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D. H. LEHMER PROBLEM OVER HALF INTERVALS

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Abstract. Let $q \geq 3$ be an odd integer and a be an integer coprime to q. Denote by $N(a, q)$ the number of pairs of integers b, c with $bc \equiv a \pmod q$, $1 \leq b, c \leq \frac{q-1}{2}$ and with b, c having different parity. The main purpose of this paper is to study the sum $\sum_{a=1}^{\prime q} \left(N(a,q) - \frac{\phi(q)}{8}\right)^2$, and obtain a sharp asymptotic formula.

1. Introduction

Let $q \geq 3$ be an odd integer and a be an integer coprime to q. For each integer b with $1 \leq b < q$ and $(b, q) = 1$, there is an unique integer c with $1 \leq c < q$ such that $bc \equiv a \pmod q$. Let $M(a, q)$ denote the number of solutions of the congruence equation $bc \equiv a \pmod q$ with $1 \leq b, c < q$ such that b, c are of opposite parity. D. H. Lehmer posed the problem to find $M(1, p)$ or at least to say some thing non-trivial about it (see problem F12 of reference [1], page 251), where p is an odd prime. Zhang [2] proved that

$$M(1, q) = \frac{\phi(q)}{2} + O\left(q^{\frac{1}{2}} d^2(q) \ln^2 q\right),$$

where $\phi(q)$ is the Euler function and $d(q)$ the Dirichlet divisor function. For the further properties of $M(a, p)$, he studied the mean square value of the error term $M(a, p) - \frac{p-1}{2}$, and obtained

$$\sum_{a=1}^{p-1} \left(M(a,p) - \frac{p-1}{2}\right)^2 = \frac{3}{4}p^2 + O\left(p \exp\left(\frac{3 \ln p}{\ln \ln p}\right)\right),$$

see reference [4]. For general odd integer q, the similar properties of $M(a, q)$ was studied in [5].

The distribution of the inverse modular q of an integer b is very irregular, so does the integer in D. H. Lehmer problem. A natural problem is how many integers with this property in a incomplete interval. How can we say some thing

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nontrival about the integers which have opposite parity with their inverses over a half interval?

It is interesting to study the D. H. Lehmer problem over a half interval $\left[1, \frac{q-1}{2}\right]$. Denote by $N(a,q)$ the number of pairs of integers b, c with $bc \equiv a \pmod q$, $1 \le b, c \le \frac{q-1}{2}$ and with b, c having different parity. The method of [2], give also the formula

$$N(a,q) = \frac{\phi(q)}{8} + O\left(q^{\frac{1}{2}} d^2(q) \ln^2 q\right).$$

In [6], the authors studied the mean square value of error term

$$E(a,q) = N(a,q) - \frac{1}{8}\phi(q)$$

in the case of $q = p$ and obtained a sharp asymptotic formula

$$\sum_{a=1}^{p-1} E^2(a,p) = \frac{9}{64}p^2 + O(p^{1+\epsilon}).$$

For general odd integer q, whether there exists an asymptotic formula for mean square value of $E(a,q)$ was posed as an open problem in [6].

In this paper, we solved the problem completely by using the properties of Dedekind sums, Cochrane sums and L-functions. That is, we will prove the following result:

Theorem. *Let $q \ge 3$ be an odd integer. Then we have the asymptotic formula*

$$\sum_{a=1}^{q}{}' E^2(a,q) = \frac{9\phi^3(q)}{64q} \prod_{p^\alpha \| q} \frac{\frac{(p+1)^3}{p(p^2+1)} - \frac{1}{p^{3\alpha-1}}}{1 + \frac{1}{p} + \frac{1}{p^2}} + O\left(q^{1+\epsilon}\right),$$

where $\sum'$ means the sum is taken only over integers coprime to q and ϵ is any fixed positive number.

2. Some lemmas

To prove Theorem, we need the following lemmas.

Lemma 1. *Let $q \ge 3$ be an odd number. For any nonprincipal character χ* mod q, *we have*

$$\sum_{a=1}^{q} a\chi(a) = \frac{\chi(2)q}{1 - 2\chi(2)} \sum_{a=1}^{\frac{q-1}{2}} \chi(a).$$

Proof. From the properties of Dirichlet character, we have

$$\sum_{a=1}^{q} 2a\chi(2a) = \sum_{a=1}^{\frac{q-1}{2}} 2a\chi(2a) + \sum_{a=\frac{q+1}{2}}^{q} 2a\chi(2a)$$

$$= \sum_{a=1}^{\frac{q-1}{2}} 2a\chi(2a) + \sum_{a=1}^{\frac{q+1}{2}} (2a-1)\chi(q+2a-1) + q\sum_{a=1}^{\frac{q+1}{2}} \chi(2a-1)$$
$$= \sum_{a=1}^{q} a\chi(a) + q\sum_{a=1}^{\frac{q+1}{2}} \chi(2a-1).$$

Noting that

$$\sum_{a=1}^{\frac{q+1}{2}} \chi(2a-1) + \sum_{a=1}^{\frac{q-1}{2}} \chi(2a) = \sum_{a=1}^{q} \chi(a) = 0,$$

we can write

$$(1-2\chi(2))\sum_{a=1}^{q} a\chi(a) = \sum_{a=1}^{q} a\chi(a) - \sum_{a=1}^{q} 2a\chi(2a) = q\sum_{a=1}^{\frac{q-1}{2}} \chi(2a) = \chi(2)q\sum_{a=1}^{\frac{q-1}{2}} \chi(a).$$

That is,

$$\sum_{a=1}^{q} a\chi(a) = \frac{\chi(2)q}{1-2\chi(2)} \sum_{a=1}^{\frac{q-1}{2}} \chi(a).$$

This proves Lemma 1. □

Lemma 2. *Let $q \geq 5$ be an odd integer and χ be a Dirichlet character modulo q such that $\chi(-1) = 1$. Then we have*

$$\sum_{a=1}^{\left[\frac{q}{4}\right]} \chi(a) = -\frac{\bar{\chi}(4)}{8q} \sum_{a=1}^{4q} a\chi\chi_4(a),$$

where χ_4 is the primitive Dirichlet character modulo 4.

Proof. See the proof of Lemma 3 in reference [6] □

Lemma 3. *Let q be an odd integer. Then for any positive integer a with $(a, q) = 1$, we have the identities*

$$E(a,q) = \frac{1}{2q^2\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \bar{\chi}(a)(\chi(4)-1)(\bar{\chi}(2)-2)^2 \left(\sum_{b=1}^{q} b\chi(b)\right)^2$$
$$- \frac{1}{32q^2\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=1 \\ \chi\neq\chi_0}} \bar{\chi}(a)\bar{\chi}(4) \left(\sum_{b=1}^{4q} b\chi\chi_4(b)\right)^2 + O\left(\frac{1}{q}\right).$$

Proof. From the orthogonality relation for character sums modulo q and the definition of $N(a,q)$ we have

(1)

$$\begin{aligned}
&N(a,q)\\
&= \frac{1}{2} \sum_{\substack{b=1 \\ bc\equiv a \pmod q}}^{\frac{q-1}{2}} \sum_{c=1}^{\frac{q-1}{2}} (1-(-1)^{b+c})\\
&= \frac{1}{2\phi(q)}\left(\sum_{\chi \bmod q} \bar{\chi}(a) \sum_{b=1}^{\frac{q-1}{2}}\sum_{c=1}^{\frac{q-1}{2}} \chi(bc) - \sum_{\chi \bmod q} \bar{\chi}(a)\left(\sum_{b=1}^{\frac{q-1}{2}}(-1)^b\chi(b)\right)^2\right)\\
&= \frac{\phi(q)}{8} + \frac{1}{2\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi\neq\chi_0}} \bar{\chi}(a)\left(\sum_{b=1}^{\frac{q-1}{2}}\chi(b)\right)^2 - \frac{1}{2\phi(q)}\sum_{\chi \bmod q}\bar{\chi}(a)\left(\sum_{b=1}^{\frac{q-1}{2}}(-1)^b\chi(b)\right)^2.
\end{aligned}$$

It is clear that

$$\sum_{b=1}^{\frac{q-1}{2}} \chi(b) = 0 \tag{2}$$

if $\chi \neq \chi_0$. Now if $q \equiv 1 \pmod 4$ and $\chi(-1) = 1$, then we can deduce that

$$\begin{aligned}
\sum_{b=1}^{\frac{q-1}{2}}(-1)^b\chi(b) &= \sum_{b=1}^{\frac{q-1}{4}}\chi(2b) - \sum_{b=1}^{\frac{q-1}{4}}\chi(q-2b+1)\\
&= \sum_{b=1}^{\frac{q-1}{4}}\chi(2b) - \sum_{b=\frac{q+3}{4}}^{\frac{q-1}{2}}\chi(2b)\\
&= \begin{cases} O(1) & \text{if } \chi=\chi_0 \text{ is a principal character;}\\ 2\chi(2)\sum_{b=1}^{\frac{q-1}{4}}\chi(b) & \text{if } \chi(-1)=1 \text{ and } \chi\neq\chi_0, \end{cases}
\end{aligned} \tag{3}$$

while if $\chi(-1) = -1$, then we have

$$\begin{aligned}
\sum_{b=1}^{\frac{q-1}{2}}(-1)^b\chi(b) &= \sum_{b=1}^{\frac{q-1}{4}}\chi(2b) + \sum_{b=1}^{\frac{q-1}{4}}\chi(q-2b+1)\\
&= \sum_{b=1}^{\frac{q-1}{4}}\chi(2b) + \sum_{b=\frac{q+3}{4}}^{\frac{q-1}{2}}\chi(2b)
\end{aligned} \tag{4}$$

$$= \chi(2) \sum_{b=1}^{\frac{q-1}{2}} \chi(b)$$

and (see Lemma 1)

$$\sum_{b=1}^{\frac{q-1}{2}} \chi(b) = \frac{\bar{\chi}(2) - 2}{q} \sum_{b=1}^{q-1} b\chi(b). \tag{5}$$

For the case of $q \equiv 3 \pmod 4$, we can also get

$$\sum_{b=1}^{\frac{q-1}{2}} (-1)^b \chi(b) = \begin{cases} O(1) & \text{if } \chi = \chi_0; \\ 2\chi(2) \sum_{b=1}^{\frac{q-3}{4}} \chi(b) & \text{if } \chi(-1) = 1 \text{ and } \chi \neq \chi_0; \\ \chi(2) \sum_{b=1}^{\frac{q-1}{2}} \chi(b) & \text{if } \chi(-1) = -1 \end{cases} \tag{6}$$

and

$$\sum_{b=1}^{\frac{q-1}{2}} \chi(b) = \begin{cases} 0 & \text{if } \chi(-1) = 1; \\ \frac{\bar{\chi}(2)-2}{q} \sum_{b=1}^{q-1} b\chi(b) & \text{if } \chi(-1) = -1. \end{cases} \tag{7}$$

Now combining (1)-(7), Lemma 1 and Lemma 2 we can write

$$\begin{aligned} N(a,q) &= \frac{\phi(q)}{8} + \frac{1}{2\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \bar{\chi}(a)(1-\chi(4)) \left(\sum_{b=1}^{\frac{q-1}{2}} \chi(b) \right)^2 \\ &\quad - \frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=1 \\ \chi\neq\chi_0}} \bar{\chi}(a)\chi(4) \left(\sum_{b=1}^{\left[\frac{q}{4}\right]} \chi(b) \right)^2 + O\left(\frac{1}{q}\right) \\ &= \frac{\phi(q)}{8} + \frac{1}{2q^2\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \bar{\chi}(a)(\chi(4)-1)(\bar{\chi}(2)-2)^2 \left(\sum_{b=1}^{q} b\chi(b) \right)^2 \\ &\quad - \frac{1}{32q^2\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=1 \\ \chi\neq\chi_0}} \bar{\chi}(a)\bar{\chi}(4) \left(\sum_{b=1}^{4q} b\chi\chi_4(b) \right)^2 + O\left(\frac{1}{q}\right), \end{aligned}$$

if $q \equiv 1 \pmod 4$.

Similarly, we have

$$N(a,q)=\frac{\phi(q)}{8}+\frac{1}{2q^2\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\bar\chi(a)(\chi(4)-1)(\bar\chi(2)-2)^2\left(\sum_{b=1}^{q}b\chi(b)\right)^2$$
$$-\frac{1}{32q^2\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1\\ \chi\neq\chi_0}}\bar\chi(a)\bar\chi(4)\left(\sum_{b=1}^{4q}b\chi\chi_4(b)\right)^2+O\left(\frac{1}{q}\right),$$

if $q\equiv 3 \pmod 4$. Now Lemma 3 can be easily obtained from the definition of $E(a,q)$. □

Lemma 4. *Let integer $q\geq 3$ and $(h,q)=1$. Denote by $S(h,q)$ the Dedekind sum*

$$S(h,q)=\sum_{a=1}^{q}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ha}{q}\right)\right),$$

where

$$((x))=\begin{cases}x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer.}\end{cases}$$

Then we have

$$S(h,q)=\frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(h)|L(1,\chi)|^2,$$

where $\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}$ denotes the summation over all Dirichlet character modulo d with $\chi(-1)=-1$, and $L(s,\chi)$ the Dirichlet L-function corresponding to χ.

Proof. See [3]. □

Lemma 5. *Let q be any odd integer with $q\geq 3$, χ be a Dirichlet character modulo q. Then for any integer $m\geq 0$, we have the identity*

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\bar\chi(2^m)\left|\sum_{r=1}^{q}r\chi(r)\right|^4=q^4\phi(q)\sum_{d|q}\sum_{l|q}\mu(d)\mu(l){\sum_{r=1}^{q}}'S\left(2^m r,\frac{q}{d}\right)S\left(r,\frac{q}{l}\right),$$

where $\mu(n)$ is Möbius function, $\sum_{r=1}^{\prime q}$ denotes the summation over all integers r with $1\leq r\leq q$ and $(r,q)=1$.

Proof. First we introduce Cochrane sum $C(h,q)$ as follows

$$C(h,q)={\sum_{a=1}^{q}}'\left(\left(\frac{\bar a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right).$$

Note that ${\sum'}_{c=1}^{q} \chi(c)\left(\left(\frac{c}{q}\right)\right) = \sum_{c=1}^{q} \chi(c)\left(\frac{c}{q}-\frac{1}{2}\right) = 0$, if $\chi(-1)=1$, then from the orthogonality relation for Dirichlet characters modulo q, we can write

(8)

$$\begin{aligned}
C(a,q) &= {\sum_{r=1}^{q}}' \left(\left(\frac{\overline{r}}{q}\right)\right)\left(\left(\frac{ar}{q}\right)\right) \\
&= \frac{1}{\phi(q)} \sum_{\chi \bmod q} \left\{\sum_{r=1}^{q} \chi(r)\left(\left(\frac{r}{q}\right)\right)\right\} \times \left\{\sum_{s=1}^{q} \chi(s)\left(\left(\frac{as}{q}\right)\right)\right\} \\
&= \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left\{\sum_{r=1}^{q} \chi(r)\left(\left(\frac{r}{q}\right)\right)\right\} \times \left\{\sum_{s=1}^{q} \overline{\chi}(a)\chi(as)\left(\left(\frac{as}{q}\right)\right)\right\} \\
&= \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(a) \left(\sum_{r=1}^{q} \chi(r)\left(\left(\frac{r}{q}\right)\right)\right)^2 \\
&= \frac{1}{q^2\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(a) \left(\sum_{r=1}^{q} r\chi(r)\right)^2,
\end{aligned}$$

where we used the fact $\sum_{r=1}^{q} \chi(r) = 0$ if χ does not equal the principal character modulo q. Now the identity

$$(9)\qquad {\sum_{a=1}^{q}}' C(2^m a,q)C(a,q) = \frac{1}{q^4\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \bar{\chi}(2^m) \left|{\sum_{r=1}^{q}}' r\chi(r)\right|^4$$

follows from (8) immediately.

On the other hand, From the definition of $C(a,q)$ we have

$$\begin{aligned}
{\sum_{a=1}^{q}}' C(2^m a,q)C(a,q) &= {\sum_{a=1}^{q}}' {\sum_{r=1}^{q}}' \left(\left(\frac{\overline{r}}{q}\right)\right)\left(\left(\frac{2^m ar}{q}\right)\right) {\sum_{s=1}^{q}}' \left(\left(\frac{\overline{s}}{q}\right)\right)\left(\left(\frac{as}{q}\right)\right) \\
&= {\sum_{r=1}^{q}}' {\sum_{s=1}^{q}}' \left(\left(\frac{\overline{r}}{q}\right)\right)\left(\left(\frac{\overline{s}}{q}\right)\right) {\sum_{a=1}^{q}}' \left(\left(\frac{2^m ar}{q}\right)\right)\left(\left(\frac{as}{q}\right)\right) \\
&= {\sum_{r=1}^{q}}' {\sum_{s=1}^{q}}' \left(\left(\frac{\overline{r}}{q}\right)\right)\left(\left(\frac{\overline{s}}{q}\right)\right) {\sum_{a=1}^{q}}' \left(\left(\frac{2^m r\overline{s}a}{q}\right)\right)\left(\left(\frac{a}{q}\right)\right).
\end{aligned}$$

Noting that $(rs,q)=1$, if a passes through a reduced residue system modulo q, so does $\overline{s}a$. Therefore, from the definition of Dedekind sums $S(h,q)$ and the identities

$$\sum_{s=1}^{q}{}' = \sum_{d|q}\mu(d)\sum_{s=1}^{q/d} \quad \text{and} \quad S(r,q) = S(\overline{r},q),$$

we have

$$\begin{aligned}
(10) \qquad &\sum_{a=1}^{q}{}' C(2^m a, q)C(a,q)\\
&= \sum_{r=1}^{q}{}'\sum_{s=1}^{q}{}'\left(\left(\frac{\overline{r}}{q}\right)\right)\left(\left(\frac{\overline{s}}{q}\right)\right)\sum_{a=1}^{q}{}'\left(\left(\frac{2^m r\overline{s}a}{q}\right)\right)\left(\left(\frac{a}{q}\right)\right)\\
&= \sum_{r=1}^{q}{}'\sum_{s=1}^{q}{}'\left(\left(\frac{\overline{r}}{q}\right)\right)\left(\left(\frac{\overline{s}}{q}\right)\right)\sum_{\ell|q}\mu(\ell)\sum_{a=1}^{\frac{q}{\ell}}\left(\left(\frac{2^m r\overline{s}a}{q/\ell}\right)\right)\left(\left(\frac{a}{q/\ell}\right)\right)\\
&= \sum_{\ell|q}\mu(\ell)\sum_{r=1}^{q}{}'\sum_{s=1}^{q}{}'\left(\left(\frac{\overline{r}}{q}\right)\right)\left(\left(\frac{\overline{s}}{q}\right)\right)S\left(2^m r\overline{s}, \frac{q}{\ell}\right)\\
&= \sum_{\ell|q}\mu(\ell)\sum_{r=1}^{q}{}'\sum_{s=1}^{q}{}'\left(\left(\frac{\overline{r}}{q}\right)\right)\left(\left(\frac{s}{q}\right)\right)S\left(2^m rs, \frac{q}{\ell}\right)\\
&= \sum_{\ell|q}\mu(\ell)\sum_{d|q}\mu(d)\sum_{r=1}^{q}{}'\sum_{s=1}^{q/d}\left(\left(\frac{2^m rs}{q/d}\right)\right)\left(\left(\frac{s}{q/d}\right)\right)S\left(\overline{r}, \frac{q}{\ell}\right)\\
&= \sum_{d|q}\sum_{\ell|q}\mu(d)\mu(\ell)\sum_{r=1}^{q}{}' S\left(2^m r, \frac{q}{d}\right)S\left(\overline{r}, \frac{q}{\ell}\right)\\
&= \sum_{d|q}\sum_{\ell|q}\mu(d)\mu(\ell)\sum_{r=1}^{q}{}' S\left(2^m r, \frac{q}{d}\right)S\left(r, \frac{q}{\ell}\right).
\end{aligned}$$

Now Lemma 5 follows from (9) and (10). □

Lemma 6. *Let u and v be odd integers with $(u,v) = d \geq 2$, χ_u^0 and χ_v^0 be the principal character modulo u and v, respectively. If $r(n) = \sum_{d|n}\chi_u^0(d)\chi_v^0\left(\frac{n}{d}\right)$, then for any integer $m \geq 0$ we have the identity*

$$\sum_{\substack{n=1\\(n,d)=1}}^{\infty}\frac{r(2^m n)r(n)}{n^2} = \frac{(3m+5)\pi^4}{72}\Pi(u,v),$$

where $\Pi(u,v) := \dfrac{\prod_{p|uv}\frac{(p^2-1)^2}{p^2(p^2+1)}}{\prod_{p|d}\frac{p^2}{p^2-1}}$.

Proof. The proof is easy once we notice that $n \mapsto r(n)$ is multiplicative and that

$$r(p^\alpha) = \begin{cases} 1, & \text{if } p|u,\ p\nmid v; \\ \alpha+1, & \text{if } p\nmid u, p\nmid v; \\ 1, & \text{if } p\nmid u,\ p|v; \\ 0, & \text{if } p|u,\ p|v, \end{cases}$$

for any positive integer α and prime p. □

Lemma 7. *Let u and v are odd integers with $(u,v) = d \geq 2$, χ_u^0 and χ_v^0 be the principal character modulo u and v, respectively. Then for any integer $m \geq 0$ we have the asymptotic formula*

$$\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(2^m)|L(1,\chi\chi_u^0)|^2|L(1,\chi\chi_v^0)|^2 = \frac{(3m+5)\pi^4}{72\cdot 2^{m+1}}\phi(d)\Pi(u,v) + O_m(d^\epsilon).$$

Proof. For convenience, we put

$$A(y,\chi) = \sum_{N<n\leq y} \chi(n)r(n),$$

where N is a parameter with $d \leq N < d^4$ and $r(n)$ was defined in Lemma 6. Then from Abel's identity we have

$$L(1,\chi\chi_u^0)L(1,\chi\chi_v^0) = \sum_{n=1}^{\infty} \frac{\chi(n)r(n)}{n} = \sum_{1\leq n\leq N} \frac{\chi(n)r(n)}{n} + \int_N^\infty \frac{A(y,\chi)}{y^2}dy.$$

Hence, we can write

$$\begin{aligned}(14)\qquad &\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(2^m)|L(1,\chi\chi_u^0)|^2|L(1,\chi\chi_v^0)|^2 \\
&= \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(2^m)\left(\sum_{1\leq n_1\leq N}\frac{\bar{\chi}(n_1)r(n_1)}{n_1}\right)\left(\sum_{1\leq n_2\leq N}\frac{\chi(n_2)r(n_2)}{n_2}\right) \\
&\quad + \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(2^m)\left(\sum_{1\leq n_1\leq N}\frac{\bar{\chi}(n_1)r(n_1)}{n_1}\right)\left(\int_N^\infty \frac{A(y,\chi)}{y^2}dy\right) \\
&\quad + \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(2^m)\left(\sum_{1\leq n_2\leq N}\frac{\chi(n_2)r(n_2)}{n_2}\right)\left(\int_N^\infty \frac{A(y,\bar{\chi})}{y^2}dy\right) \\
&\quad + \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(2^m)\left(\int_N^\infty \frac{A(y,\bar{\chi})}{y^2}dy\right)\left(\int_N^\infty \frac{A(y,\chi)}{y^2}dy\right)\end{aligned}$$

$$:= M_1 + M_2 + M_3 + M_4.$$

Now we shall calculate each term in the expression (14).

(i) From the orthogonality of Dirichlet characters we can write

(15)

$$M_1 = \frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{1\le n_1\le N\\ 2^m n_2\equiv n_1}} \sideset{}{'}\sum_{\substack{1\le n_2\le N\\ (\bmod d)}} \frac{r(n_1)r(n_2)}{n_1 n_2} - \frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{1\le n_1\le N\\ 2^m n_2\equiv n_1}} \sideset{}{'}\sum_{\substack{1\le n_2\le N\\ (\bmod d)}} \frac{r(n_1)r(n_2)}{n_1 n_2}.$$

For convenience, we split the sum over n_1 or n_2 into following cases:

- $d \le n_1 \le N, \frac{d}{2^m} \le n_2 \le N$;
- $d \le n_1 \le N, 1 \le n_2 \le \frac{d}{2^m} - 1$;
- $1 \le n_1 \le d-1, \frac{d}{2^m} \le n_2 \le N$;
- $1 \le n_1 \le d-1, 1 \le n_2 \le \frac{d}{2^m} - 1$.

So we have

$$\begin{aligned}
&\frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{d\le n_1\le N\\ 2^m n_2\equiv n_1}} \sideset{}{'}\sum_{\substack{\frac{d}{2^m}\le n_2\le N\\ (\bmod d)}} \frac{r(n_1)r(n_2)}{n_1 n_2}\\
&\ll \phi(d) \sum_{1\le s_1\le \frac{N}{d}} \sum_{1\le s_2\le \frac{2^m N}{d}} \sideset{}{'}\sum_{\substack{l_1=1\\ l_2\equiv l_1}}^{d-1} \sideset{}{'}\sum_{\substack{l_2=1\\ (\bmod d)}}^{d-1} \frac{r(s_1 d + l_1) r(s_2 d + l_2)}{(s_1 d + l_1)(s_2 d + l_2)}\\
&\ll \phi(d) \sum_{1\le s_1\le \frac{N}{d}} \sum_{1\le s_2\le \frac{2^m N}{d}} \sideset{}{'}\sum_{l_1=1}^{d-1} \frac{[(s_1 d + l_1)(s_2 d + l_1)]^\epsilon}{(s_1 d + l_1)(s_2 d + l_1)}\\
&\ll \frac{\phi(d)}{d} \sum_{1\le s_1\le \frac{N}{d}} \sum_{1\le s_2\le \frac{2^m N}{d}} \frac{[(s_1 d + 1)(s_2 d + 1)]^\epsilon}{s_1 s_2}\\
&\ll_m d^\epsilon,
\end{aligned}$$

$$\begin{aligned}
&\frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{d\le n_1\le N\\ 2^m n_2\equiv n_1}} \sideset{}{'}\sum_{\substack{1\le n_2\le \frac{d}{2^m}-1\\ (\bmod d)}} \frac{r(n_1)r(n_2)}{n_1 n_2}\\
&\ll \phi(d) \sum_{1\le r\le \frac{N}{d}} \sum_{1\le n_2\le \frac{d}{2^m n}-1} (r n_2 d)^{\epsilon-1}\\
&\ll d^\epsilon
\end{aligned}$$

and

$$\frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{1\le n_1\le d-1\\ 2^m n_2\equiv n_1}} \sideset{}{'}\sum_{\substack{\frac{d}{2^m}\le n_2\le N\\ (\bmod d)}} \frac{r(n_1)r(n_2)}{n_1 n_2} \ll d^\epsilon,$$

where we have used the estimate $r(n) \ll n^\epsilon$.

For the case $1 \le n_1 \le d-1,\ 1 \le n_2 \le \frac{d}{2^m} - 1$, the solution of the congruence $2^m n_2 \equiv n_1 \pmod{d}$ is $2^m n_2 = n_1$. Hence,

$$\begin{aligned}
&\frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{1\le n_1\le d-1 \\ 2^m n_2\equiv n_1}} \sideset{}{'}\sum_{\substack{1\le n_2\le \frac{d}{2^m}-1 \\ \pmod d}} \frac{r(n_1)r(n_2)}{n_1 n_2} \\
&= \frac{\phi(d)}{2^{m+1}} \sideset{}{'}\sum_{1\le n_2\le \frac{d}{2^m}-1} \frac{r(2^m n_2)r(n_2)}{n_2^2} \\
&= \frac{\phi(d)}{2^{m+1}} \sideset{}{'}\sum_{n=1}^{\infty} \frac{r(2^m n)r(n)}{n^2} + O_m(d^\epsilon).
\end{aligned}$$

Now from Lemma 6, we can immediately get

$$\frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{1\le n_1\le N \\ 2^m n_2\equiv n_1}} \sideset{}{'}\sum_{\substack{1\le n_2\le N \\ \pmod d}} \frac{r(n_1)r(n_2)}{n_1 n_2} = \frac{(3m+5)\pi^4}{72\cdot 2^{m+1}}\phi(d)\Pi(u,v) + O_m(d^\epsilon). \tag{16}$$

Similarly, we can also get the estimate

$$\begin{aligned}
&\frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{1\le n_1\le N \\ 2^m n_2\equiv -n_1}} \sideset{}{'}\sum_{\substack{1\le n_2\le N \\ \pmod d}} \frac{r(n_1)r(n_2)}{n_1 n_2} \\
&= \frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{1\le n_1\le N \\ 2^m n_2+n_1=d}} \sideset{}{'}\sum_{1\le n_2\le N} \frac{r(n_1)r(n_2)}{n_1 n_2} + \frac{\phi(d)}{2} \sideset{}{'}\sum_{\substack{1\le n_1\le N \\ 2^m n_2+n_1=ld,}} \sideset{}{'}\sum_{\substack{1\le n_2\le N \\ l\ge 2}} \frac{r(n_1)r(n_2)}{n_1 n_2} \\
&\ll \phi(d) \sum_{1\le n\le d-1} \frac{2^m r(n) r\left(\frac{d-n}{2^m}\right)}{n(d-n)} + \phi(d) \sideset{}{'}\sum_{1\le n_1\le N} \sum_{l=\left[\frac{n_1}{d}\right]+2}^{\left[\frac{N+n_1}{d}\right]} \frac{2^m r(n_1) r\left(\frac{ld-n_1}{2^m}\right)}{ldn_1 - n_1^2} \\
&\ll_m \frac{\phi(d)}{d} \sum_{1\le n\le d-1} \frac{(n(d-n))^\epsilon}{n} + \frac{\phi(d)}{d} \sideset{}{'}\sum_{1\le n_1\le N} \sum_{l=\left[\frac{n_1}{d}\right]+2}^{\left[\frac{N+n_1}{d}\right]} \frac{n_1^\epsilon (ld-n_1)^\epsilon}{ln_1 - \frac{n_1^2}{d}} \\
&\ll_m d^\epsilon + \frac{\phi(d)d^\epsilon}{d} \sum_{n_1=1}^{N} \sum_{l=1}^{N} \frac{n_1^\epsilon l^\epsilon}{ln_1} \ll_m d^\epsilon.
\end{aligned} \tag{17}$$

Then from (15), (16) and (17), we have

$$M_1 = \frac{(3m+5)\pi^4}{72\cdot 2^{m+1}}\phi(d)\Pi(u,v) + O_m(d^\epsilon). \tag{18}$$

(ii) Noting the partition identity

$$A(y,\chi) = \sum_{n\le\sqrt{y}} \chi(n)\chi_u^0(n) \sum_{m\le y/n} \chi(m)\chi_v^0(m)$$

$$
\begin{aligned}
&+ \sum_{m\le\sqrt{y}} \chi(m)\chi_v^0(m) \sum_{n\le y/m} \chi(n)\chi_u^0(n)\\
&- \sum_{n\le\sqrt{N}} \chi(n)\chi_u^0(n) \sum_{m\le N/n} \chi(m)\chi_v^0(m)\\
&- \sum_{m\le\sqrt{N}} \chi(m)\chi_v^0(m) \sum_{n\le N/m} \chi(n)\chi_u^0(n)\\
&- \left(\sum_{n\le\sqrt{y}} \chi(n)\chi_u^0(n)\right)\left(\sum_{n\le\sqrt{y}} \chi(n)\chi_v^0(n)\right)\\
&+ \left(\sum_{n\le\sqrt{N}} \chi(n)\chi_u^0(n)\right)\left(\sum_{n\le\sqrt{N}} \chi(n)\chi_v^0(n)\right).
\end{aligned}
$$

Applying Cauchy inequality and the estimates for character sums

$$
\begin{aligned}
\sum_{\chi\ne\chi_0}\left|\sum_{N\le n\le M}\chi(n)\right|^2 &= \sum_{\chi\ne\chi_0}\left|\sum_{N\le n\le M\le N+d}\chi(n)\right|^2\\
&= \phi(d)\sum_{N\le n\le M\le N+d}\chi_0(n) - \left|\sum_{N\le n\le M\le N+d}\chi_0(n)\right|^2\\
&\le \frac{\phi^2(d)}{4}
\end{aligned}
$$

and note that the identity

$$
\sum_{N\le n\le M}\chi(n)\chi_u^0(n) = \sum_{d|u}\mu(d)\chi(d)\sum_{N/d\le n\le M/d}\chi(n),
$$

we have

$$
\begin{aligned}
\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} |A(y,\chi)|^2 &\ll \sqrt{y}\sum_{n\le\sqrt{y}}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\left|\sum_{m\le y/n}\chi(m)\chi_u^0(m)\right|^2\\
&\quad + \sqrt{y}\sum_{m\le\sqrt{y}}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\left|\sum_{n\le y/m}\chi(n)\chi_v^0(n)\right|^2\\
&\quad + \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\left|\sum_{n\le\sqrt{y}}\chi(n)\chi_u^0(n)\right|^2\times\left|\sum_{n\le\sqrt{y}}\chi(n)\chi_v^0(n)\right|^2\\
&\ll yd^{2+\epsilon}.
\end{aligned}
\tag{19}
$$

Then from Cauchy inequality and (19) we can write

$$M_2 \ll \sum_{1\le n_1\le N} n_1^{\epsilon-1}\int_N^\infty \frac{1}{y^2}\left(\sum_{\chi(-1)=-1}|A(y,\chi)|\right)dy \tag{20}$$
$$\ll N^\epsilon \int_N^\infty \frac{d^{\frac{3}{2}+\epsilon}\sqrt{y}}{y^2}dy \ll \frac{d^{\frac{3}{2}+\epsilon}}{N^{\frac{1}{2}-\epsilon}}.$$

(iii) Similar to (ii), we can also get

$$M_3 \ll \frac{d^{\frac{3}{2}+\epsilon}}{N^{\frac{1}{2}-\epsilon}}. \tag{21}$$

(iv) By the same argument as in (ii), and noting the absolute convergence of the integrals, we can write

(22)

$$M_4 \le \int_N^\infty\int_N^\infty \frac{1}{y^2z^2}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}|A(y,\bar\chi)||A(z,\chi)|dydz$$
$$\ll \int_N^\infty \frac{1}{y^2}\int_N^\infty \frac{1}{z^2}\left(\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}|A(y,\bar\chi)|^2\right)^{\frac{1}{2}}\left(\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}|A(z,\chi)|^2\right)^{\frac{1}{2}}dydz$$
$$\ll \left(\int_N^\infty \frac{1}{y^2}\left(\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}|A(y,\chi)|^2\right)^{\frac{1}{2}}dy\right)^2$$
$$\ll \left(\int_N^\infty \frac{d^{1+\epsilon}}{y^{\frac{3}{2}}}dy\right)^2 \ll \frac{d^{2+\epsilon}}{N}.$$

Now, taking $N=d^3$, combining (14)-(22) we obtain the asymptotic formula

$$\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(2^m)|L(1,\chi\chi_u^0)|^2|L(1,\chi\chi_v^0)|^2 = \frac{(3m+5)\pi^4}{72\cdot 2^{m+1}}\phi(d)\Pi(u,v)+O_m(d^\epsilon).$$

This proves Lemma 7. □

Lemma 8. *Let p be a prime, α and β are non-negative integer with $\beta\ge\alpha$. Then we have*

$$\sum_{u|p^\beta}\sum_{v|p^\alpha}\frac{u^2v^2}{\phi(u)\phi(v)}\phi(d)\Pi(u,v)$$

$$= p^{3\alpha}\frac{\left(1+\frac{1}{p}\right)^2-\frac{1}{p^{3\alpha+1}}}{1+\frac{1}{p}+\frac{1}{p^2}}+\frac{(p^2-1)^2p^{2\alpha}(p^\beta-p^\alpha)}{(p-1)^2(p^2+1)},$$

where $d=(u,v)$ denotes the greatest common divisors of d_1 and d_2.

Proof. See Lemma 5 of [5]. □

Lemma 9. *Let q be any odd integer with $q\geq 3$, χ be a Dirichlet character modulo q. Then for any integer $m\geq 0$, we have the asymptotic formulas*

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\bar{\chi}(2^m)\left|\sum_{r=1}^{q} r\chi(r)\right|^4=\frac{(3m+5)q^3\phi^4(q)}{72\cdot 2^{m+1}}\prod_{p^\alpha\| q}\frac{\frac{(p+1)^3}{p(p^2+1)}-\frac{1}{p^{3\alpha-1}}}{1+\frac{1}{p}+\frac{1}{p^2}}+O_m\left(q^{6+\epsilon}\right)$$

and

$$\sum_{\substack{\chi \bmod 4q\\ \chi(-1)=-1}}\left|\sum_{r=1}^{4q} r\chi(r)\right|^4=\frac{488}{9}q^3\phi^4(q)\prod_{p^\alpha\| q}\frac{\frac{(p+1)^3}{p(p^2+1)}-\frac{1}{p^{3\alpha-1}}}{1+\frac{1}{p}+\frac{1}{p^2}}+O_m\left(q^{6+\epsilon}\right).$$

Proof. We only prove the first formula, the second one can be proved by the same method. From Lemma 5 and Lemma 4 we have

$$\begin{aligned}
(23)\qquad &\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\bar{\chi}(2^m)\left|\sum_{r=1}^{q} r\chi(r)\right|^4\\
&=q^4\phi(q)\sum_{d|q}\sum_{\ell|q}\mu(d)\mu(\ell){\sum_{a=1}^{q}}' S\left(2^m a,\frac{q}{d}\right)S\left(a,\frac{q}{\ell}\right)\\
&=q^4\phi(q)\sum_{d|q}\sum_{\ell|q}\mu(d)\mu(\ell){\sum_{a=1}^{q}}'\left(\frac{d}{\pi^2 q}\sum_{u|\frac{q}{d}}\frac{u^2}{\phi(u)}\sum_{\substack{\chi \bmod u\\ \chi(-1)=-1}}\chi(2^m a)\left|L(1,\chi)\right|^2\right)\\
&\quad\times\left(\frac{\ell}{\pi^2 q}\sum_{v|\frac{q}{\ell}}\frac{v^2}{\phi(v)}\sum_{\substack{\chi \bmod v\\ \chi(-1)=-1}}\chi(a)\left|L(1,\chi)\right|^2\right)\\
&=\frac{q^2\phi(q)}{\pi^4}\sum_{d|q}\sum_{\ell|q}\mu(d)\mu(\ell)d\ell\sum_{u|\frac{q}{d}}\sum_{v|\frac{q}{\ell}}\frac{u^2v^2}{\phi(u)\phi(v)}\\
&\quad\times\sum_{\substack{\chi_1 \bmod u\\ \chi_1(-1)=-1}}\sum_{\substack{\chi_2 \bmod v\\ \chi_2(-1)=-1}}{\sum_{a=1}^{q}}'\chi_1(2^m a)\chi_2(a)\left|L(1,\chi_1)\right|^2\left|L(1,\chi_2)\right|^2.
\end{aligned}$$

For each $\chi_1 \bmod u$, it is clear that there exists one and only one $q_1|u$ with a unique primitive character $\chi_{q_1}^1 \bmod q_1$ such that $\chi_1=\chi_{q_1}^1\chi_u^0$. Similarly, we also have $\chi_2 = \chi_{q_2}^2 \chi_v^0$, here $q_2|v$, and $\chi_{q_2}^2$ is a primitive character modulo q_2. Note that $u|q$ and $v|q$, from the orthogonality of Dirichlet characters we have

$$\sideset{}{'}\sum_{a=1}^{q} \chi_1(a)\chi_2(a) = \sum_{a=1}^{q} \left[\chi_{q_1}^1(a)\chi_q^0(a)\right] \left[\chi_{q_2}^2(a)\chi_q^0(a)\right] = \begin{cases} \phi(q), & \text{if } q_1 = q_2 \text{ and } \chi_{q_1}^1 = \overline{\chi_{q_2}^2}; \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

Let $d_1 = (u, v)$. If $q_1 = q_2$ and $\chi_{q_1}^1 = \overline{\chi_{q_2}^2}$, then $\chi_{q_1}^1 \chi_{d_1}^0$ is also a character mod d_1. So from (23), (24) and Lemma 7 we get

$$\begin{aligned} &\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \bar{\chi}(2^m) \left| \sum_{r=1}^{q} r\chi(r) \right|^4 \\ &= \frac{q^2\phi^2(q)}{\pi^4} \sum_{d|q} \sum_{\ell|q} \mu(d)\mu(\ell) d\ell \sum_{u|\frac{q}{d}} \sum_{v|\frac{q}{\ell}} \frac{u^2 v^2}{\phi(u)\phi(v)} \\ &\quad \sum_{\substack{\chi \bmod (u,v) \\ \chi(-1)=-1}} \chi(2^m) \left| L(1, \chi\chi_u^0) \right|^2 \left| L(1, \chi\chi_v^0) \right|^2 \\ &= \frac{q^2\phi^2(q)}{\pi^4} \sum_{d|q} \sum_{\ell|q} \sum_{u|\frac{q}{d}} \sum_{v|\frac{q}{\ell}} \frac{\mu(d)\mu(\ell) d\ell u^2 v^2}{\phi(u)\phi(v)} \\ &\quad \times \left\{ \frac{(3m+5)\pi^4}{72 \cdot 2^{m+1}} \phi((u,v))\Pi(u,v) + O_m\left((u,v)^{\epsilon}\right) \right\} \\ &= \frac{(3m+5)q^2\phi^2(q)}{72 \cdot 2^{m+1}} \sum_{d|q} \sum_{\ell|q} \sum_{u|\frac{q}{d}} \sum_{v|\frac{q}{\ell}} \frac{\mu(d)\mu(\ell) d\ell u^2 v^2}{\phi(u)\phi(v)} \phi((u,v))\Pi(u,v) \\ &\quad + O_m\left(q^{6+\epsilon}\right). \end{aligned} \tag{25}$$

For any multiplicative functions $f(u)$ and $g(v)$, there holds

$$\begin{aligned} &\sum_{d|q} \sum_{\ell|q} \mu(d)\mu(\ell) d\ell \sum_{u|\frac{q}{d}} \sum_{v|\frac{q}{\ell}} f(u)g(v) \\ &= \prod_{p^\alpha \| q} \left[\sum_{u|p^\alpha} \sum_{v|p^\alpha} f(u)g(v) - 2p \sum_{u|p^{\alpha-1}} \sum_{v|p^\alpha} f(u)g(v) + p^2 \sum_{u|p^{\alpha-1}} \sum_{v|p^{\alpha-1}} f(u)g(v) \right]. \end{aligned}$$

Now from Lemma 8 and the identity

$$p^{3\alpha} \frac{\left(1 + \frac{1}{p}\right)^2 - \frac{1}{p^{3\alpha+1}}}{1 + \frac{1}{p} + \frac{1}{p^2}} + p^2 \cdot p^{3\alpha-3} \frac{\left(1 + \frac{1}{p}\right)^2 - \frac{1}{p^{3\alpha-2}}}{1 + \frac{1}{p} + \frac{1}{p^2}}$$

$$- 2p \left[p^{3\alpha-3} \frac{\left(1+\frac{1}{p}\right)^2 - \frac{1}{p^{3\alpha-2}}}{1+\frac{1}{p}+\frac{1}{p^2}} + \frac{(p^2-1)^2 p^{2\alpha-2}(p^\alpha - p^{\alpha-1})}{(p-1)^2(p^2+1)} \right]$$
$$= \frac{p^{3\alpha+1}\left(1-\frac{1}{p}\right)^2}{1+\frac{1}{p}+\frac{1}{p^2}} \left(\frac{(p+1)^3}{(p^2+1)p^2} - \frac{1}{p^{3\alpha}} \right)$$

we get

$$\sum_{d|q}\sum_{\ell|q}\sum_{u|\frac{q}{d}}\sum_{v|\frac{q}{\ell}} \frac{\mu(d)\mu(\ell)d\ell u^2v^2}{\phi(u)\phi(v)}\phi((u,v))\Pi(u,v)$$
$$= \prod_{p^\alpha\|q} \left\{ \sum_{d|p^\alpha}\sum_{\ell|p^\alpha} \mu(d)d\mu(\ell)\ell \sum_{u|\frac{p^\alpha}{d}}\sum_{v|\frac{p^\alpha}{\ell}} \frac{uv}{\phi(u)\phi(v)}\phi((u,v))\Pi(u,v) \right\} \tag{26}$$
$$= q\phi^2(q) \prod_{p^\alpha\|q} \frac{\frac{(p+1)^3}{p(p^2+1)} - \frac{1}{p^{3\alpha-1}}}{1+\frac{1}{p}+\frac{1}{p^2}}.$$

Now Lemma 9 follows immediately from (25) and (26). □

Lemma 10. *Let $q > 2$ be an odd integer. Then we have*

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=1 \\ \chi\neq\chi_0}} \left| \sum_{b=1}^{4q} b\chi\chi_4(b) \right|^4 = \sum_{\substack{\chi \bmod 4q \\ \chi(-1)=-1}} \left| \sum_{b=1}^{4q} b\chi(b) \right|^4 - 256 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |1-\chi(2)|^4 \left| \sum_{b=1}^{q} b\chi(b) \right|^4 .$$

Proof. Let χ_4^0 denotes the principal character modulo 4. There holds

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=1 \\ \chi\neq\chi_0}} \left| \sum_{b=1}^{4q} b\chi\chi_4(b) \right|^4 = \sum_{\substack{\chi \bmod 4q \\ \chi(-1)=-1}} \left| \sum_{b=1}^{4q} b\chi(b) \right|^4 - \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left| \sum_{b=1}^{4q} b\chi\chi_4^0(b) \right|^4 . \tag{27}$$

For the inner summation in the second part, we have

$$\sum_{b=1}^{4q} b\chi\chi_4^0(b) = \sum_{\substack{b=1 \\ (b,2)=1}}^{4q} b\chi(b)$$
$$= 2\sum_{\substack{b=1 \\ (b,2)=1}}^{q} b\chi(b) + 2\sum_{\substack{b=1 \\ 2|b}}^{q} b\chi(b) + 4q\sum_{\substack{b=1 \\ 2|b}}^{q} \chi(b) + 2q\sum_{\substack{b=1 \\ (b,2)=1}}^{q} \chi(b)$$
$$= 2\sum_{b=1}^{q} b\chi(b) + 2q\sum_{\substack{b=1 \\ 2|b}}^{q} \chi(b)$$

$$= 2\sum_{b=1}^{q} b\chi(b) + 2\chi(2)\sum_{b=1}^{\frac{q-1}{2}} \chi(b).$$

Now from Lemma 1, we have

$$\sum_{b=1}^{4q} b\chi\chi_4^0(b) = (4 - 4\chi(2))\sum_{b=1}^{q} b\chi(b).$$

Now combining (27), we get the lemma. □

3. Proof of Theorem

In this section we will complete the proof of Theorem. From Lemma 3, we can write

$$\sideset{}{'}\sum_{a=1}^{q} E^2(a,q) = \frac{1}{4q^4\phi^2(q)}\sideset{}{'}\sum_{a=1}^{q}\left[\sum_{\substack{\chi \mod q\\ \chi(-1)=-1}} \bar{\chi}(a)(\chi(4)-1)(\bar{\chi}(2)-2)^2\left(\sum_{b=1}^{q} b\chi(b)\right)^2 \right.$$
$$\left. - \frac{1}{16}\sum_{\substack{\chi \mod q\\ \chi(-1)=1\\ \chi\neq\chi_0}} \bar{\chi}(a)\bar{\chi}(4)\left(\sum_{b=1}^{4q} b\chi\chi_4(b)\right)^2 + O(q^2)\right]^2.$$

Noting that

$$\sideset{}{'}\sum_{a=1}^{q} \chi_1\chi_2(a) = 0,$$

if $\chi_1(-1) = -1$ and $\chi_2(-1) = 1$. So from the orthogonality of Dirichlet characters and Lemma 10, we can write

$$\sideset{}{'}\sum_{a=1}^{q} E^2(a,q)$$
$$= \frac{1}{4q^4\phi^2(q)}\sideset{}{'}\sum_{a=1}^{q}\left[\left(\sum_{\substack{\chi \mod q\\ \chi(-1)=-1}} \bar{\chi}(a)(\chi(4)-1)(\bar{\chi}(2)-2)^2\left(\sum_{b=1}^{q} b\chi(b)\right)^2\right)^2\right.$$
$$\left. + \left(\frac{1}{16}\sum_{\substack{\chi \mod q\\ \chi(-1)=1\\ \chi\neq\chi_0}} \bar{\chi}(a)\bar{\chi}(4)\left(\sum_{b=1}^{4q} b\chi\chi_4(b)\right)^2\right)^2\right] + O\left(\frac{1}{q}\right)$$

$$= \frac{1}{4q^4\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=-1}} \left|(\chi(4)-1)(\bar{\chi}(2)-2)^2\right|^2 \left|\sum_{b=1}^{q} b\chi(b)\right|^4$$

$$+ \frac{1}{4\cdot 16^2 q^4\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=1 \\ \chi\neq\chi_0}} \left|\sum_{b=1}^{4q} b\chi\chi_4(b)\right|^4 + O\left(\frac{1}{q}\right)$$

$$= \frac{1}{4q^4\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=-1}} [58 - 4\chi(16) + 20\chi(8) - 25\chi(4) - 20\chi(2) - 20\bar{\chi}(2)$$

$$- 25\bar{\chi}(4) + 20\bar{\chi}(8) - 4\bar{\chi}(16)] \left|\sum_{r=1}^{q} r\chi(r)\right|^4$$

$$- \frac{1}{4q^4\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=-1}} [6 - 4\chi(2) - 4\bar{\chi}(2) + \chi(4) + \bar{\chi}(4)] \left|\sum_{r=1}^{q} r\chi(r)\right|^4$$

$$+ \frac{1}{4\cdot 16^2 q^4\phi(q)} \sum_{\substack{\chi \mod 4q \\ \chi(-1)=-1}} \left|\sum_{b=1}^{4q} b\chi(b)\right|^4 + O\left(\frac{1}{q}\right).$$

Now from Lemma 9, we can easily get

$$\sum_{a=1}^{q}{}' E^2(a,q) = \frac{9\phi^3(q)}{64q} \prod_{p^\alpha \| q} \frac{\frac{(p+1)^3}{p(p^2+1)} - \frac{1}{p^{3\alpha-1}}}{1 + \frac{1}{p} + \frac{1}{p^2}} + O\left(q^{1+\epsilon}\right).$$

This completes the proof of Theorem.

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