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WEAK AND STRONG CONVERGENCE THEOREMS FOR AN ASYMPTOTICALLY k-STRICT PSEUDO-CONTRACTION AND A MIXED EQUILIBRIUM PROBLEM

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Abstract. We introduce two iterative algorithms for finding a common element of the set of fixed points of an asymptotically k-strict pseudo-contraction and the set of solutions of a mixed equilibrium problem in a Hilbert space. We obtain some weak and strong convergence theorems by using the proposed iterative algorithms. Our results extend and improve the corresponding results of Tada and Takahashi [16] and Kim and Xu [8, 9].

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\varphi : C \to \mathbb{R}$ be a real valued function and $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction, i.e., $\Theta(u,u) = 0$ for each $u \in C$. Now we concern the following mixed equilibrium problem (MEP) which is to find $x^* \in C$ such that

$$\text{(MEP)} \qquad \Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C.$$

In particular, if $\varphi \equiv 0$, this problem reduces to the equilibrium problem (EP), which is to find $x^* \in C$ such that

$$\text{(EP)} \qquad \Theta(x^*, y) \geq 0, \quad \forall y \in C.$$

Denote the set of solutions of (MEP) by Ω and the set of solutions of (EP) by Γ. The mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems and the equilibrium problems as special cases; see, e.g., [1, 3, 4, 10, 22]. Some methods have been proposed to solve the equilibrium problems and the mixed equilibrium problems, see, e.g., [2, 5, 6, 7, 10, 12, 14, 15, 16, 17, 18, 19, 20, 21].

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On the other hand, recently, Kim and Xu [8, 9] introduced some iterative methods for solving fixed point problems of asymptotically nonexpansive mappings and asymptotically k-strict pseudo-contractions, respectively. The corresponding iterative algorithms are as follows. The first one introduced in [8] is:

$$\text{(KX1)} \qquad \begin{cases} x_0 \in C \ \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where $T : C \to C$ is an asymptotically nonexpansive mapping and $\theta_n = (1 - \alpha_n)(k_n^2 - 1)(\text{diam C})^2 \to 0$ as $n \to \infty$. And the second one introduced in [9] is:

$$\text{(KX2)} \qquad \begin{cases} x_0 \in C \ \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ \qquad\qquad + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where $T : C \to C$ is an asymptotically k-strict pseudo-contraction and $\theta_n = \Delta_n^2(1-\alpha_n)\gamma_n \to 0 (n \to \infty), \Delta_n = \sup\{\|x_n - z\| : z \in F(T)\} < \infty$. Subsequently, Kim and Xu proved that the iterative algorithm (KX1) and (KX2) are strongly convergent. For more details, see [8, 9]. However, we note that the $(n+1)$th iterate x_{n+1} is defined as the projection of the initial guess x_0 onto the intersection of two closed convex subsets C_n and Q_n. Therefore, an interesting problem is how to construct appropriately C_n and Q_n such that the computations become easier.

Motivated by the above works, in this paper we introduce two iterative algorithms for finding a common element of the set of fixed points of an asymptotically k-strict pseudo-contraction and the set of solutions of a mixed equilibrium problem in a Hilbert space. We obtain some weak and strong convergence theorems by using the proposed iterative algorithms. Our results extend and improve the corresponding results of Tada and Takahashi [16], Kim and Xu [8, 9].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \le \|x - y\|$$

for all $y \in C$. Such a P_C is called the metric projection of H onto C. We know that P_C is nonexpansive. Further, for $x \in H$ and $x^* \in C$,

$$x^* = P_C(x) \Leftrightarrow \langle x - x^*, x^* - y\rangle \geq 0 \text{ for all } y \in C. \tag{1}$$

Recall that a mapping $T : C \to C$ is said to be an asymptotically k-strict pseudo-contraction if, there exists a constant $k \in [0, 1)$ satisfying

$$\|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2 \tag{2}$$

for all $x, y \in C$ and all integers $n \geq 1$, where $\gamma_n \geq 0$ for all n and such that $\gamma_n \to 0$ as $n \to \infty$. Note that if $k = 0$, then T is an asymptotically nonexpansive mapping, that is, there exists a sequence $\{\gamma_n\}$ of nonnegative numbers with $\gamma_n \to 0$ such that

$$\|T^n x - T^n y\| \leq (1 + \gamma_n)\|x - y\|^2$$

for all $x, y \in C$ and all integers $n \geq 1$.

In the sequel, we will use $F(T)$ to denote the set of fixed points of T.

For given sequence $\{x_n\} \subset C$, let $\omega_w(x_n) = \{x : \exists x_{n_j} \to x \text{ weakly}\}$ denote the weak ω-limit set of $\{x_n\}$.

In this paper, for solving the mixed equilibrium problems for an equilibrium bifunction $\Theta : C \times C \to \mathbb{R}$, we assume that Θ satisfies the following conditions:

(H1) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$;
(H2) for each fixed $y \in C$, $x \mapsto \Theta(x, y)$ is concave and upper semicontinuous;
(H3) for each $x \in C$, $y \mapsto \Theta(x, y)$ is convex.

A mapping $\eta : C \times C \to H$ is called Lipschitz continuous, if there exists a constant $\lambda > 0$ such that

$$\|\eta(x, y)\| \leq \lambda\|x - y\|, \ \forall x, y \in C.$$

A differentiable function $K : C \to \mathbb{R}$ on a convex set C is called:

(i) η-convex if

$$K(y) - K(x) \geq \langle K^{'}(x), \eta(y, x)\rangle, \ \forall x, y \in C,$$

where $K^{'}$ is the Frechet derivative of K at x;

(ii) η-strongly convex if there exists a constant $\sigma > 0$ such that

$$K(y) - K(x) - \langle K^{'}(x), \eta(y, x)\rangle \geq (\sigma/2)\|x - y\|^2, \ \forall x, y \in C.$$

Let C be a nonempty closed convex subset of a real Hilbert space H, $\varphi : C \to \mathbb{R}$ be real-valued function and $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction. Let r be a positive number. For a given point $x \in C$, the auxiliary problem for (MEP) consists of finding $y \in C$ such that

$$\Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K^{'}(y) - K^{'}(x), \eta(z, y)\rangle \geq 0, \ \forall z \in C.$$

Let $S_r : C \to C$ be the mapping such that for each $x \in C$, $S_r(x)$ is the solution set of the auxiliary problem, i.e., $\forall x \in C$,

$$S_r(x) = \left\{ y \in C : \Theta(y, z) + \varphi(y) + \frac{1}{r}\langle K^{'}(y) - K^{'}(x), \eta(z, y)\rangle \geq 0, \forall z \in C \right\}.$$

We need the following important and interesting result for proving our main results.

Lemma 2.1 ([21]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction satisfying conditions* (H1)-(H3). *Assume that*

(i) *$\eta : C \times C \to H$ is Lipschitz continuous with constant $\lambda > 0$ such that*
  (a) *$\eta(x, y) + \eta(y, x) = 0, \ \forall x, y \in C$,*
  (b) *$\eta(\cdot, \cdot)$ is affine in the first variable,*
  (c) *for each fixed $y \in C, x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology*;

(ii) *$K : C \to \mathbb{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative $K^{'}$ is sequentially continuous from the weak topology to the strong topology*;

(iii) *for each $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K^{'}(y) - K^{'}(x), \eta(z_x, y)\rangle < 0.$$

Then there hold the following:

(I) *S_r is single-valued*;

(II) *S_r is nonexpansive if $K^{'}$ is Lipschitz continuous with constant $\nu > 0$ such that $\sigma \geq \lambda\nu$ and*

$$\langle K^{'}(x_1) - K^{'}(x_2), \eta(u_1, u_2)\rangle \geq \langle K^{'}(u_1) - K^{'}(u_2), \eta(u_1, u_2)\rangle, \ \ \forall (x_1, x_2) \in C \times C$$

where $u_i = S_r(x_i)$ for $i = 1, 2$;

(III) $F(S_r) = \Omega$;

(IV) *Ω is closed and convex.*

We also need the following lemmas.

Lemma 2.2. *Let H be a real Hilbert space. There hold the following well-known identities*:

(i) $\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \ \ \forall x, y \in H$.

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2 \ \ \forall t \in [0, 1], \forall x, y \in H$.

Lemma 2.3 ([9]). *Assume C is a closed convex subset of a real Hilbert space H and let $T : C \to C$ be an asymptotically k-strict pseudo-contraction. Then the following conclusions hold*:

(i) *for each* $n \geq 1$, T^n *satisfies the Lipschitz condition*:

$$\|T^n x - T^n y\| \leq L_n \|x - y\| \ \ \forall x, y \in C,$$

where $L_n = \frac{k+\sqrt{1+\gamma_n(1-k)}}{1-k}$;

(ii) *the mapping* $I - T$ *is demiclosed at zero, that is, if* $\{x_n\}$ *is a sequence in* C *such that* $x_n \to x^*$ *weakly and* $(I - T)x_n \to 0$ *strongly, then* $(I - T)x^* = 0$;

(iii) *the fixed point set* $F(T)$ *of* T *is closed and convex so that the projection* $P_{F(T)}$ *is well-defined.*

Lemma 2.4 ([11])**.** *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* $\{x_n\}$ *be a sequence in* H *and* $u \in H$. *Let* $q = P_C u$. *If* $\{x_n\}$ *is such that* $\omega_w(x_n) \subset C$ *and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\| \ \ \textit{for all} \ \ n,$$

then $x_n \to q$.

3. Main results

In this section, we first introduce the following new iterative algorithm.

Algorithm 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, $\varphi : C \to \mathbb{R}$ *be a lower semicontinuous and convex real valued function,* $\Theta : C \times C \to \mathbb{R}$ *be an equilibrium bifunction and* $T : C \to C$ *be an asymptotically* k*-strict pseudo-contraction. Assume that* $F(T) \cap \Omega$ *is nonempty and bounded. Let* r *be a positive parameter and* $\delta \in (k, 1)$ *be a constant. Let* $\{\alpha_n\}$ *be a sequence in* $[0, 1]$. *Define the sequences* $\{x_n\}$ *and* $\{y_n\}$ *by the following manner*:

$$(3) \quad \begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ \Theta(y_n, x) + \varphi(x) - \varphi(y_n) + \dfrac{1}{r}\langle K^{'}(y_n) - K^{'}(x_n), \eta(x, y_n)\rangle \geq 0, \forall x \in C, \\ z_n = (1 - \alpha_n)x_n + \alpha_n[\delta y_n + (1 - \delta)T^n y_n], \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where $\theta_n = \gamma_n \Delta_n^2$, $\Delta_n = \sup\{\|x_n - p\| : p \in F(T) \cap \Omega\} < \infty$.

Now we give a strong convergence result concerning iterative Algorithm 3.1 as follows.

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\varphi : C \to \mathbb{R}$ *be a lower semicontinuous and convex functional. Let* $\Theta : C \times C \to \mathbb{R}$ *be an equilibrium bifunction satisfying conditions* (H1)-(H3) *and let* $T : C \to C$ *be an asymptotically* k*-strict pseudo-contraction. Assume that* $F(T) \cap \Omega$ *is nonempty and bounded. Assume that*:

(i) $\eta : C \times C \to H$ *is Lipschitz continuous with constant* $\lambda > 0$ *such that*;

(a) $\eta(x, y) + \eta(y, x) = 0, \ \forall x, y \in C$,
(b) $\eta(\cdot, \cdot)$ *is affine in the first variable,*
(c) *for each fixed* $y \in C$, $x \mapsto \eta(y, x)$ *is sequentially continuous from the weak topology to the weak topology;*

(ii) $K : C \to \mathbb{R}$ *is* η*-strongly convex with constant* $\sigma > 0$ *and its derivative* $K^{'}$ *is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant* $\nu > 0$ *such that* $\sigma \geq \lambda\nu$;

(iii) *for each* $x \in C$; *there exist a bounded subset* $D_x \subset C$ *and* $z_x \in C$ *such that, for any* $C \ni y \notin D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K^{'}(y) - K^{'}(x), \eta(z_x, y)\rangle < 0;$$

(iv) $\alpha_n \in [a, 1]$ *for some* $a \in (0, 1)$.

Then the sequence $\{x_n\}$ *generated iteratively by* (3) *converges strongly to* $P_{F(T)\cap\Omega}x_0$ *provided* S_r *is firmly nonexpansive.*

Proof. First, we show that the sequence $\{x_n\}$ is well-defined. It is obvious that C_n and Q_n are closed and convex. Let $p \in F(T) \cap \Omega$. From $y_n = S_r x_n$, we have

$$\|y_n - p\| = \|S_r x_n - S_r p\| \leq \|x_n - p\|. \tag{4}$$

From Lemma 2.2, (2) and (4), we obtain

$$\begin{aligned} \|z_n - p\|^2 &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|\delta(y_n - p) + (1 - \delta)(T^n y_n - p)\|^2 \\ &= (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n[\delta\|y_n - p\|^2 + (1 - \delta)\|T^n y_n - p\|^2 \\ &\quad - \delta(1 - \delta)\|T^n y_n - y_n\|^2] \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\{\delta\|y_n - p\|^2 + (1 - \delta)[(1 + \gamma_n)\|y_n - p\|^2 \\ &\quad + k\|y_n - T^n y_n\|^2] - \delta(1 - \delta)\|y_n - T^n y_n\|^2\} \\ &= (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\{\|y_n - p\|^2 + (1 - \delta)\gamma_n\|y_n - p\|^2 \\ &\quad + (1 - \delta)(k - \delta)\|y_n - T^n y_n\|^2\} \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n(1 + \gamma_n)\|y_n - p\|^2 \\ &\leq (1 + \gamma_n)\|x_n - p\|^2. \end{aligned} \tag{5}$$

Hence, we have $\|z_n - p\|^2 \leq \|x_n - p\|^2 + \theta_n$. This implies that $p \in C_n$; thus,

$$F(T) \cap \Omega \subset C_n \tag{6}$$

for every $n \geq 0$. Next we show by induction that $F(T) \cap \Omega \subset C_n \cap Q_n$ for each $n \geq 0$. Since $F(T) \cap \Omega \subset C_0$ and $Q_0 = C$, we get

$$F(T) \cap \Omega \subset C_0 \cap Q_0.$$

Suppose that $F(T)\cap\Omega \subset C_k\cap Q_k$ for $k \in \mathbb{N}$. Then, there exists $x_{n+1} \in C_k\cap Q_k$ such that

$$x_{n+1} = P_{C_k\cap Q_k}x_0.$$

Therefore, for each $z \in C_k \cap Q_k$, we have

$$\langle x_{k+1} - z, x_0 - x_{k+1}\rangle \geq 0.$$

Note that $F(T) \cap \Omega \subset C_k \cap Q_k$. Hence, for any $z \in F(T) \cap \Omega$ we have

$$\langle x_{k+1} - z, x_0 - x_{k+1}\rangle \geq 0,$$

therefore $z \in Q_{k+1}$. So, we get

$$F(T) \cap \Omega \subset Q_{k+1}.$$

From this and (6), we have

$$F(T) \cap \Omega \subset C_{k+1} \cap Q_{k+1}.$$

This denotes that the sequence $\{x_n\}$ is well-defined.

Since $F(T) \cap \Omega$ is a nonempty closed convex subset of C, there exists a unique $z^{'} \in F(T) \cap \Omega$ such that $z^{'} = P_{F(T)\cap\Omega}x_0$. From $x_{n+1} = P_{C_n\cap Q_n}x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|$$

for all $z \in C_n \cap Q_n$. Since $z^{'} \in F(T) \cap \Omega \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x_0\| \leq \|z^{'} - x_0\| \tag{7}$$

for every $n \geq 0$. Therefore, $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. Since $x_n = P_{Q_n}x_0$ and $x_{n+1} \in Q_n$, we get

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|.$$

This together with the boundedness of $\{\|x_n - x_0\|\}$ implies that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. The fact that $x_{n+1} \in Q_n$ implies that $\langle x_{n+1} - x_n, x_n - x_0\rangle \geq 0$. Applying Lemma 2.2, we obtain

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \\ &\to 0. \end{aligned} \tag{8}$$

From $x_{n+1} \in C_n$, we have

$$\begin{aligned} \|x_n - z_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \\ &\leq (2 + \gamma_n)\|x_n - x_{n+1}\| \\ &\to 0. \end{aligned}$$

For $p \in F(T) \cap \Omega$, noting that S_r is firmly nonexpansive, we have

$$\begin{aligned}\|y_n - p\|^2 &= \|S_r x_n - S_r p\|^2 \\ &\le \|S_r x_n - S_r p, x_n - p\rangle \\ &= \langle y_n - p, x_n - p\rangle \\ &= \frac{1}{2}\{\|y_n - p\|^2 + \|x_n - p\|^2 - \|x_n - y_n\|^2\},\end{aligned}$$

and hence,

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - y_n\|^2.$$

From (5), we have

$$\begin{aligned}\|z_n - p\|^2 &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n(1+\gamma_n)\|y_n - p\|^2 \\ &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n(1+\gamma_n)\{\|x_n - p\|^2 - \|x_n - y_n\|^2\} \\ &= (1+\alpha_n\gamma_n)\|x_n - p\|^2 - \alpha_n(1+\gamma_n)\|x_n - y_n\|^2,\end{aligned}$$

that is,

(9)
$$\begin{aligned}\|x_n - y_n\|^2 &\le \frac{1}{\alpha_n(1+\gamma_n)}\{\|x_n - p\|^2 - \|z_n - p\|^2\} + \frac{\gamma_n\|x_n - p\|^2}{1+\gamma_n} \\ &\le \frac{1}{\alpha_n(1+\gamma_n)}\|x_n - z_n\|\{\|x_n - p\| + \|z_n - p\|\} + \frac{\gamma_n\|x_n - p\|^2}{1+\gamma_n} \\ &\to 0.\end{aligned}$$

Combining (8) and (9), we have

$$\lim_{n\to\infty}\|y_{n+1} - y_n\| = 0.$$

By the fact $\alpha_n(1-\delta)(T^n y_n - y_n) = z_n - (1-\alpha_n)x_n - \alpha_n y_n$, we get

$$\|\alpha_n(1-\delta)(T^n y_n - y_n)\| \le \|z_n - x_n\| + \alpha_n\|x_n - y_n\| \to 0,$$

which implies that

$$\|T^n y_n - y_n\| \to 0.$$

Next we show that

$$\lim_{n\to\infty}\|y_n - Ty_n\| = 0.$$

As a matter of fact, we have

$$\begin{aligned}\|y_n - Ty_n\| &\le \|y_n - T^n y_n\| + \|T^n y_n - T^{n+1}y_n\| + \|T^{n+1}y_n - Ty_n\| \\ &\le (1+L_1)\|y_n - T^n y_n\| + \|T^n y_n - T^{n+1}y_n\|.\end{aligned}$$

Note that

$$\begin{aligned}\|T^n y_n - T^{n+1}y_n\| &\le \|T^n y_n - y_n\| + \|y_n - y_{n+1}\| + \|y_{n+1} - T^{n+1}y_{n+1}\| \\ &\quad + \|T^{n+1}y_{n+1} - T^{n+1}y_n\| \\ &\le \|T^n y_n - y_n\| + (1+L_{n+1})\|y_n - y_{n+1}\| \\ &\quad + \|y_{n+1} - T^{n+1}y_{n+1}\|.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|y_n - Ty_n\| \le (2+L_1)\|y_n - T^n y_n\| + (1+L_{n+1})\|y_n - y_{n+1}\| \\ + \|y_{n+1} - T^{n+1}y_{n+1}\| \to 0.\end{aligned} \tag{10}$$

Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ which converges weakly to w. From (7), we also obtain that $Ty_{n_i} \to w$ weakly. Next we show that $w \in \Omega$. Since $y_n = S_r x_n$, we derive

$$\Theta(y_n, x) + \varphi(x) - \varphi(y_n) + \frac{1}{r}\langle K'(y_n) - K'(x_n), \eta(x, y_n)\rangle \ge 0, \quad \forall x \in C.$$

From the monotonicity of Θ, we have

$$\frac{1}{r}\langle K'(y_n) - K'(x_n), \eta(x, y_n)\rangle + \varphi(x) - \varphi(y_n) \ge -\Theta(y_n, x) \ge \Theta(x, y_n),$$

and hence

$$\left\langle \frac{K'(y_{n_i}) - K'(x_{n_i})}{r}, \eta(x, y_{n_i}) \right\rangle + \varphi(x) - \varphi(y_{n_i}) \ge \Theta(x, y_{n_i}).$$

Since $\frac{K'(y_{n_i}) - K'(x_{n_i})}{r} \to 0$ and $y_{n_i} \to w$ weakly, from the weak lower semicontinuity of φ and $\Theta(x, y)$ in the second variable y, we have

$$\Theta(x, w) + \varphi(w) - \varphi(x) \le 0$$

for all $x \in C$. For $0 < t \le 1$ and $x \in C$, let $x_t = tx + (1-t)w$. Since $x \in C$ and $w \in C$, we have $x_t \in C$ and hence $\Theta(x_t, w) + \varphi(w) - \varphi(x_t) \le 0$. From the convexity of equilibrium bifunction $\Theta(x, y)$ in the second variable y, we have

$$\begin{aligned}0 &= \Theta(x_t, x_t) + \varphi(x_t) - \varphi(x_t) \\ &\le t\Theta(x_t, x) + (1-t)\Theta(x_t, w) + t\varphi(x) + (1-t)\varphi(w) - \varphi(x_t) \\ &\le t[\Theta(x_t, x) + \varphi(x) - \varphi(x_t)],\end{aligned}$$

and hence $\Theta(x_t, x) + \varphi(x) - \varphi(x_t) \ge 0$. Then, we have

$$\Theta(w, x) + \varphi(x) - \varphi(w) \ge 0$$

for all $x \in C$ and hence $w \in \Omega$.

We shall prove that $w \in F(T)$. As a matter of fact, Lemma 2.3 and (10) guarantee that every weak limit point of $\{y_n\}$ is a fixed point of T. That is, $\omega_w(y_n) \subset F(T)$. Hence, $w \in F(T)$.

Therefore, we have

$$w \in F(T) \cap \Omega.$$

This fact, the inequality (7) and Lemma 2.4 ensure the strong convergence of $\{x_n\}$ to $P_{F(T)\cap\Omega}x_0$. This completes the proof. □

As a direct consequence of Theorem 3.1, we obtain the following.

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction satisfying conditions* (H1)-(H3) *and let $T : C \to C$ be an asymptotically nonexpansive mapping such that $F(T) \cap \Omega \neq \emptyset$. Assume that*:

(i) *$\eta : C \times C \to H$ is Lipschitz continuous with constant $\lambda > 0$ such that*;
  (a) *$\eta(x, y) + \eta(y, x) = 0, \ \forall x, y \in C$,*
  (b) *$\eta(\cdot, \cdot)$ is affine in the first variable,*
  (c) *for each fixed $y \in C$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology*;

(ii) *$K : C \to \mathbb{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative $K^{'}$ is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu > 0$ such that $\sigma \geq \lambda\nu$*;

(iii) *for each $x \in C$; there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that, for any $C \ni y \notin D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K^{'}(y) - K^{'}(x), \eta(z_x, y)\rangle < 0;$$

(iv) *$\alpha_n \in [a, 1]$ for some $a \in (0, 1)$.*

Then the sequence $\{x_n\}$ generated iteratively by (3) *converges strongly to $P_{F(T)\cap\Omega}x_0$ provided S_r is firmly nonexpansive.*

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be an asymptotically k-strict pseudo-contraction such that $F(T)$ is nonempty and bounded. Let $\delta \in (k, 1)$ be a constant and $\{\alpha_n\}$ be a sequence in $[0, 1]$. Assume $\alpha_n \in [a, 1]$ for some $a \in (0, 1)$. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ generated iteratively by*

$$(11) \qquad \begin{cases} z_n = (1 - \alpha_n)x_n + \alpha_n[\delta x_n + (1 - \delta)T^n x_n], \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where $\theta_n = \gamma_n\Delta_n^2$, $\Delta_n = \sup\{\|x_n - p\| : p \in F(T)\} < \infty$. Then the sequence $\{x_n\}$ defined by (11) *converges strongly to $P_{F(T)}x_0$.*

Proof. Set $\varphi(x) = 0$ and $\Theta(x, y) = 0$ for all $x, y \in C$ and put $r = 1$. Take $K(x) = \frac{\|x\|^2}{2}$ and $\eta(y, x) = y - x$ for all $x, y \in C$. Then we have $y_n = x_n$. Hence, by the similar argument as that in the proof of Theorem 3.1, we can obtain the desired result. This completes the proof. □

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be an asymptotically nonexpansive mappings such that $F(T)$ is nonempty and bounded. Let $\delta \in (k, 1)$ be a constant and $\{\alpha_n\}$ be* *a sequence in* $[0,1]$. *Assume* $\alpha_n \in [a,1]$ *for some* $a \in (0,1)$. *Then the sequence* $\{x_n\}$ *defined by* (11) *converges strongly to* $P_{F(T)}x_0$.

Now we give another iterative algorithm as follows.

Algorithm 3.2. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, $\varphi : C \to \mathbb{R}$ *be a lower semicontinuous and convex real valued function,* $\Theta : C \times C \to \mathbb{R}$ *be an equilibrium bifunction and* $T : C \to C$ *be an asymptotically* k*-strict pseudo-contraction. Let* r *be a positive parameter and* $\delta \in (k,1)$ *be a constant. Let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$. *Define the sequences* $\{x_n\}$ *and* $\{y_n\}$ *by the following manner*:

$$(12)\quad \begin{cases} x_0 \in C \ \ \text{chosen arbitrarily,} \\ \Theta(y_n, x) + \varphi(x) - \varphi(y_n) + \dfrac{1}{r}\langle K'(y_n) - K'(x_n), \eta(x, y_n)\rangle \geq 0, \forall x \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n[\delta y_n + (1-\delta)T^n y_n]. \end{cases}$$

Finally we state and prove a weak convergence theorem concerning Algorithm 3.2.

Theorem 3.2. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\varphi : C \to \mathbb{R}$ *be a lower semicontinuous and convex functional. Let* $\Theta : C \times C \to \mathbb{R}$ *be an equilibrium bifunction satisfying conditions* (H1)-(H3) *and let* $T : C \to C$ *be an asymptotically* k*-strict pseudo-contraction such that* $F(T) \cap \Omega \neq \emptyset$. *Assume that*:

- (i) $\eta : C \times C \to H$ *is Lipschitz continuous with constant* $\lambda > 0$ *such that*;
  - (a) $\eta(x,y) + \eta(y,x) = 0, \ \ \forall x, y \in C,$
  - (b) $\eta(\cdot,\cdot)$ *is affine in the first variable,*
  - (c) *for each fixed* $y \in C$, $x \mapsto \eta(y,x)$ *is sequentially continuous from the weak topology to the weak topology*;
- (ii) $K : C \to \mathbb{R}$ *is* η*-strongly convex with constant* $\sigma > 0$ *and its derivative* K' *is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant* $\nu > 0$ *such that* $\sigma \geq \lambda\nu$;
- (iii) *for each* $x \in C$, *there exist a bounded subset* $D_x \subset C$ *and* $z_x \in C$ *such that, for any* $C \ni y \notin D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0;$$

- (iv) $\alpha_n \in [a,b]$ *for some* $a, b \in (0,1)$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then the sequence $\{x_n\}$ *generated iteratively by* (12) *converges weakly to* $w \in P_{F(T)\cap\Omega}$ *provided* S_r *is firmly nonexpansive, where* $w = \lim_{n\to\infty} P_{F(T)\cap\Omega}(x_n)$.

Proof. By Lemma 2.1, $\{x_n\}$ and $\{y_n\}$ are all well-defined. Let $p \in F(T) \cap \Omega$, from $y_n = S_r x_n$, we have

$$\|y_n - p\| = \|S_r x_n - S_r p\| \leq \|x_n - p\|.$$

Therefore, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|\delta(y_n-p)+(1-\delta)(T^ny_n-p)\|^2\\ &=(1-\alpha_n)\|x_n-p\|^2+\alpha_n[\delta\|y_n-p\|^2+(1-\delta)\|T^ny_n-p\|^2\\ &\quad-\delta(1-\delta)\|y_n-T^ny_n\|^2]\\ &\le(1-\alpha_n)\|x_n-p\|^2+\alpha_n\{\delta\|y_n-p\|^2+(1-\delta)[(1+\gamma_n)\\ &\quad\times\|y_n-p\|^2+k\|y_n-T^ny_n\|^2]-\delta(1-\delta)\|y_n-T^ny_n\|^2\}\\ &=(1-\alpha_n)\|x_n-p\|^2+\alpha_n\{[1+(1-\delta)\gamma_n]\|y_n-p\|^2\\ &\quad+(1-\delta)(k-\delta)\|y_n-Ty_n\|^2\}\\ &\le(1-\alpha_n)\|x_n-p\|^2+\alpha_n(1+\gamma_n)\|y_n-p\|^2\\ &\le(1+\gamma_n)\|x_n-p\|^2.\end{aligned}$$

This implies that $\lim_{n\to\infty}\|x_n-p\|$ exists. Hence, $\{x_n\}$, $\{y_n\}$ are all bounded and

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

Next, for $p\in F(T)\cap\Omega$, noting S_r is firmly nonexpansive, we have

$$\begin{aligned}\|y_n-p\|^2&=\|S_rx_n-S_rp\|^2\\ &\le\|S_rx_n-S_rp,x_n-p\rangle\\ &=\langle y_n-p,x_n-p\rangle\\ &=\frac{1}{2}\{\|y_n-p\|^2+\|x_n-p\|^2-\|x_n-y_n\|^2\},\end{aligned}$$

and hence,

$$\|y_n-p\|^2\le\|x_n-p\|^2-\|x_n-y_n\|^2.$$

Therefore, we have

$$\begin{aligned}\|x_{n+1}-p\|^2&\le(1-\alpha_n)\|x_n-p\|^2+\alpha_n(1+\gamma_n)\|y_n-p\|^2\\ &\le(1-\alpha_n)\|x_n-p\|^2+\alpha_n(\|x_n-p\|^2-\|x_n-y_n\|^2)\\ &\quad+\alpha_n\gamma_n\|y_n-p\|^2\\ &\le\|x_n-p\|^2-\alpha_n\|x_n-y_n\|^2+\alpha_n\gamma_n\|y_n-p\|^2.\end{aligned}$$

So, we obtain

$$\|x_n-y_n\|^2\le\frac{1}{\alpha_n}\{\|x_n-p\|^2-\|x_{n+1}-p\|^2\}+\gamma_n\|y_n-p\|^2\to0.$$

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to w. From $\|x_n-y_n\|\to0$, we also have that $y_{n_i}\to w$ weakly. First, as in the proof of Theorem 3.1, we can show $w\in\Omega$. Let us show that $w\in F(T)$.

Let $p \in F(T) \cap \Omega$. Since $\alpha_n(1-\delta)T^n y_n = x_{n+1} - (1-\alpha_n)x_n - \alpha\delta y_n$, we have

$$\begin{aligned}\|\alpha_n(1-\delta)(T^n y_n - x_n)\| &= \|x_{n+1} - x_n + \alpha_n\delta(x_n - y_n)\| \\ &\le \|x_{n+1} - x_n\| + \alpha_n\|x_n - y_n\| \\ &\to 0,\end{aligned}$$

which implies that

$$\|T^n y_n - x_n\| \to 0.$$

Hence

$$\|T^n y_n - y_n\| \le \|T^n y_n - x_n\| + \|y_n - x_n\| \to 0.$$

Repeating the similar argument as Theorem 3.1, we can obtain

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0.$$

From this and $y_{n_i} \to w$ weakly, we obtain $w \in F(T)$. Then, $w \in F(T) \cap \Omega$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \to w^{'}$ weakly. Then, we have

$$w^{'} \in F(T) \cap \Omega.$$

If $w \ne w^{'}$, from the opial theorem [20], we get

$$\begin{aligned}\lim_{n\to\infty} \|x_n - w\| &= \liminf_{i\to\infty} \|x_{n_i} - w\| \\ &< \liminf_{i\to\infty} \|x_{n_i} - w^{'}\| \\ &= \lim_{n\to\infty} \|x_n - w^{'}\| \\ &= \liminf_{j\to\infty} \|x_{n_j} - w^{'}\| \\ &< \liminf_{j\to\infty} \|x_{n_j} - w\| \\ &= \lim_{n\to\infty} \|x_n - w\|.\end{aligned}$$

This is a contradiction. So we have $w = w^{'}$. This implies that

$$x_n \to F(T) \cap \Omega \quad \text{weakly}.$$

Let $z_n = P_{F(T)\cap\Omega}(x_n)$. Since $w \in F(T) \cap \Omega$, we have $\langle x_n - z_n, z_n - w\rangle \ge 0$. Hence, we have that $\{z_n\}$ converges strongly to some $w_0 \in F(T) \cap \Omega$. Since $\{x_n\}$ converges weakly to w, we have

$$\langle w - w_0, w_0 - w\rangle \ge 0.$$

Therefore, we obtain

$$w = w_0 = \lim_{n\to\infty} P_{F(T)\cap\Omega}(x_n).$$

This completes the proof. □

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction satisfying conditions* (H1)-(H3) *and let $T : C \to C$ be an asymptotically nonexpansive mapping such that $F(T) \cap \Omega \neq \emptyset$. Assume that*:

- (i) *$\eta : C \times C \to H$ is Lipschitz continuous with constant $\lambda > 0$ such that*;
  - (a) *$\eta(x, y) + \eta(y, x) = 0, \ \forall x, y \in C$,*
  - (b) *$\eta(\cdot, \cdot)$ is affine in the first variable,*
  - (c) *for each fixed $y \in C$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology*;
- (ii) *$K : C \to \mathbb{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu > 0$ such that $\sigma \geq \lambda\nu$;*
- (iii) *for each $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that, for any $C \ni y \notin D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0;$$

- (iv) *$\alpha_n \in [a, b]$ for some $a, b \in (0, 1)$ and $\sum_{n=0}^{\infty} \gamma_n < \infty$.*

Then the sequence $\{x_n\}$ generated iteratively by (12) *converges weakly to $w \in P_{F(T)\cap\Omega}$ provided S_r is firmly nonexpansive, where $w = \lim_{n\to\infty} P_{F(T)\cap\Omega}(x_n)$.*

Corollary 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction satisfying conditions* (H1)-(H3) *such that $\Omega \neq \emptyset$. Assume that*:

- (i) *$\eta : C \times C \to H$ is Lipschitz continuous with constant $\lambda > 0$ such that*;
  - (a) *$\eta(x, y) + \eta(y, x) = 0, \ \forall x, y \in C$,*
  - (b) *$\eta(\cdot, \cdot)$ is affine in the first variable,*
  - (c) *for each fixed $y \in C$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology*;
- (ii) *$K : C \to \mathbb{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu > 0$ such that $\sigma \geq \lambda\nu$;*
- (iii) *for each $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that, for any $C \ni y \notin D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0.$$

Let the sequences $\{x_n\}$ *and* $\{y_n\}$ *be generated by*

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ \Theta(y_n, x) + \varphi(x) - \varphi(y_n) + \dfrac{1}{r}\langle K^{'}(y_n) - K^{'}(x_n), \eta(x, y_n)\rangle \geq 0, \forall x \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n, \end{cases}$$

where r *is a positive parameter and* $\{\alpha_n\}$ *is a sequence* $\in [a, b]$ *for some* $a, b \in (0, 1)$.
Then the sequence $\{x_n\}$ *converges weakly to* $w \in \Omega$ *provided* S_r *is firmly nonexpansive, where* $w = \lim_{n\to\infty} P_\Omega(x_n)$.

Proof. Taking $T = I$ in Theorem 3.2, we can obtain our desired result. This completes the proof. □

Corollary 3.6. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $T : C \to C$ *be an asymptotically* k*-strict pseudo-contraction such that* $F(T) \neq \emptyset$. *Let* $\delta \in (k, 1)$ *be a constant and* $\{\alpha_n\}$ *be a real sequence in* $[0, 1]$. *Assume that* $\alpha_n \in [a, b]$ *for some* $a, b \in (0, 1)$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$. *For given* $x_0 \in C$ *arbitrarily, let the sequence* $\{x_n\}$ *generated iteratively by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[\delta y_n + (1 - \delta)T^n y_n].$$

Then $\{x_n\}$ *converges weakly to* $w \in P_{F(T)}$, *where* $w = \lim_{n\to\infty} P_{F(T)}(x_n)$.

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