined by

$$T_{b,k}^{\Phi,*}f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} e^{i\Phi(x,y)}[b(x)-b(y)]^k \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \right|, \tag{1.4}$$

we will also establish the following result.

Theorem 1.2. *Under the same assumptions as in Theorem 1.1, we have*

$$\|T_{b,k}^{\Phi,*}f\|_p \le C_p \left(1 + \|\Omega\|_{L(\log^+ L)^{k+1}(S^{n-1})}\right) \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k \|f\|_p, \qquad 1 < p < \infty,$$

where C_p is independent of the coefficients of the polynomials $\{P_j : 0 \le j \le l\}$.

This paper is organized as follows. In Section 2, we shall give some auxiliary lemmas. The proof of Theorem 1.1 will be given in Section 3. Finally, we will prove Theorem 1.2 in Section 4. We would remark that our some ideas in the proof of the main theorem are taken from [2, 10, 12]. Throughout this paper, we always use the letter C to denote a positive constant that may vary at each occurrence but is independent of the essential variable.

2. Auxiliary lemmas

In this section, we give some auxiliary lemmas, which will be needed in the proof of our main result.

Lemma 2.1 (see [6])**.** *Let Ω, b, k be as in Theorem 1.1. The the maximal operator $M_{b,k}^{\Omega}$ defined by*

$$M_{b,k}^{\Omega}f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |b(x)-b(y)|^k |\Omega(x-y)f(y)| dy$$

satisfies

$$\left\| M_{b,k}^{\Omega} f \right\|_p \le C \left(1 + \|\Omega\|_{L(\log^+ L)^{k+1}(S^{n-1})}\right) \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k \|f\|_p.$$

Lemma 2.2 (see [6])**.** *Let $\widetilde{\Omega} \in L^\infty(S^{n-1})$ be homogeneous of degree zero, k be a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Define the operator $M_{\widetilde{\Omega};\,b,k}$ by*

$$M_{\widetilde{\Omega};\,b,k}f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |b(x)-b(y)|^k |\widetilde{\Omega}(x-y)f(y)| dy$$

and let

$$\lambda_{\widetilde{\Omega},k} = \inf\left\{\lambda > 0 : \quad \frac{\|\widetilde{\Omega}\|_1}{\lambda}\log^k\left(2 + \frac{\|\widetilde{\Omega}\|_\infty}{\lambda}\right) \le 1\right\}.$$

Then $M_{\widetilde{\Omega};b,k}$ *is bounded on* $L^p(\mathbb{R}^n)$ *with bound* $C\lambda_{\widetilde{\Omega},k}\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$ *for all* $1 < p < \infty$.

Lemma 2.3 (see [6])**.** *Let* Ω, b, k *be as in Theorem 1.1. Then the* k*-th order commutator of singular integral operator* $\overline{T}_{b,k}$ *defined by*

$$\overline{T}_{b,k}f(x) = \mathrm{p.v}\int_{\mathbb{R}^n}[b(x) - b(y)]^k\frac{\Omega(x-y)}{|x-y|^n}f(y)dy$$

and the corresponding maximal operator defined by

$$\overline{T}^*_{b,k}f(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon}[b(x) - b(y)]^k\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\right|$$

are bounded on $L^p(\mathbb{R}^n)$ *for* $1 < p < \infty$*, with norm bounded by*

$$C(1 + \|\Omega\|_{L(\log^+L)^{k+1}(S^{n-1})})\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Lemma 2.4. *Let* Ω, b, k, Φ *be as in Theorem 1.1. If* $T^\Phi_{b,k}$ *is bounded on* $L^p(\mathbb{R}^n)$ *for* $1 < p < \infty$*, with norm bounded by* $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$*. Then for any* $\varepsilon > 0$*, the truncated operator*

$$T^\Phi_{\varepsilon,b,k}f(x) = \mathrm{p.v.}\int_{|x-y|\le\varepsilon}e^{i\Phi(x,y)}[b(x) - b(y)]^k\frac{\Omega(x-y)}{|x-y|^n}f(y)dy$$

is also bounded on $L^p(\mathbb{R}^n)$ *with bound* $C(1+\|\Omega\|_{L(\log^+L)^{k+1}(S^{n-1})})\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$.

Proof. Decompose $\mathbb{R}^n$ as $\mathbb{R}^n = \cup_d I_d$, where each I_d is a cube having side length $\varepsilon/8n$ and these cubes $\{I_d\}$ have disjoint interiors. Set $f_d = f\chi_{I_d}$. Since the support of $T^\Phi_{\varepsilon,b,k}f_d$ is contained in a fixed multiple of I_d, the supports of the various terms $T^\Phi_{\varepsilon,b,k}f_d$ have bounded overlaps and so we have

$$\|T^\Phi_{\varepsilon;b,k}f\|^p_p \le C\sum_d\|T^\Phi_{\varepsilon;\,b,k}f_d\|^p_p.$$

Thus we may assume that $\mathrm{supp}(f) \subset I$ for some cube I with side length $\varepsilon/8n$ and center at x_0. Write

$$\begin{aligned}&\int_{\mathbb{R}^n}|T^\Phi_{\varepsilon;\,b,k}f(x)|^pdx\\ &= \left(\int_{|x-x_0|\le\varepsilon/4n} + \int_{\varepsilon/4n<|x-x_0|\le3\varepsilon} + \int_{3\varepsilon<|x-x_0|}\right)|T^\Phi_{\varepsilon;\,b,k}f(x)|^pdx.\end{aligned}$$

Since $|x - x_0| < \varepsilon/4n$ and $|y - x_0| \le \varepsilon/8n$ imply $|x - x_0| \le \varepsilon$, we have $T^\Phi_{\varepsilon;\,b,k}f(x) = T^\Phi_{b,k}f(x)$. Thus, for the first term, by the L^p-boundedness of $T^\Phi_{b,\,k}$, the result holds in this case. When $\varepsilon/4n < |x-x_0| \le 3\varepsilon$, by $|y-x_0| \le \varepsilon/8n$, we have $c_0\varepsilon \le |x-y| \le c_1\varepsilon$ for some constants c_0 and c_1. Therefore

$$|T^{\Phi}_{\varepsilon;b,k}f(x)| \le \int_{c_0\varepsilon\le|x-y|\le c_1\varepsilon} |b(x)-b(y)|^k \frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|dy \le CM^{\Omega}_{b,k}f(x).$$

By Lemma 2.1, we get

$$\|T^{\Phi}_{\varepsilon;b,k}f\|_p \le C\|M^{\Omega}_{b,k}f\|_p \le C\left(1+\|\Omega\|_{L(\log^+ L)^{k+1}(S^{n-1})}\right)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p,$$

which is the estimate for the second term. When $3\varepsilon < |x-x_0|$, we get $T^{\Phi}_{\varepsilon;b,k}f(x) = 0$ and complete the proof of Lemma 2.4. □

3. Proof of Theorem 1.1

Employing the ideas of [2, 10], which originated from [12], we shall use induction on

$$d(\Phi) := \inf \max_{0\le j\le l}\{d_j + m_j\},$$

where the infimum is taken over all representations of the form $\Phi(x,y) = \sum_{j=0}^{l} P_j(x)\phi_j(y-x)$ with d_j is the degree of P_j and m_j is the degree of homogeneity of ϕ_j. Following the notation and the procedure in the proof of [2, Theorem 1.1], we proceed now to the proof as follows.

It is clear that if $d(\Phi) = 0$, then $|T^{\Phi}_{b,k}f(x)| = |\overline{T}_{b,k}f(x)|$. Therefore, Theorem 1.1 directly follows from Lemma 2.3. Now we assume that Theorem 1.1 holds for all Φ with $d(\Phi) \le N$. As the same as in [2], given $\Phi(x,y) = \sum_{j=0}^{l} P_j(x)\phi_j(y-x)$ with $d(\Phi) = N+1$, let $j_1, j_2, \ldots, j_k$ be all $0 \le j \le l$ with $d_j + m_j = N+1$. For $1 \le s \le k$, let

$$h_s(x) = \sum_{|\alpha_{j_s}|=d_{j_s}} a_{\alpha_{j_s}} x^{\alpha_{j_s}} \quad \text{and} \quad H(x,y) = \sum_{s=1}^{k} h_s(x)\phi_{j_s}(y-x).$$

Without loss of generality, we may assume that $\deg(\phi_{j_s}) = m_{j_s} > 0$. A straightforward calculation shows that

$$H(x,y) = \sum_{\mu=1}^{M} \lambda_\mu \psi_\mu(x,y) \tag{3.1}$$

for some integers $M > 0$, constants $\{\lambda_\mu : 1 \le \mu \le M\}$ with

$$\sum_{\mu=1}^{M} |\lambda_\mu| = \sum_{s=1}^{k} \sum_{|\alpha_{j_s}|=d_{j_s}} |a_{\alpha_{j_s}}|, \tag{3.2}$$

and functions ψ_μ, $1 \le \mu \le M$, of the form $x^\alpha \eta(y-x)$ for some multi-index α and a homogeneous function η of degree $N+1-|\alpha|$ which is real analytic on S^{n-1}. Then

$$\Phi(x,y) = \sum_{\mu=1}^{M} \lambda_\mu \psi_\mu(x,y) + \sum_{0\le j\le l,\, d_j+m_j\le N} P_j(x)\phi_j(y-x). \tag{3.3}$$

Now set $\delta = (\sum_{\mu=1}^{M} |\lambda_\mu|)^{1/(N+1)}$,

$$\Phi_\delta(x,y) = \sum_{\mu=1}^{M} \lambda_\mu \delta^{-(N+1)} \psi_\mu(x,y) + \sum_{0\le j\le l,\, d_j+m_j\le N} P_j(\delta^{-1}x)\phi_j(\delta^{-1}(y-x)),$$

and $f_\delta(x) = f(\delta^{-1}x)$. It is easy to see that the following hold:

$$\Phi(x,y) = \Phi_\delta(\delta x, \delta y), \tag{3.4}$$

$$\sum_{\mu=1}^{M} \lambda_\mu \delta^{-(N+1)} = 1, \tag{3.5}$$

$$\|T_{b,k}^{\Phi} f\|_p = \delta^{-n/p} \|T_{b,k}^{\Phi_\delta} f_\delta\|_p. \tag{3.6}$$

Notice that $\delta^{-n/p}\|f_\delta\|_p = \|f\|_p$ and $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = \|b(\delta^{-1}\cdot)\|_{\mathrm{BMO}(\mathbb{R}^n)}$, in order to complete the proof of our theorem, by (3.6) we need only to show that

$$\|T_{b,k}^{\Phi_\delta} f\|_p \le C\left(1 + \|\Omega\|_{L(\log^+ L)^{k+1}(S^{n-1})}\right) \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k \|f\|_p \tag{3.7}$$

for all $1 < p < \infty$, where C is a constant independent of δ and the coefficients of the polynomials P_j. Write

$$T_{b,k}^{\Phi_\delta} f(x) = T_{b,\,k}^{\Phi_\delta,0} f(x) + T_{b,\,k}^{\Phi_\delta,\infty} f(x),$$

where

$$T_{b,\,k}^{\Phi_\delta,\infty} f(x) = \int_{|x-y|\ge 1} e^{i\Phi_\delta(x,y)} [b(x)-b(y)]^k \frac{\Omega(x-y)}{|x-y|^n} f(y)dy.$$

To prove (3.7), it suffices to show that

$$\|T_{b,\,k}^{\Phi_\delta,0} f\|_p \le C\left(1 + \|\Omega\|_{L(\log^+ L)^{k+1}(S^{n-1})}\right) \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k \|f\|_p \tag{3.8}$$

and

$$\|T_{b,\,k}^{\Phi_\delta,\infty} f\|_p \le C\left(1 + \|\Omega\|_{L(\log^+ L)^{k+1}(S^{n-1})}\right) \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k \|f\|_p \tag{3.9}$$

for all $1 < p < \infty$, where C is a constant independent of δ and the coefficients of the polynomials P_j.

At first, we prove (3.8). For $h \in \mathbb{R}^n$, let

$$\begin{aligned} \Psi_{h,\delta}(x,y) &= \sum_{\mu=1}^{M} \lambda_\mu \delta^{-(d+1)} \{\psi_\mu(x,y) - \psi_\mu(x-h, y-h)\} \\ &\quad + \sum_{0\le j\le l,\, d_j+m_j\le N} P_j(\delta^{-1}x)\phi_j(\delta^{-1}(y-x)). \end{aligned} \tag{3.10}$$

Since $\Psi_{h,\delta}$ satisfies the induction assumption, by Lemma 2.4 we have

$$\|T_{b,\,k}^{\Psi_{h,\delta},0} f\|_p \le C\left(1 + \|\Omega\|_{L(\log^+ L)^{k+1}(S^{n-1})}\right) \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k \|f\|_p \tag{3.11}$$

for all $1 < p < \infty$, where C is a constant independent of δ and the coefficients of the polynomials P_j, and hence of h. By (3.11) and the fact that

$$T_{b,k}^{\Phi_\delta,0} f(x) = T_{b,k}^{\Psi_{h,\delta},0} f(x) + (T_{b,k}^{\Phi_\delta,0} - T_{b,k}^{\Psi_{h,\delta},0})(f)(x),$$

to prove (3.8), we need only to prove

(3.12)

$$\|(T_{b,k}^{\Phi_\delta,0} - T_{b,k}^{\Psi_{h,\delta},0})(f)\|_p \le C\left(1 + \|\Omega\|_{L(\log^+ L)^{k+1}(S^{n-1})}\right) \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k \|f\|_p$$

for all $1 < p < \infty$, where C is as in (3.11).

Note that

$$\begin{aligned}
&\left| e^{i\Phi_\delta(x,y)} - e^{i\Psi_{h,\delta}(x,y)} \right| \\
&\le C \left| \sum_{\mu=1}^{M} \lambda_\mu \delta^{-(N+1)} \psi_\mu(x-h, y-h) \right| = C\delta^{-(N+1)} |H(x-h, y-h)| \\
&= C\delta^{-(N+1)} \left| \sum_{s=1}^{k} \sum_{|\alpha_{j_s}| = d_{j_s}} a_{\alpha_{j_s}} (x-h)^{\alpha_{j_s}} \phi_{j_s}(y-x) \right| \\
&\le C \max_{1\le s\le k} \|\phi_{j_s}\|_{L^\infty(S^{n-1})} \sum_{s=1}^{k} \sum_{|\alpha_{j_s}| = d_{j_s}} |a_{\alpha_{j_s}}| |x-h|^{|\alpha_{j_s}|} \delta^{-(N+1)} |y-x|^{m_{j_s}},
\end{aligned}$$

by (3.2) and (3.5), for $|x-h| < 1/4$, we have

$$\begin{aligned}
&\left| (T_{b,k}^{\Phi_\delta,0} - T_{b,k}^{\Psi_{h,\delta},0})(f)(x) \right| \\
&= \left| \int_{|x-y|<1} \left[e^{i\Phi_\delta(x,y)} - e^{i\Psi_{h,\delta}(x,y)} \right] [b(x) - b(y)]^k \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \right| \\
&\le C \sum_{s=1}^{k} \sum_{|\alpha_{j_s}| = d_{j_s}} |a_{\alpha_{j_s}}| \delta^{-(N+1)} \int_{|x-y|<1} |x-h|^{|\alpha_{j_s}|} |x-y|^{m_{j_s} - n} \\
&\quad \times |\Omega(x-y)[b(x) - b(y)]^k f(y)| dy \\
&\le C \sum_{s=1}^{k} \sum_{|\alpha_{j_s}| = d_{j_s}} |a_{\alpha_{j_s}}| \delta^{-(N+1)} \int_{|x-y|<1} |x-y|^{m_{j_s} - n} \\
&\quad \times |\Omega(x-y)[b(x) - b(y)]^k f(y)| dy \\
&\le C M_{b,k}^{\Omega} \widetilde{f}(x),
\end{aligned}$$

where $\widetilde{f}(x) = f(x)\chi_{\{|x-h|<5/4\}}(x)$. Therefore, it follows from Lemma 2.1 that

$$\begin{aligned}
&\int_{|x-h|<1/4} \left| (T_{b,k}^{\Phi_\delta,0} - T_{b,k}^{\Psi_{h,\delta},0})(f)(x) \right|^p dx \\
&\le C\left(1 + \|\Omega\|_{L(\log^+ L)^{k+1}(S^{n-1})}\right) \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k \int_{|y-h|<5/4} |f(y)|^p dy
\end{aligned}$$

holds for all $h \in \mathbb{R}^n$, with bound independent of h. Integrating the above inequality with respect to h yields (3.12). This completes the proof of (3.8).

It remains to prove (3.9). Let $E_0 = \{x' \in S^{n-1} : |\Omega(x')| \le 2\}$ and $E_l = \{x' \in S^{n-1} : 2^l < |\Omega(x') \le 2^{l+1}\}$ for positive integer l. Denote by Ω_l the restriction of Ω on E_l, that is, $\Omega_l(x') = \Omega(x')\chi_{E_l}(x')$. Then

$$\Omega(x') = \sum_{l=0}^{\infty} \Omega_l(x'), \tag{3.13}$$

and our hypothesis on Ω now shows that $\sum_{l\ge 1} l^{k+1}\|\Omega_l\|_1 < \|\Omega\|_{L(\log^+ L)^{k+1}(S^{n-1})} < \infty$. By (3.13), we have

$$T_{b,k}^{\Phi_\delta,\infty} f(x) = \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} T_{b,k;j}^{\Phi_\delta;l} f(x), \tag{3.14}$$

where

$$T_{b,k;j}^{\Phi_\delta;l} f(x) = \int_{2^j \le |x-y| < 2^{j+1}} e^{i\Phi_\delta(x,y)}[b(x)-b(y)]^k \frac{\Omega_l(x-y)}{|x-y|^n} f(y)dy.$$

Consequently,

$$\|T_{b,k}^{\Phi_\delta,\infty} f\|_p \le \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} \|T_{b,k;j}^{\Phi_\delta,l} f\|_p. \tag{3.15}$$

Invoking Lemma 2.2, we have the following crude estimates

$$\|T_{b,k;j}^{\Phi_\delta;l} f\|_p \le C\|M_{\Omega_l;b,k} f\|_p \le C\lambda_{\Omega_l,k}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k \|f\|_p, \quad 1<p<\infty. \tag{3.16}$$

Next we shall establish a refined L^2-estimate on $T_{b,k;j}^{\Phi_\delta;l} f$. Precisely, we shall show that there exists a positive constant $\varepsilon = \varepsilon(n,N)$ such that

$$\|T_{b,k;j}^{\Phi_\delta;l} f\|_2 \le C2^{-\varepsilon j}\|\Omega_l\|_\infty \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k \|f\|_2. \tag{3.17}$$

To prove (3.17), we turn our attention to the following operators

$$\widetilde{T}_{b,k;j}^{\Phi_\delta;l} f(x) = \int_{1\le |x-y|<2} e^{i\Phi_\delta(2^j x, 2^j y)}[b(x)-b(y)]^k \frac{\Omega_l(x-y)}{|x-y|^n} f(y)dy,$$

and

$$\widetilde{T}_j^{\Phi_\delta;l} f(x) = \int_{1\le |x-y|<2} e^{i\Phi_\delta(x,y)} \frac{\Omega_l(x-y)}{|x-y|^n} f(y)dy.$$

By dilation-invariance, it is easy to see that the proof of (3.17) can be reduced to showing that

$$\|\widetilde{T}_{b,k;j}^{\Phi_\delta;l} f\|_2 \le C2^{-\varepsilon j}\|\Omega_l\|_\infty \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k \|f\|_2. \tag{3.18}$$

As in the proof of Lemma 2.4, we decompose $\mathbb{R}^n$ into $\mathbb{R}^n = \cup_d I_d$, where I_d is a cube with side length 1 and the cubes have disjoint interiors. Set $f_d = f\chi_{I_d}$. Similarly to the proof of Lemma 2.4, we have

$$\|\widetilde{T}_{b,k;j}^{\Phi_\delta;l} f\|_2^2 \le C\sum_d \|\widetilde{T}_{b,k;j}^{\Phi_\delta;l} f_d\|_2^2.$$

Thus we may assume $\mathrm{supp}(f) \subset I$ for cube I with side length 1. Choose $\varphi \in C_0^\infty(\mathbb{R}^n)$, $0 \le \varphi \le 1$, φ is identically one on $50nI$ and vanishes outside $100nI$. Write $\overline{I} = 100nI$ and $\widetilde{b}(x) = (b(x) - m_{\overline{I}}(b))\varphi(x)$, where $m_{\overline{I}}(b)$ is the mean value of b on $\overline{I}$. When $y \in I$ and x in the support of $\widetilde{T}^{\Phi_\delta;\,l}_{b,k;\,j} f$, we have

$$(b(x) - b(y))^k = \sum_{m=0}^{k} (-1)^{k-m} C_k^m \widetilde{b}^m(x)\widetilde{b}^{k-m}(y).$$

Consequently,

$$\widetilde{T}^{\Phi_\delta;\,l}_{b,k;\,j} f(x) = \sum_{m=0}^{k} (-1)^{k-m} C_k^m \widetilde{b}^m(x) \widetilde{T}^{\Phi_\delta;\,l}_{j}(\widetilde{b}^{k-m} f)(x). \tag{3.19}$$

We first claim that for $1 < q < 2$, there exists $\varepsilon > 0$ such that

$$\|\widetilde{T}^{\Phi_\delta;\,l}_{j} f\|_{q'} \le C2^{-\varepsilon j}\|\Omega_l\|_\infty \|f\|_q, \qquad 1/q + 1/q' = 1. \tag{3.20}$$

Indeed, let us consider the operator

$$T^{\Phi_\delta;\,l}_{j} f(x) = \int_{2^j \le |x-y| < 2^{j+1}} e^{i\Phi_\delta(x,y)} \frac{\Omega_l(x-y)}{|x-y|^n} f(y)dy.$$

From [2, p. 577, (3.24)-(3.25)], it is easy to see that for some $\theta > 0$,

$$\|T^{\Phi_\delta;\,l}_{j} f\|_2 \le C2^{-\theta j}\|\Omega_l\|_\infty \|f\|_2,$$

and

$$\|T^{\Phi_\delta;\,l}_{j} f\|_\infty \le C\|\Omega_l\|_\infty \|f\|_1.$$

Then (3.20) is obtained by the interpolation and the dilation-invariance.

Now we estimate $\|\widetilde{T}^{\Phi_\delta;\,l}_{b,k;\,j} f\|_2$. Choose $q \in (1, 2)$, p_0, $p_1 \in (1, \infty)$ such that $1/q' + 1/p_0 = 1/2$ and $1/q = 1/2 + 1/p_1$. Notice that for each fixed integer m, $0 \le m \le k$, $\mathrm{supp}(\widetilde{T}^{\Phi_\delta;\,l}_{j}(\widetilde{b}^{k-m} f)) \subset 20nI$, by Hölder's inequality and (3.20) we have

$$\begin{aligned}
\|\widetilde{b}^m \widetilde{T}^{\Phi_\delta;\,l}_{j}(\widetilde{b}^{k-m} f)\|_2 &\le \|\widetilde{b}^m\|_{p_0} \|\widetilde{T}^{\Phi_\delta;\,l}_{j}(|\widetilde{b}^{k-m} f|)\|_{q'} \\
&\le C2^{-\varepsilon j}\|\Omega_l\|_\infty \|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)} \|\widetilde{b}^{k-m} f\|_q \\
&\le C2^{-\varepsilon j}\|\Omega_l\|_\infty \|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)} \|\widetilde{b}^{k-m}\|_{p_1} \|f\|_2 \\
&\le C2^{-\varepsilon j}\|\Omega_l\|_\infty \|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_2.
\end{aligned}$$

Summing over m, we obtain (3.18) and complete the proof of (3.17).

Note that $\lambda_{\Omega_l,\,k} \le C\|\Omega_l\|_\infty$ (see [6, Lemma 3]), it follows from interpolation between (3.16) and (3.17) that

$$\|T^{\Phi_\delta;\,l}_{b,k;\,j} f\|_p \le C2^{-\beta j}\|\Omega_l\|_\infty \|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_p \tag{3.21}$$

for any $1 < p < \infty$ and some $\beta > 0$.

Let r be a large positive integer such that $r > 2\beta^{-1}$. Combining (3.15), (3.16) and (3.21) gives

$$\begin{aligned}
&\|T_{b,k}^{\Phi_\delta,\infty}f\|_p \\
&\le \sum_{j=0}^{\infty}\|T_{b,k;\,j}^{\Phi_\delta,\,0}f\|_p+\sum_{l>0}^{\infty}\sum_{j>rl}\|T_{b,k;\,j}^{\Phi_\delta,\,l}f\|_p+\sum_{l>0}^{\infty}\sum_{j\le rl}^{\infty}\|T_{b,k;\,j}^{\Phi_\delta,\,l}f\|_p \\
&\le C\sum_{j=0}^{\infty}2^{-\beta j}\|\Omega_0\|_\infty\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k\|f\|_p+C\sum_{l>0}\sum_{j>rl}2^{-\beta j}\Omega_l\|_\infty\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k\|\|f\|_p \\
&\quad +C\sum_{l>0}\sum_{j\le rl}\lambda_{\Omega_l,k}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k\|f\|_p \\
&\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k\|f\|_p+C\sum_{l>0}2^l\sum_{j>rl}2^{-\beta j}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k\|f\|_p \\
&\quad +C\sum_{l>0}l\lambda_{\Omega_l,k}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k\|f\|_p \\
&\le C\left(1+\|\Omega\|_{L(\log^+L)^{k+1}(S^{n-1})}\right)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k\|f\|_p,
\end{aligned}$$

where the last inequality follows from the following fact

$$\sum_{l>0}l\lambda_{\Omega_l,k}\le C\sum_{l>0}l^{k+1}\|\Omega_l\|_1+C\sum_{l>0}l2^{-l}\le C\left(\|\Omega\|_{L(\log^+L)^{k+1}(S^{n-1})}+1\right)<\infty.$$

This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

As in the proof of Theorem 1.1, we shall use induction on

$$d(\Phi):=\inf\max_{0\le j\le l}\{d_j+m_j\},$$

where the infimum is taken over all representations of the form $\Phi(x,y)=\sum_{j=0}^{l}P_j(x)\phi_j(y-x)$ with d_j is the degree of P_j and m_j is the degree of homogeneity of ϕ_j.

Obviously, if $d(\Phi)=0$, then $T_{b,\,k}^{\Phi,\,*}f(x)=\overline{T}_{b,k}^{*}f(x)$. Therefore, by Lemma 2.3, Theorem 1.2 holds for all Φ with $d(\Phi)=0$. Now we assume that Theorem 1.2 holds for all Φ with $d(\Phi)\le N$. Given $\Phi(x,y)=\sum_{j=0}^{l}P_j(x)\phi_j(y-x)$ with $d(\Phi)=N+1$, let $\Phi_\delta(x,y)$ be as before. By the same arguments as in the proof of Theorem 1.1, we need only to prove

$$\|T_{b,\,k}^{\Phi_\delta,\,*}f\|_p\le C\left(1+\|\Omega\|_{L(\log^+L)^{k+1}(S^{n-1})}\right)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k\|f\|_p,\qquad 1<p<\infty,$$

where the definition of $T_{b,k}^{\Phi_\delta,*}f(x)$ is as in (1.4) replaced $\Phi(x,y)$ by $\Phi_\delta(x,y)$. Similarly to the proof of Theorem 4 in [10], we write

$$\begin{aligned}
T_{b,k}^{\Phi_\delta,*}f(x) \le \sup_{0<\varepsilon<1}&\left|\int_{|x-y|>\varepsilon} e^{i\Phi_\delta(x,y)}[b(x)-b(y)]^k\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\right|\\
&+\sup_{\varepsilon\ge 1}\left|\int_{|x-y|>\varepsilon} e^{i\Phi_\delta(x,y)}[b(x)-b(y)]^k\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\right|\\
\le \sup_{0<\varepsilon<1}&\left|\int_{1>|x-y|>\varepsilon} e^{i\Phi_\delta(x,y)}[b(x)-b(y)]^k\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\right|\\
&+\left|\int_{|x-y|\ge 1} e^{i\Phi_\delta(x,y)}[b(x)-b(y)]^k\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\right|\\
&+\sup_{\varepsilon>1}\left|\int_{|x-y|>\varepsilon} e^{i\Phi_\delta(x,y)}[b(x)-b(y)]^k\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\right|\\
:= T_{b,k;0}^{\Phi_\delta,*}&f(x)+\left|T_{b,k}^{\Phi_\delta,\infty}f(x)\right|+T_{b,k;\infty}^{\Phi_\delta,*}f(x).
\end{aligned}$$

Then by (3.9), it suffices to prove that both $T_{b,k;0}^{\Phi_\delta,*}$ and $T_{b,k;\infty}^{\Phi_\delta,*}$ are bounded on $L^p(\mathbb{R}^n)$ for all $1<p<\infty$, with bound $C(1+\|\Omega\|_{L(\log^+L)^{k+1}(S^{n-1})})\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k$.

By the method similar to proving (3.8), we can easily prove the desired conclusion on $T_{b,k;0}^{\Phi_\delta,*}$.

Now we estimate $\|T_{b,k;\infty}^{\Phi_\delta,*}f\|_p$. For each fixed $\varepsilon>1$, we have unique $J\in\mathbb{N}$ such that $2^{J-1}\le\varepsilon<2^J$. Thus

$$\begin{aligned}
&T_{b,k;\infty}^{\Phi_\delta,*}f(x)\\
&\le \sup_{J\in\mathbb{N}}\int_{2^{J-1}\le|x-y|<2^J}|b(x)-b(y)|^k\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|dy\\
&\quad+\sum_{J\in\mathbb{N}}\sum_{j=J+1}^{\infty}\left|\int_{2^{j-1}\le|x-y|<2^j} e^{i\Phi_\delta(x,y)}[b(x)-b(y)]^k\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\right|\\
&\le CM_{b,k}^{\Omega}f(x)+\sum_{j=0}^{\infty}\left|\int_{2^{j-1}\le|x-y|<2^j} e^{i\Phi(x,y)}[b(x)-b(y)]^k\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\right|.
\end{aligned}$$

By Lemma 2.1 and the method similar to proving (3.9), we obtain

$$\|T_{b,k;\infty}^{\Phi_\delta,*}f\|_p\le C\left(1+\|\Omega\|_{L(\log^+L)^{k+1}(S^{n-1})}\right)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k\|f\|_p,$$

which completes the proof of Theorem 1.2. □

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