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SOME RESULTS ON $(LCS)_n$-MANIFOLDS

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Abstract. The object of the present paper is to study $(LCS)_n$-manifolds. Several interesting results on a $(LCS)_n$-manifold are obtained. Also the generalized Ricci recurrent $(LCS)_n$-manifolds are studied. The existence of such a manifold is ensured by several non-trivial new examples.

1. Introduction

Recently the present author [6] introduced the notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) with an example. The present paper deals with a study of various types of $(LCS)_n$-manifolds. After preliminaries, in Section 3 we study the fundamental results of $(LCS)_n$-manifolds and proved that in such a manifold the Ricci operator commutes with the structure tensor ϕ. Section 4 is devoted to the study of conformally flat $(LCS)_n$-manifolds and it is proved that such a $(LCS)_n$-manifold is η-Einstein as well as a manifold of quasi constant curvature. The notion of generalized Ricci recurrent manifold was introduced by De, Guha, and Kamilya [2] in 1995. Section 5 is concerned with generalized Ricci recurrent $(LCS)_n$-manifolds and in the last section we investigate the existence of such a manifold and found various new examples of both in even and odd dimensions.

2. $(LCS)_n$-manifolds

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type (0, 2) such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, \dots, +)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp., ≤ 0, $= 0$, > 0) [5]. The category to which a given vector falls is called its causal character.

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Definition 2.1. In a Lorentzian manifold (M, g) a vector field P defined by

$$g(X, P) = A(X)$$

for any $X \in \chi(M)$ is said to be a concircular vector field if

$$(\nabla_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\},$$

where α is a non-zero scalar and ω is a closed 1-form.

Let M^n be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X), \tag{2.2}$$

the equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0) \tag{2.3}$$

for all vector fields X, Y, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.4}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.5}$$

then from (2.3) and (2.5) we have

$$\phi X = X + \eta(X)\xi, \tag{2.6}$$

from which it follows that ϕ is a symmetric (1, 1) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M^n together with the unit timelike concircular vector field ξ, its associated 1-form η and (1, 1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (*briefly* $(LCS)_n$-*manifold*) [6]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [4]. In a $(LCS)_n$-manifold, the following relations hold [6]:

(2.7)
a) $\eta(\xi) = -1$, b) $\phi\xi = 0$, c) $\eta(\phi X) = 0$, d) $g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$,

$$\eta(R(X, Y)Z) = (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.8}$$

$$S(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{2.9}$$

$$R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\} \tag{2.11}$$

for all vector fields X, Y, Z, where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold.

3. Fundamental results of $(LCS)_n$-manifolds

Proposition 3.1. *A $(LCS)_n$-manifold of constant curvature is a manifold of constant curvature $(\alpha^2 - \rho)$.*

Proof. If a $(LCS)_n$-manifold is of constant curvature k, say, then we have

$$R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y],$$

which yields by setting $Z = \xi$ that

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y].$$

This implies by virtue of (2.10) that $k = (\alpha^2 - \rho)$. Hence the proposition is proved. □

Lemma 3.1. *In a $(LCS)_n$-manifold, the following relation holds*:

$$(X\rho) = d\rho(X) = \beta\eta(X) \tag{3.1}$$

for any vector field X and β is a certain scalar function.

Proof. From (2.4), it follows that

$$\nabla(d\alpha)(Y,X) = \nabla_X(d\alpha)(Y) = X(Y\alpha) - ((\nabla_X Y)\alpha)$$

which implies that

$$\nabla(d\alpha)(X,Y) = (d\alpha)(Y,X). \tag{3.2}$$

Also

$$\nabla(d\alpha)(Y,X) = Y(d\alpha(X)) - d\alpha(\nabla_Y X),$$

which implies by virtue of (2.3) and (2.4) that

$$\nabla(d\alpha)(Y,X) = (Y\rho)\eta(X) + \rho\alpha[g(X,Y) + \eta(X)\eta(Y)].$$

This implies by virtue of (2.2) that

$$(X\rho)\eta(Y) = (Y\rho)\eta(X),$$

which yields

$$(X\rho) = \beta\eta(X),$$

where $\beta = -(\xi\rho)$ is a scalar function. Hence the result holds. □

Lemma 3.2. *Let $M^n(\phi, \xi, \eta, g)$ be a $(LCS)_n$-manifold. Then for any X, Y, Z on M^n, the following relation holds*:

$$R(X,Y)\phi Z - \phi R(X,Y)Z = (\alpha^2 - \rho)[\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi + \eta(Z)\{\eta(X)Y - \eta(Y)X\}]. \tag{3.3}$$

Proof. From (2.3)-(2.7), (2.11) and the Ricci identity we can easily get (3.3). □

Lemma 3.3. *Let (M^n, g) be a $(LCS)_n$-manifold. Then*

$$g(\phi R(\phi X, \phi Y)Z, \phi W) = g(R(X,Y)Z, W) + (\alpha^2 - \rho)[\{g(Y,W)\eta(Z) - g(Y,Z)\eta(W)\}\eta(X) + \{g(X,W)\eta(Z) - g(X,Z)\eta(W)\}\eta(Y)] \tag{3.4}$$

for any vector field $X,\ Y,\ Z,\ W$ on M^n.

Proof. Using (2.6), (2.8) and $\eta(\phi X) = 0$, we can calculate

$$\begin{aligned} g(\phi R(\phi X, \phi Y)Z, \phi W) &= g(R(\phi X, \phi Y)Z, W) = g(R(Z,W)\phi X, \phi Y) \\ &= g(\phi R(Z,W)X, \phi Y) + (\alpha^2 - \rho)[g(W, \phi Y)\eta(X)\eta(Z) \\ &\quad - g(Z, \phi Y)\eta(X)\eta(W)]. \end{aligned}$$

The relation (3.4) follows from this and

$$g(R(Z,W)X, Y) = g(R(X,Y)Z, W).$$

□

Lemma 3.4. *Let (M^n, g) be a $(LCS)_n$-manifold. Then for any X, Y, Z on M^n, the following relation holds*:

$$g(R(\phi X, \phi Y)\phi Z, \phi W) = g(R(X,Y)Z, W) + (\alpha^2 - \rho)[\{g(Y,W)\eta(Z) - g(Y,Z)\eta(W)\}\eta(X) + \{g(X,W)\eta(Z) - g(X,Z)\eta(W)\}\eta(Y)]. \tag{3.5}$$

Proof. Replacing X, Y by $\phi X, \phi Y$ respectively in (3.3) and taking the inner product on both sides by ϕW we get

$$g(R(\phi X, \phi Y)\phi Z, \phi W) = g(\phi R(\phi X, \phi Y)Z, \phi W). \tag{3.6}$$

Using (3.4) in (3.6) we obtain (3.5). □

Theorem 3.1. *Let (M^n, g) be a $(LCS)_n$-manifold. Then the Ricci operator Q commutes with ϕ.*

Proof. To prove the result, we shall show that

$$Q\phi = \phi Q. \tag{3.7}$$

From (3.2), it follows that

$$\phi R(\phi X, \phi Y)\phi Z = R(X,Y)Z + (\alpha^2 - \rho)[\eta(X)\{\eta(Z)Y - g(Y,Z)\xi\} + \eta(Y)\{\eta(Z)X - g(X,Z)\xi\}]. \tag{3.8}$$

We now consider the following two cases:

(i) $\dim M = n = \text{odd} = 2m + 1$,

(ii) $\dim M = n = \text{even} = 2m + 2$.

Case (i): If $n = 2m+1$, let $\{e_i, \phi e_i, \xi\}$, $i = 1, 2, \ldots, m$ be an orthonormal frame at any point of the manifold. Then putting $Y = Z = e_i$ in (3.8) and taking summation over i and using $\eta(e_i) = 0$, we get

$$\sum_{i=1}^{m} \epsilon_i \phi R(\phi X, \phi e_i)\phi e_i = \sum_{i=1}^{m} \epsilon_i R(X, e_i)e_i - m(\alpha^2 - \rho)\eta(X)\xi, \tag{3.9}$$

where $\epsilon_i = g(e_i, e_i)$.

Again setting $Y = Z = \phi e_i$ in (3.8) and taking summation over i and then using $\eta \circ \phi = 0$ and (2.1) we get

$$\sum_{i=1}^{m} \epsilon_i \phi R(\phi X, e_i)e_i = \sum_{i=1}^{m} \epsilon_i R(X, \phi e_i)\phi e_i - m(\alpha^2 - \rho)\eta(X)\xi. \tag{3.10}$$

Adding (3.9) and (3.10) and using the definition of the Ricci operator, we obtain

$$\phi(Q\phi X - R(\phi X, \xi)\xi) = QX - R(X, \xi)\xi - 2m(\alpha^2 - \rho)\eta(X)\xi.$$

Using (2.10) and $\phi\xi = 0$ in the above relation we have

$$\phi Q \phi X = QX - 2m(\alpha^2 - \rho)\eta(X)\xi.$$

Operating both sides by ϕ and using (2.1), symmetry of Q, $\phi\xi = 0$ and (2.9) we get (3.7).

Case (ii): If $n = 2m+2$, let $\{e_i, \phi e_i\}$, $i = 1, 2, \ldots, m+1$ be an orthonormal frame such that each e_i is orthogonal to ξ, i.e., $\eta(e_i) = 0$. Then putting $Y = Z = e_i$ in (3.8) and taking summation over i and using $\eta(e_i) = 0$, we get

$$\sum_{i=1}^{m+1} \epsilon_i \phi R(\phi X, \phi e_i)\phi e_i = \sum_{i=1}^{m+1} \epsilon_i R(X, e_i)e_i - (m+1)(\alpha^2 - \rho)\eta(X)\xi, \tag{3.11}$$

where $\epsilon_i = g(e_i, e_i)$.

Again replacing Y and Z by ϕe_i in (3.8) and taking summation over i and then using $\eta(e_i) = 0$ and (2.1), it follows that

$$\sum_{i=1}^{m+1} \epsilon_i \phi R(\phi X, e_i)e_i = \sum_{i=1}^{m+1} \epsilon_i R(X, \phi e_i)\phi e_i - (m+1)(\alpha^2 - \rho)\eta(X)\xi. \tag{3.12}$$

Adding (3.11) and (3.12) and then proceeding similarly as in Case (i) we can easily obtain (3.7). This proves the theorem. □

Proposition 3.2. *In a $(LCS)_n$-manifold the relation*

$$S(\phi X, \phi Y) = (n-1)(\alpha^2 - \rho)g(X, Y) + S(X, Y) \tag{3.13}$$

holds.

Proof. The proposition follows from Theorem 3.1. □

4. Conformally flat $(LCS)_n$-manifolds

This section deals with conformally flat $(LCS)_n$ $(n \geq 4)$ manifolds.

Definition 4.1. A $(LCS)_n$-manifold is said to be η-Einstein if its Ricci tensor S of type (0, 2) is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a, b are the smooth functions over the manifold such that b is non-zero.

Theorem 4.1. *A conformally flat $(LCS)_n$ $(n \geq 4)$ manifold is an η-Einstein manifold.*

Proof. If a $(LCS)_n$ $(n \geq 4)$ manifold is conformally flat, then its curvature tensor is given by

$$\text{(4.1)} \quad R(X,Y)Z = \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y].$$

Setting $Z = \xi$ in (4.1) and then using (2.9) and (2.10) we obtain

$$\text{(4.2)} \quad (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y] = \frac{1}{n-2}[(n-1)(\alpha^2-\rho)\{\eta(Y)X - \eta(X)Y\} + \eta(Y)QX - \eta(X)QY] - \frac{r}{(n-1)(n-2)}[\eta(Y)X - \eta(X)Y].$$

Again replacing Y by ξ in (4.2) we obtain by virtue of (2.9) that

$$\text{(4.3)} \qquad QX = \Big[\frac{r}{n-1} - (\alpha^2 - \rho)\Big]X - \Big[\frac{r}{n-1} - n(\alpha^2-\rho)\Big]\eta(X)\xi$$

which can also be written as

$$\text{(4.4)} \quad S(X,Y) = \Big[\frac{r}{n-1} - (\alpha^2 - \rho)\Big]g(X,Y) - \Big[\frac{r}{n-1} - n(\alpha^2-\rho)\Big]\eta(X)\eta(Y)$$

which implies that the manifold is η-Einstein. □

Corollary 4.1. *A $(LCS)_3$ manifold is an η-Einstein manifold.*

Proof. Since in a 3-dimensional Lorentzian manifold, the Weyl conformal curvature tensor vanishes, it follows that (4.1) holds for $n = 3$ and hence it can be easily shown that a $(LCS)_3$ manifold is always an η-Einstein manifold. □

Definition 4.2. A Riemannian manifold (M^n, g) $(n \geq 4)$ is said to be of quasi-constant curvature if it is conformally flat and its curvature tensor $\tilde{R}$ of type (0, 4) has the following form:

$$\text{(4.5)} \qquad \tilde{R}(X,Y,Z,W) = a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + b[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)],$$

where A is a 1-form and a, b are scalars of which $b \neq 0$.

This notion of *quasi-constant curvature* was introduced by Chen and Yano [1].

Theorem 4.2. *A conformally flat $(LCS)_n$ $(n \geq 4)$ manifold is of quasi-constant curvature.*

Proof. By virtue of (4.3) and (4.4), the relation (4.1) takes the form

$$\begin{aligned} \tilde{R}(X,Y,Z,W) &= \tilde{a}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad + \tilde{b}[g(X,W)\eta(Y)\eta(Z) - g(Y,W)\eta(X)\eta(Z) \\ &\quad + g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W)], \end{aligned} \tag{4.6}$$

where $\tilde{a} = \frac{1}{n-2}[\frac{r}{n-1} - 2(\alpha^2 - \rho)]$ and $\tilde{b} = \frac{1}{n-2}[\frac{r}{n-1} - n(\alpha^2 - \rho)]$ are smooth functions. Here $\tilde{b} \neq 0$ as for $\tilde{b} = 0$, (4.4) yields that the manifold is Einstein, but the manifold under consideration is η-Einstein. Hence comparing (4.5) and (4.6), the theorem is proved. □

5. Generalized Ricci recurrent $(LCS)_n$-manifold

Definition 5.1. A $(LCS)_n$-manifold is said to be generalized Ricci recurrent [2] if its Ricci tensor S of type (0, 2) satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(X)g(Y,Z), \tag{5.1}$$

where A and B are two non-zero 1-forms such that $A(X) = g(X,P)$ and $B(X) = g(X,L)$, P and L being associated vector fields of the 1-form A and B, respectively.

Theorem 5.1. *In a generalized Ricci recurrent $(LCS)_n$ $(n \geq 4)$ manifold, the 1-form A and B are related by*

$$B(X) = (n-1)[(2\alpha\rho - \beta)\eta(X) - (\alpha^2 - \rho)A(X)]. \tag{5.2}$$

Proof. In a generalized Ricci recurrent $(LCS)_n$-manifold, we have the relation (5.1). Setting $Z = \xi$ in (5.1) we have

$$(\nabla_X S)(Y,\xi) = [(\alpha^2 - \rho)A(X) + B(X)]\eta(Y). \tag{5.3}$$

Again

$$(\nabla_X S)(Y,\xi) = \nabla_X S(Y,\xi) - S(\nabla_X Y,\xi) - S(Y,\nabla_X \xi)$$

which yields by virtue of (2.3), (2.4), (2.9), and (3.1) that

$$(\nabla_X S)(Y,\xi) = (n-1)[(2\alpha\rho - \beta)\eta(X)\eta(Y) + \alpha(\alpha^2 - \rho)g(X,Y)] - \alpha S(X,Y). \tag{5.4}$$

From (5.3) and (5.4), it follows that

$$\begin{aligned} \alpha S(X,Y) &= (n-1)[(2\alpha\rho - \beta)\eta(X)\eta(Y) + \alpha(\alpha^2 - \rho)g(X,Y) \\ &\quad - (\alpha^2 - \rho)A(X)\eta(Y)] - B(X)\eta(Y). \end{aligned} \tag{5.5}$$

Replacing Y by ξ in (5.5) we obtain (5.2). This proves the theorem. □

Theorem 5.2. *A generalized Ricci recurrent $(LCS)_n$-manifold is Einstein if and only if $\beta = 2\alpha\rho$.*

Proof. In a generalized Ricci recurrent $(LCS)_n$-manifold we have the relation (5.5). Hence setting $Y = \phi Y$ in (5.5) and then using (2.7) we have

$$S(X,Y) = (n-1)(\alpha^2 - \rho)g(X,Y). \tag{5.6}$$

If the manifold under consideration is Einstein, then (5.6) implies $\alpha^2 - \rho =$ constant and hence $2\alpha\rho - \beta = 0$. Conversely, if $2\alpha\rho - \beta = 0$, then $\nabla_X(\alpha^2 - \rho) = 0$. Consequently $\alpha^2 - \rho =$ constant. □

Theorem 5.3. *In an Einstein generalized Ricci recurrent $(LCS)_n$-manifold the associated 1-forms are linearly dependent and the vector fields of the associated 1-forms are of opposite direction for $\alpha^2 - \rho > 0$.*

Proof. In a generalized Ricci recurrent $(LCS)_n$-manifold we have the relation (5.5). If such a manifold is Einstein, then $\alpha^2 - \rho$ is constant and hence $2\alpha\rho - \beta = 0$. Consequently (5.2) reduces to

$$B(X) + kA(X) = 0, \tag{5.7}$$

where $k = (n-1)(\alpha^2 - \rho) =$ constant. This proves the theorem. □

Theorem 5.4. *A generalized Ricci recurrent $(LCS)_n$ $(n \geq 4)$ manifold is Ricci symmetric if and only if $\beta = 2\alpha\rho$.*

Proof. In a generalized Ricci recurrent $(LCS)_n$-manifold we have the relation (5.6) from which it follows that

$$(\nabla_X S)(Y,Z) = (n-1)(2\alpha\rho - \beta)\eta(X)g(Y,Z). \tag{5.8}$$

If in a generalized Ricci recurrent $(LCS)_n$-manifold $\alpha^2 - \rho$ is constant, then the relation (5.7) holds. Hence using (5.7) in (5.1) we get

$$(\nabla_X S)(Y,Z) = A(X)[S(Y,Z) - kg(Y,Z)]. \tag{5.9}$$

This implies by virtue of (5.6) that

$$(\nabla_X S)(Y,Z) = 0. \tag{5.10}$$

Conversely, if (5.10) holds, then (5.8) implies that $2\alpha\rho - \beta = 0$ and hence $\alpha^2 - \rho =$ constant. This proves the theorem. □

Definition 5.2. The Ricci tensor of a generalized Ricci recurrent $(LCS)_n$-manifold is said to be η-parallel if it satisfies

$$(\nabla_Z S)(\phi X, \phi Y) = 0 \tag{5.11}$$

for all vector fields X, Y and Z on M.

The notion of Ricci η-parallelity was first introduced by M. Kon [3] for the Sasakian manifolds.

Theorem 5.5. *The Ricci tensor of a generalized Ricci recurrent $(LCS)_n$ $(n \geq 4)$ manifold is η-parallel if and only if the manifold is Einstein.*

Proof. The Ricci tensor of a generalized Ricci recurrent $(LCS)_n$-manifold is η-parallel if and only if the following relation holds [6]

$$(5.12)\qquad \begin{aligned}(\nabla_Z S)(X,Y) = \alpha[&S(Y,Z)\eta(X) + S(X,Z)\eta(Y)]\\ &- (n-1)[(2\alpha\rho - \beta)\eta(X)\eta(Y)\eta(Z)\\ &+ \alpha(\alpha^2 - \rho)\{g(X,Z)\eta(Y) + g(Y,Z)\eta(X)\}].\end{aligned}$$

Again in a generalized Ricci recurrent $(LCS)_n$-manifold, the relations (5.5) and (5.6) hold. Therefore in view of (5.6), (5.8) and (5.12) we obtain $2\alpha\rho - \beta = 0$ and hence $\alpha^2 - \rho =$ constant. Consequently (5.6) implies that the manifold under consideration is Einstein. Conversely, if $2\alpha\rho - \beta = 0$, then $\nabla_X(\alpha^2 - \rho) = 0$. Thus if a generalized Ricci recurrent $(LCS)_n$-manifold is Einstein, then we have $\alpha^2 - \rho =$ constant and hence the relation (5.10) holds, which implies that

$$(\nabla_Z S)(\phi X, \phi Y) = 0$$

for all X, Y and Z on M. Therefore the Ricci tensor of the manifold under consideration is η-parallel. Thus the theorem is proved. □

6. Examples of $(LCS)_n$-manifolds

Example 6.1. We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Let $\{e_1, e_2, e_3\}$ be linearly independent global frame on M given by

$$e_1 = e^{-z}\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \quad e_2 = e^{-z}\frac{\partial}{\partial y}, \quad e_3 = e^{-2z}\frac{\partial}{\partial z}.$$

Let g be the Lorentzian metric defined by $g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0$, $g(e_1,e_1) = g(e_2,e_2) = 1$, $g(e_3,e_3) = -1$. Let η be the 1-form defined by $\eta(U) = g(U, e_3)$ for any $U \in \chi(M)$. Let ϕ be the (1, 1) tensor field defined by $\phi e_1 = e_1$, $\phi e_2 = e_2$, $\phi e_3 = 0$. Then using the linearity of ϕ and g we have $\eta(e_3) = -1$, $\phi^2 U = U + \eta(U)e_3$ and $g(\phi U, \phi W) = g(U,W) + \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[e_1, e_2] = -e^{-z}e_2, \quad [e_1, e_3] = e^{-2z}e_1, \quad [e_2, e_3] = e^{-2z}e_2.$$

Taking $e_3 = \xi$ and using Koszul formula for the Lorentzian metric g, we can easily calculate

$$\begin{aligned}
&\nabla_{e_1} e_3 = e^{-2z}e_1, && \nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_1 = e^{-2z}e_3,\\
&\nabla_{e_2} e_3 = e^{-2z}e_2, && \nabla_{e_2} e_2 = e^{-2z}e_3 - e^{-z}e_1, && \nabla_{e_2} e_1 = e^{-2z}e_2,\\
&\nabla_{e_3} e_3 = 0, && \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_1 = 0.
\end{aligned}$$

From the above it can be easily seen that (ϕ, ξ, η, g) is a $(LCS)_3$ structure on M. Consequently $M^3(\phi, \xi, \eta, g)$ is a $(LCS)_3$-manifold with $\alpha = e^{-2z} \neq 0$ such that $(X\alpha) = \rho\eta(X)$, where $\rho = 2e^{-4z}$. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(e_2, e_3)e_3 = e^{-4z}e_2, \quad R(e_1, e_3)e_3 = e^{-4z}e_1, \quad R(e_1, e_2)e_2 = e^{-4z}e_1 - e^{-2z}e_1,$$

$$R(e_2, e_3)e_2 = e^{-4z}e_3, \quad R(e_1, e_3)e_1 = e^{-4z}e_3, \quad R(e_1, e_2)e_1 = -e^{-4z}e_2 + e^{-2z}e_2$$

and the components which can be obtained from these by the symmetry properties from which, we can easily calculate the non-vanishing components of the Ricci tensor S as follows:

$$S(e_1, e_1) = 2e^{-4z} - e^{-2z}, \quad S(e_2, e_2) = 2e^{-4z} - e^{-2z}, \quad S(e_3, e_3) = 2e^{-4z}.$$

Since $\{e_1, e_2, e_3\}$ is a frame field for $(LCS)_3$-manifold, any vector field $X, Y \in \chi(M)$ can be written as

$$X = a_1e_1 + b_1e_2 + c_1e_3$$

and

$$Y = a_2e_1 + b_2e_2 + c_2e_3,$$

where $a_i, b_i, c_i \in \mathbb{R}^+$ (= the set of positive real numbers), $i = 1, 2, 3$, such that $c_1c_2 \neq a_1a_2 + b_1b_2$. Hence

$$S(X, Y) = 2(a_1a_2 + b_1b_2 + c_1c_2)e^{-4z} - (a_1a_2 + b_1b_2)e^{-2z}$$

and

$$g(X, Y) = a_1a_2 + b_1b_2 - c_1c_2.$$

By virtue of the above we have the following:

$$(\nabla_{e_1}S)(X, Y) = (a_1c_2 + a_2c_1)(e^{-4z} - 4e^{-6z}),$$

$$(\nabla_{e_2}S)(X, Y) = (b_1c_2 + b_2c_1)(e^{-4z} - 4e^{-6z})$$

and

$$(\nabla_{e_3}S)(X, Y) = 0.$$

We shall show that this $(LCS)_3$-manifold is a generalized Ricci recurrent, i.e., it satisfies the relation (5.1). Let us now consider the 1-forms

$$A(e_1) = \frac{(a_1c_2 + a_2c_1)}{2(a_1a_2 + b_1b_2 + c_1c_2)},$$

$$A(e_2) = \frac{(b_1c_2 + b_2c_1)}{2(a_1a_2 + b_1b_2 + c_1c_2)},$$

$$A(e_3) = 0,$$

$$B(e_1) = \frac{e^{-2z}(a_1c_2 + a_2c_1)[(a_1a_2 + b_1b_2)(1 - 8e^{-4z}) - 8c_1c_2e^{-4z}]}{2(a_1a_2 + b_1b_2 + c_1c_2)(a_1a_2 + b_1b_2 - c_1c_2)},$$

$$B(e_2) = \frac{e^{-2z}(b_1c_2 + b_2c_1)[(a_1a_2 + b_1b_2)(1 - 8e^{-4z}) - 8c_1c_2e^{-4z}]}{2(a_1a_2 + b_1b_2 + c_1c_2)(a_1a_2 + b_1b_2 - c_1c_2)},$$

$$B(e_3) = 0$$

at any point $x \in M$. In our M^3, (5.1) reduces with these 1-forms to the following equations:

$$\begin{aligned}
&\text{(i)}\ \ (\nabla_{e_1} S)(X,Y) = A(e_1)S(X,Y) + B(e_1)g(X,Y),\\
&\text{(ii)}\ \ (\nabla_{e_2} S)(X,Y) = A(e_2)S(X,Y) + B(e_2)g(X,Y),\\
&\text{(iii)}\ \ (\nabla_{e_3} S)(X,Y) = A(e_3)S(X,Y) + B(e_3)g(X,Y).
\end{aligned}$$

This shows that the manifold under consideration is a generalized Ricci recurrent $(LCS)_3$-manifold which is neither Ricci-symmetric nor Ricci-recurrent. Hence we can state the following:

Theorem 6.1. *There exists a generalized Ricci recurrent $(LCS)_3$-manifold which is neither Ricci-symmetric nor Ricci-recurrent.*

Example 6.2. We consider the 4-dimensional manifold $M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_4 \neq 0\}$, where (x_1, x_2, x_3, x_4) are the standard coordinates in $\mathbb{R}^4$. Let $\{e_1, e_2, e_3, e_4\}$ be linearly independent global frame on M given by

$$e_1 = x_4\left(\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right), e_2 = x_4\frac{\partial}{\partial x_2}, e_3 = x_4\left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right), e_4 = (x_4)^3\frac{\partial}{\partial x_4}.$$

We define ϕ, ξ, η, g by $\phi e_1 = e_1$, $\phi e_2 = e_2$, $\phi e_3 = e_3$, $\phi e_4 = 0$, $\xi = (x_4)^3\frac{\partial}{\partial x_4}$, $\eta(X) = g(X, e_4)$ for any $X \in \chi(M)$, $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$, $g(e_4, e_4) = -1$, $g(e_i, e_j) = 0$ for $i \neq j$, $i, j = 1, 2, 3, 4$.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1, e_2] = -x_4 e_2,\ [e_1, e_4] = -(x_4)^2 e_1,\ [e_2, e_4] = -(x_4)^2 e_2,\ [e_3, e_4] = -(x_4)^2 e_3.$$

Taking $e_4 = \xi$ and using Koszul formula for the Lorentzian metric g, we can easily calculate

$$\nabla_{e_1} e_4 = -(x_4)^2 e_1, \quad \nabla_{e_2} e_1 = x_4 e_2, \quad \nabla_{e_1} e_1 = -(x_4)^2 e_4, \quad \nabla_{e_2} e_4 = -(x_4)^2 e_2,$$

$$\nabla_{e_3} e_4 = -(x_4)^2 e_3, \qquad \nabla_{e_3} e_3 = -(x_4)^2 e_4. \qquad \nabla_{e_2} e_2 = -(x_4)^2 e_4 - x_4 e_1.$$

From the above it can be easily seen that (ϕ, ξ, η, g) is an $(LCS)_4$ structure on M. Consequently $M^4(\phi, \xi, \eta, g)$ is an $(LCS)_4$-manifold with $\alpha = -(x_4)^2 \neq 0$ such that $(X\alpha) = \rho\eta(X)$, where $\rho = 2(x_4)^4$.

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$\begin{aligned}
&R(e_1, e_4)e_1 = (x_4)^4 e_4,\ R(e_2, e_4)e_2 = (x_4)^4 e_4, \quad R(e_3, e_4)e_3 = (x_4)^4 e_4,\\
&R(e_1, e_3)e_3 = (x_4)^4 e_1,\ R(e_1, e_3)e_1 = -(x_4)^4 e_3,\ R(e_2, e_3)e_2 = -(x_4)^4 e_3,\\
&R(e_1, e_4)e_4 = (x_4)^4 e_1,\ R(e_2, e_4)e_4 = (x_4)^4 e_2,\ R(e_1, e_2)e_2 = [(x_4)^4 - (x_4)^2]e_1,\\
&R(e_2, e_3)e_3 = (x_4)^4 e_2,\ R(e_3, e_4)e_4 = (x_4)^4 e_3,\ R(e_1, e_2)e_1 = -[(x_4)^4 - (x_4)^2]e_2
\end{aligned}$$

and the components which can be obtained from these by the symmetry properties from which, we can easily calculate the non-vanishing components of the Ricci tensor S as follows:

$$S(e_1, e_1) = 3(x_4)^4 - (x_4)^2, \quad S(e_3, e_3) = 3(x_4)^4,$$
$$S(e_2, e_2) = 3(x_4)^4 - (x_4)^2, \quad S(e_4, e_4) = 3(x_4)^4.$$

Since $\{e_1, e_2, e_3, e_4\}$ is a frame field for $(LCS)_4$-manifold, any vector field $X, Y \in \chi(M)$ can be written as

$$X = a_1 e_1 + b_1 e_2 + c_1 e_3 + d_1 e_4$$

and

$$Y = a_2 e_1 + b_2 e_2 + c_2 e_3 + d_2 e_4,$$

where $a_i, b_i, c_i, d_i \in \mathbb{R}^+$ (= the set of positive real numbers), $i = 1, 2, 3, 4$, such that $d_1 d_2 \neq a_1 a_2 + b_1 b_2 + c_1 c_2$. Hence

$$S(X, Y) = 3(a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2)(x_4)^4 - (a_1 a_2 + b_1 b_2)(x_4)^2$$

and

$$g(X, Y) = a_1 a_2 + b_1 b_2 + c_1 c_2 - d_1 d_2.$$

By virtue of the above we have the following:

$$(\nabla_{e_1} S)(X, Y) = (x_4)^4 (a_1 d_2 + a_2 d_1)[6(x_4)^2 - 1],$$
$$(\nabla_{e_2} S)(X, Y) = (x_4)^4 (b_1 d_2 + b_2 d_1)[6(x_4)^2 - 1],$$
$$(\nabla_{e_3} S)(X, Y) = 3(c_1 d_2 + c_2 d_1)(x_4)^6, \quad \text{and}$$
$$(\nabla_{e_4} S)(X, Y) = 0.$$

We shall now show that this $(LCS)_4$-manifold is a generalized Ricci recurrent, i.e., it satisfies the relation (5.1). Let us now consider the 1-forms

$$A(e_1) = -\frac{(a_1 d_2 + a_2 d_1)}{3(a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2)},$$
$$A(e_2) = -\frac{(b_1 d_2 + b_2 d_1)}{3(a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2)},$$
$$A(e_3) = -\frac{(x_4)^2 (c_1 d_2 + c_2 d_1)}{(a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2)}, \quad A(e_4) = 0,$$
$$B(e_1) = \frac{(x_4)^2 (a_1 d_2 + a_2 d_1)[(a_1 a_2 + b_1 b_2)\{18(x_4)^4 - 1\} + 18(c_1 c_2 + d_1 d_2)(x_2)^4]}{3(a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2)(a_1 a_2 + b_1 b_2 + c_1 c_2 - d_1 d_2)},$$
$$B(e_2) = \frac{(x_4)^2 (b_1 d_2 + b_2 d_1)[(a_1 a_2 + b_1 b_2)\{18(x_4)^4 - 1\} + 18(c_1 c_2 + d_1 d_2)(x_4)^4]}{3(a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2)(a_1 a_2 + b_1 b_2 + c_1 c_2 - d_1 d_2)},$$
$$B(e_3) = \frac{(x_4)^4 (c_1 d_2 + c_2 d_1)(a_1 a_2 + b_1 b_2)}{(a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2)(a_1 a_2 + b_1 b_2 + c_1 c_2 - d_1 d_2)}, \quad B(e_4) = 0$$

at any point $x \in M$. In our M^4, (5.1) reduces with these 1-forms to the following equations:

$$\begin{aligned}
&\text{(i)}\ \ (\nabla_{e_1} S)(X,Y) = A(e_1)S(X,Y) + B(e_1)g(X,Y),\\
&\text{(ii)}\ \ (\nabla_{e_2} S)(X,Y) = A(e_2)S(X,Y) + B(e_2)g(X,Y),\\
&\text{(iii)}\ \ (\nabla_{e_3} S)(X,Y) = A(e_3)S(X,Y) + B(e_3)g(X,Y),\\
&\text{(iv)}\ \ (\nabla_{e_4} S)(X,Y) = A(e_4)S(X,Y) + B(e_4)g(X,Y).
\end{aligned}$$

This shows that the manifold under consideration is a generalized Ricci recurrent $(LCS)_4$-manifold which is neither Ricci-symmetric nor Ricci-recurrent. This leads to the following:

Theorem 6.2. *There exists a generalized Ricci recurrent $(LCS)_4$-manifold which is neither Ricci-symmetric nor Ricci-recurrent.*

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