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SPECIFIC EXAMPLES OF EXPONENTIAL WEIGHTS

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Abstract. Let $Q \in \mathbb{C}^2 : \mathbb{R} \to [0, \infty)$ be an even function. Then we will consider the exponential weights $w(x) = \exp(-Q(x))$ in the weight class from [2]. In the paper, we will give some relations among exponential weights in this class and introduce a new weight subclass. In addition, we will investigate some properties of the typical and specific weights in these weight classes.

1. Introduction and results

Let $Q \in C^2 : \mathbb{R} \to [0, \infty)$ be an even function and $w(x) = \exp(-Q(x))$ be such that for all $n = 0, 1, 2, \ldots,$

$$\int_0^\infty x^n w^2(x)dx < \infty.$$

A function $f : \mathbb{R} \to [0, \infty)$ is said to be quasi-increasing (quasi-decreasing) if there exists $C > 0$ such that $f(x) \le Cf(y)(f(x) \ge Cf(y))$ for $0 < x < y$. For any two sequences $\{b_n\}_{n=1}^\infty$ and $\{c_n\}_{n=1}^\infty$ of nonzero real numbers (or functions), we write $b_n \lesssim c_n$ if there exists a constant $C > 0$ independent of n(or x) such that $b_n \le Cc_n$ for n large enough and $b_n \sim c_n$ if $b_n \lesssim c_n$ and $b_n \gtrsim c_n$.

Throughout the sections, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t. The same symbol does not necessarily denote the same constant in different occurrences.

We shall be interested in the following subclass of weights from [2].

Definition 1.1. Let $Q(x) : \mathbb{R} \to [0, \infty)$ be even and satisfy the following properties:
(a) $Q'(x)$ is continuous in $\mathbb{R}$, with $Q(0) = 0$.
(b) $Q''(x)$ exists and is positive in $\mathbb{R}\backslash\{0\}$.
(c)

$$\lim_{x\to\infty} Q(x) = \infty.$$

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(d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0$$

is quasi-increasing in $(0, \infty)$ with

$$T(x) \geq \Lambda > 1, \quad x \in \mathbb{R}\backslash\{0\}.$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e.} \quad x \in \mathbb{R}\backslash\{0\}. \tag{1.1}$$

Then we write $w(x) \in \mathcal{F}(C^2)$. If there also exist a compact subinterval $J(\ni 0)$ of $\mathbb{R}$ and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R}\backslash J,$$

then we write $w(x) \in \mathcal{F}(C^2+)$.

Remark 1.2. (a) The simplest of the above definition is when $T(x)$ is bounded in $\mathbb{R}$. This is the so-called Freud weight case. Typical example then would be

$$Q(x) = |x|^\alpha, \quad \alpha > 1.$$

(b) A more general example satisfying the above conditions is

$$Q_{l,\alpha}(x) := \exp_l(|x|^\alpha) - \exp_l(0),$$

where $\alpha > 1$ and $l \geq 0$. Here we let $\exp_0(x) := x$ and for $l \geq 1$, $\exp_l(x) := \exp(\exp(\cdots(\exp(;))\cdots))$ denotes the l th iterated exponential. In particular, $\exp_l(x) = \exp(\exp_{l-1}(x))$. We estimate the details of these examples in Section 3.

For the future works such as the differential relation of orthogonal polynomials with respect to the exponential weights, we need further assumptions with respect to $Q(x)$ (see [1, 3]). In the following, we introduce a new weight subclass of the weight class in Definition 1.1.

Definition 1.3. Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$ and ν be a positive integer. Assume that $Q(x)$ is ν-times continuously differentiable on $\mathbb{R}$ and satisfies the followings;

(i) $Q^{(\nu+1)}(x)$ exists and $Q^{(i)}(x)$, $i = 0, 1, \ldots, \nu+1$ are nonnegative for $x > 0$.

(ii) There exist positive constants $C_i > 0$ such that for $x \in \mathbb{R}\backslash\{0\}$

$$|Q^{(i+1)}(x)| \leq C_i |Q^{(i)}(x)| \frac{|Q'(x)|}{Q(x)}, \quad i = 2, \ldots, \nu. \tag{1.2}$$

(iii) There exist constants $0 \leq \delta < 1$ and $c_1 > 0$ such that on $(0, c_1]$

$$Q^{(\nu+1)}(x) \leq C \left(\frac{1}{x}\right)^\delta. \tag{1.3}$$

Then we write $w(x) \in \mathcal{F}_\nu(C^2+)$.

The following theorems give some relations of theses exponential weights.

Theorem 1.4. *Let $w(x) = \exp(-R(x)) \in \mathcal{F}(C^2)$. Then*

(a) *if $Q(x) = \exp(R(x)) - 1$, then $w(x) = \exp(-Q(x))$ belongs to $\mathcal{F}(C^2)$. Moreover, if $w(x) = \exp(-R(x)) \in \mathcal{F}(C^2+)$, then $w(x) = \exp(-Q(x))$ belongs to $\mathcal{F}(C^2+)$.*

(b) *if $Q(x) = (1+|x|)^{R(x)} - 1$, then $w(x) = \exp(-Q(x))$ belongs to $\mathcal{F}(C^2+)$.*

(c) *if $Q(x) = (1 + R(x))^{R(x)} - 1$, then $w(x) = \exp(-Q(x))$ belongs to $\mathcal{F}(C^2+)$.*

Remark 1.5. For all cases of Theorem 1.4

$$\frac{Q''(x)}{Q'(x)} \Big/ \frac{Q'(x)}{Q(x)} \to 1, \qquad \text{as } x \to \infty.$$

Theorem 1.6. *Let ν be a positive integer and $w(x) = \exp(-R(x)) \in \mathcal{F}_\nu(C^2+)$. Then $w(x) = \exp(-Q(x))$ belongs to $\mathcal{F}_\nu(C^2+)$, where $Q(x) = \exp(R(x)) - 1$.*

In the following section, we will prove the results in Section 1. In addition, we will investigate some properties of the typical and specific weights in the weight classes of Definitions 1.1 and 1.3.

2. Proofs of theorems

In this section we will prove the theorems of Section 1.

Lemma 2.1. *For $t > 1$, $\dfrac{t \ln t}{t-1}$ is increasing and*

$$\frac{t \ln t}{t-1} > 1. \tag{2.4}$$

Proof of Theorem 1.4. First, we start by letting

$$T_R(x) := \frac{xR'(x)}{R(x)} \geq \Lambda_R > 1 \tag{2.5}$$

since $w(x) = \exp(-R(x)) \in \mathcal{F}(C^2)$.

(a) Let $S(x) = \exp(R(x))$ and $x > 0$. Then since

$$Q'(x) = R'(x)S(x), \qquad \text{and} \qquad Q''(x) = \left(R''(x) + R'^2(x)\right) S(x),$$

by the conditions of $R(x)$ and $S(x)$ we see that (a), (b), and (c) of Definition 1.1 are satisfied. For $T(x)$, we know that $T_R(x)$ is quasi-increasing and $S(x)\ln S(x)/(S(x)-1)$ is increasing by Lemma 2.1. Therefore, since

$$T(x) = \frac{xQ'(x)}{Q(x)} = \frac{xR'(x)S(x)}{S(x)-1} = T_R(x)\frac{S(x)\ln S(x)}{S(x)-1},$$

$T(x)$ is quasi-increasing. By (2.4) and (2.5) we have

$$T(x) = T_R(x)\frac{S(x)\ln S(x)}{S(x)-1} > T_R(x) \geq \Lambda_R > 1.$$

On the other hand, we have

$$\frac{Q''(x)}{Q'(x)} = \frac{R''(x) + R'^2(x)}{R'(x)} \lesssim \frac{R'(x)}{R(x)} + R'(x) \lesssim \frac{Q'(x)}{Q(x)}, \tag{2.6}$$

because we know from (2.4) that

$$\frac{Q'(x)}{Q(x)} = \frac{R'S(x)}{S(x)-1} > \frac{R'(x)}{R(x)} \quad \text{and} \quad \frac{Q'(x)}{Q(x)} = \frac{R'S(x)}{S(x)-1} > R'(x). \tag{2.7}$$

Therefore, we have $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2)$. Now, suppose $w(x) = \exp(-R(x)) \in \mathcal{F}(C^2+)$. Then there exists a positive constant c such that for $x \geq c > 0$, we have by (2.6) and (2.7)

$$\frac{Q'(x)}{Q(x)} \sim R'(x) \leq \frac{R'(x)}{R(x)} + R'(x) \lesssim \frac{R''(x)}{R'(x)} + R'(x) = \frac{Q''(x)}{Q'(x)}.$$

Therefore, $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$.

(b) Let $S(x) = (1+|x|)^{R(x)}$ and $x > 0$. Then

$$Q'(x) = S'(x) = S(x)f(x) \quad \text{and} \quad Q''(x) = \left(f'(x) + f^2(x)\right) S(x),$$

where

$$f(x) = R'(x)\ln(1+x) + \frac{R(x)}{1+x}. \tag{2.8}$$

Therefore, by the conditions of $R(x)$ and $f(x)$ we see that (a), (b), and (c) of Definition 1.1 are satisfied. To prove $T(x)$ is quasi-increasing in $(0,\infty)$, we let

$$\begin{aligned} T(x) = \frac{xQ'(x)}{Q(x)} &= \frac{S(x)}{S(x)-1}\left(xR'(x)\ln(1+x) + x\frac{R(x)}{1+x}\right) \\ &= T_R(x)\frac{S(x)\ln S(x)}{S(x)-1} + \frac{S(x)R(x)}{S(x)-1}\frac{x}{1+x} \\ &:= A(x) + B(x). \end{aligned}$$

Since for $x > 0$

$$0 < \frac{x}{(1+x)\ln(1+x)} < 1,$$

from Lemma 2.1 we have for $0 < x < 1$

$$0 < B(x) = \frac{S(x)\ln S(x)}{S(x)-1}\frac{x}{(1+x)\ln(1+x)} < \frac{S(1)\ln S(1)}{S(1)-1} \tag{2.9}$$

and since for $x \geq 1$

$$\frac{S(x)}{S(x)-1}\frac{x}{1+x} \sim 1,$$

we have

$$B(x) \sim R(x). \tag{2.10}$$

Now, let $0 < x \leq y$. Then we know that from Lemma 2.1 and (2.5)

$$A(y) \gtrsim A(x) \geq \Lambda_R > 1. \tag{2.11}$$

If $0 < x < 1$, then we have from (2.9) and (2.11)

$$T(x) = A(x) + B(x) \leq A(x) + \frac{S(1)\ln S(1)}{S(1)-1} \lesssim A(y) \lesssim T(y).$$

If $x \geq 1$, then since $R(x)$ is increasing, we have by (2.10) and (2.11)

$$\begin{aligned} T(x) &= A(x) + B(x) \sim A(x) + R(x) \\ &\lesssim A(y) + R(y) \sim A(y) + B(y) = T(y). \end{aligned}$$

Therefore, we have for $0 < x \leq y$

$$T(x) \lesssim T(y),$$

that is, $T(x)$ is quasi-increasing for $x > 0$. Moreover, we have

$$T(x) \geq A(x) \geq \Lambda_R > 1.$$

On the other hand, we can see that $Q''(x) > 0$, because we know from (2.8)

$$f'(x) = R''(x)\ln(1+x) + \frac{R(x)\,(2T_R(x)-1) + 2R'(x)}{(1+x)^2} > 0. \tag{2.12}$$

For $0 < x < 1$, we know that

$$\ln(1+x) \sim x \quad \text{and} \quad \frac{1}{1+x} \sim 1. \tag{2.13}$$

Since for $0 < x < 1$

$$\ln S(x) = R(x)\ln(1+x) \sim xR(x),$$

there exist positive constants C_1 and C_2 such that

$$\exp(C_1 xR(x)) \leq S(x) \leq \exp(C_2 xR(x)).$$

Therefore, we have

$$\exp(C_1 xR(x)) - 1 \;\sim\; \exp(C_2 xR(x)) - 1 \;\sim\; xR(x).$$

So, we obtain that for $0 < x < 1$

$$S(x) - 1 \sim xR(x). \tag{2.14}$$

From (2.8), (2.12), and (2.13), we obtain that for $0 < x < 1$

$$f(x) \sim xR'(x) + R(x) \tag{2.15}$$

and

$$f'(x) \sim xR''(x) + 2xR'(x) - R(x) + 2R'(x). \tag{2.16}$$

Then we have by (2.14) and (2.15)

$$\begin{aligned} \frac{Q'(x)}{Q(x)} &= f(x) + \frac{1}{S(x)-1}f(x) \\ &\sim f(x) + \frac{xR'(x)+R(x)}{xR(x)} \sim f(x) + \frac{R'(x)}{R(x)} + \frac{1}{x} \end{aligned} \tag{2.17}$$

and by (2.15) and (2.16)

$$\begin{aligned}\frac{Q''(x)}{Q'(x)} &= f(x)+\frac{f'(x)}{f(x)} \\ &\sim f(x)+\frac{xR''(x)+2xR'(x)-R(x)+2R'(x)}{xR'(x)+R(x)} \\ &\lesssim f(x)+\frac{R''(x)}{R'(x)}+\frac{1}{x}\lesssim f(x)+\frac{R'(x)}{R(x)}+\frac{1}{x}.\end{aligned} \tag{2.18}$$

Therefore, we have for $0<x<1$

$$\frac{Q''(x)}{Q'(x)}\lesssim\frac{Q'(x)}{Q(x)}.$$

Now, we consider $x\geq 1$. Then

$$\ln(1+x)\sim\ln x \quad\text{and}\quad \frac{1}{1+x}\sim\frac{1}{x}. \tag{2.19}$$

From (2.8), and (2.12), and (2.19), we obtain that

$$f(x)\sim R'(x)\ln x+\frac{R(x)}{x}\ \gtrsim\frac{R'(x)}{R(x)} \tag{2.20}$$

and

$$f'(x)\sim R''(x)\ln x+\frac{R'(x)}{x}. \tag{2.21}$$

Therefore, we have by (2.17)

$$\frac{Q'(x)}{Q(x)}\sim f(x)$$

and by (2.18), (2.20) and (2.21),

$$\begin{aligned}f(x)\lesssim\frac{Q''(x)}{Q'(x)} &\sim f(x)+\frac{R''(x)\ln x+R'(x)/x}{R'(x)\ln x+R(x)/x} \\ &\lesssim f(x)+\frac{R''(x)}{R'(x)}+\frac{R'(x)}{R(x)}\lesssim f(x)+\frac{R'(x)}{R(x)}\lesssim f(x).\end{aligned}$$

Thus, we have for $x\geq 1$,

$$\frac{Q''(x)}{Q'(x)}\sim\frac{Q'(x)}{Q(x)}\sim f(x).$$

Therefore, we obtain the result.

(c) This part is similar to the proof of (b). Let $S(x)=(1+R(x))^{R(x)}$ and $x>0$. Then

$$Q'(x)=S'(x)=S(x)f(x) \quad\text{and}\quad Q''(x)=\left(f'(x)+f^2(x)\right)S(x),$$

where

$$f(x)=R'(x)\ln(1+R(x))+\frac{R'(x)R(x)}{1+R(x)}. \tag{2.22}$$

So, we have

$$T(x)=\frac{xQ'(x)}{Q(x)} = T_R(x)\frac{S(x)\ln S(x)}{S(x)-1}+\frac{S(x)xR'(x)}{S(x)-1}\frac{R(x)}{1+R(x)} := A(x)+B(x).$$

Since for $x>0$

$$0<\frac{xR'(x)}{(1+R(x))\ln(1+R(x))}<T_R(x),$$

from Lemma 2.1 we have for $0<x<1$

$$0<B(x)=\frac{S(x)\ln S(x)}{S(x)-1}\frac{xR'(x)}{(1+R(x))\ln(1+R(x))}\lesssim\frac{S(1)\ln S(1)}{S(1)-1}T_R(1)$$

and since for $x\geq 1$

$$\frac{S(x)}{S(x)-1}\frac{R(x)}{1+R(x)}\sim 1,$$

we have

$$B(x)\sim xR'(x).$$

Then by the same argument as the proof of Theorem 1.4 (b) we can show that $T(x)$ is quasi-increasing for $x>0$ and

$$T(x)\geq \Lambda_R>1.$$

Also, we can see that $Q''(x)>0$ because

(2.23)

$$f'(x)=R''(x)\ln(1+R(x))+\frac{R''(x)R(x)+2R'^2(x)+R''(x)R^2(x)+R'^2(x)R(x)}{(1+R(x))^2}>0.$$

Let x_0 be the positive constant satisfying $R(x_0)=1$. Then for $0<x<x_0$, we know that

$$\ln(1+R(x))\sim R(x) \quad\text{and}\quad \frac{1}{1+R(x)}\sim 1 \tag{2.24}$$

and similarly to the proof of (b),

$$S(x)-1\sim R^2(x). \tag{2.25}$$

From (2.22), (2.23), and (2.24), we obtain that for $0<x<x_0$

$$f(x)\sim R(x)R'(x) \tag{2.26}$$

and

$$f'(x)\sim R''(x)R(x)+R'^2(x)+R'^2(x)R(x). \tag{2.27}$$

Then we have by (2.25) and (2.26)

$$\frac{Q'(x)}{Q(x)} = f(x)+\frac{1}{S(x)-1}f(x) \sim f(x)+\frac{R(x)R'(x))}{R^2(x)}\sim f(x)+\frac{R'(x)}{R(x)} \tag{2.28}$$

and by (2.26) and (2.27)

$$\begin{aligned} \frac{Q''(x)}{Q'(x)} &= f(x)+\frac{f'(x)}{f(x)} \\ &\sim f(x)+\frac{R''(x)R(x)+R'^2(x)+R'(x)^2R(x)}{R(x)R'(x)} \\ &\sim f(x)+\frac{R'(x)}{R(x)}. \end{aligned} \tag{2.29}$$

Therefore, we have for $0<x<x_0$

$$\frac{Q''(x)}{Q'(x)} \sim \frac{Q'(x)}{Q(x)}.$$

We consider $x \geq x_0$. Then

$$\ln(1+R(x)) \sim \ln R(x) \quad \text{and} \quad \frac{1}{1+R(x)} \sim \frac{1}{R(x)}. \tag{2.30}$$

From (2.22), (2.23), and (2.30), we obtain that for $x \geq x_0$

$$f(x) \sim R'(x)\ln R(x) \gtrsim \frac{R'(x)}{R(x)} \tag{2.31}$$

and

$$f'(x) \sim R''(x)\ln R(x)+\frac{R'^2(x)}{R(x)}. \tag{2.32}$$

Then we obtain by (2.28)

$$\frac{Q'(x)}{Q(x)} \sim f(x)$$

and by (2.29), (2.31), and (2.32),

$$\begin{aligned} f(x) \lesssim \frac{Q''(x)}{Q'(x)} &\sim f(x)+\frac{R''(x)\ln R(x)+R'^2(x)/R(x)}{R'(x)\ln R(x)} \\ &\lesssim f(x)+\frac{R''(x)}{R'(x)}+\frac{R'(x)}{R(x)\ln R(x)} \lesssim f(x)+\frac{R'(x)}{R(x)} \lesssim f(x). \end{aligned}$$

Thus, we have for $x \geq x_0$,

$$\frac{Q''(x)}{Q'(x)} \sim \frac{Q'(x)}{Q(x)} \sim f(x).$$

Therefore, we obtain the result. □

Proof of Theorem 1.6. Let $S(x)=\exp(R(x))$. Then since for $k=1,2,\ldots,\nu+1$,

$$Q^{(k)}(x)=\sum_{i=0}^{k-1}\binom{k-1}{i}R^{(i+1)}(x)S^{(k-1-i)}(x), \tag{2.33}$$

by the inductive method, we have that $Q^{(k)}(x)$, $k = 0, 1, \dots, \nu$ are continuous on $\mathbb{R}$, $Q^{(k)}(x) > 0$, $k = 0, 1, \dots, \nu + 1$, and

$$\begin{aligned}
\frac{Q^{(k+1)}(x)}{Q^{(k)}(x)} &= \frac{\sum_{i=0}^{k} \binom{k}{i} R^{(i+1)}(x) S^{(k-i)}(x)}{\sum_{i=0}^{k-1} \binom{k-1}{i} R^{(i+1)}(x) S^{(k-1-i)}(x)} = \frac{\sum_{i=0}^{k} \binom{k}{i} R^{(i+1)}(x) S^{(k-i)}(x)}{\sum_{i=1}^{k} \binom{k-1}{i-1} R^{(i)}(x) S^{(k-i)}(x)} \\
&= \frac{R'(x) S^{(k)}(x)}{\sum_{i=1}^{k} \binom{k-1}{i-1} R^{(i)}(x) S^{(k-i)}(x)} + \frac{\sum_{i=1}^{k} \binom{k}{i} R^{(i+1)}(x) S^{(k-i)}(x)}{\sum_{i=1}^{k} \binom{k-1}{i-1} R^{(i)}(x) S^{(k-i)}(x)} \\
&\lesssim \frac{S^{(k)}(x)}{S^{(k-1)}(x)} + \sum_{i=1}^{k} \frac{R^{(i+1)}(x)}{R^{(i)}(x)} \lesssim R'(x) + \sum_{i=1}^{k} \frac{R^{(i+1)}(x)}{R^{(i)}(x)} \\
&\lesssim R'(x) + \frac{R'(x)}{R(x)} \lesssim \frac{Q'(x)}{Q(x)}, \quad k = 1, 2, \dots, \nu.
\end{aligned}$$

For $x \in (0, c]$, since $R^{(\nu+1)}(x) \lesssim x^{-\delta}$, $0 \le \delta < 1$, we have by (2.33)

$$\begin{aligned}
Q^{(\nu+1)}(x) &= \sum_{i=0}^{\nu} \binom{\nu}{i} R^{(i+1)}(x) S^{(\nu-i)}(x) \\
&= R^{(\nu+1)}(x) S(x) + \sum_{i=0}^{\nu-1} \binom{\nu}{i} R^{(i+1)}(x) S^{(\nu-i)}(x) \\
&\lesssim R^{(\nu+1)}(x) O(1) + O(1) \lesssim x^{-\delta}
\end{aligned}$$

because $R^{(i)}(x)$ and $S^{(i)}(x)$, $i = 0, 1, \dots, \nu$ are increasing. Therefore, the result is proved. □

Proof of Remark 1.5. Case of $Q(x) = \exp(R(x)) - 1$:

$$\frac{Q''(x)}{Q'(x)} \Big/ \frac{Q'(x)}{Q(x)} = 1 + O(1/R(x)).$$

Case of $Q(x) = (1 + R(x))^{R(x)} - 1$: Here, $S(x) = (1 + R(x))^{R(x)}$ and $f(x) = R'(x) \ln(1 + R(x)) + \frac{R'(x)R(x)}{1+R(x)}$. Then since

$$\frac{S''(x)}{S'(x)} \Big/ \frac{S'(x)}{S(x)} = 1 + \frac{f'}{f^2} = 1 + O\left(\frac{1}{R(x) \ln R(x)}\right) \to 1, \quad \text{as } x \to \infty,$$

we have

$$\lim_{x \to \infty} \frac{Q''(x)}{Q'(x)} \Big/ \frac{Q'(x)}{Q(x)} = \lim_{x \to \infty} \frac{S''(x)}{S'(x)} \Big/ \frac{S'(x)}{S(x)} = 1.$$

Case of $Q(x) = (1 + |x|)^{R(x)} - 1$: The proof of this case is similar to the above case. □

3. Examples of exponential weights

Now we will consider some typical examples of $\mathcal{F}(C^2+)$. Define for $\alpha > 1$ and $l \geq 1$,

$$Q_{l,\alpha}(x) := \exp_l(|x|^\alpha) - \exp_l(0). \tag{3.34}$$

More precisely, define for $\alpha + m > 1$, $m \geq 0$, $l \geq 1$ and $\alpha \geq 0$,

$$Q_{l,\alpha,m}(x) := |x|^m(\exp_l(|x|^\alpha) - \alpha^* \exp_l(0)), \tag{3.35}$$

where $\alpha^* = 0$ if $\alpha = 0$, otherwise $\alpha^* = 1$ and define

$$Q_\alpha(x) := (1 + |x|)^{|x|^\alpha} - 1, \ \alpha > 1. \tag{3.36}$$

In the following, we consider the exponential weights with the exponents $Q_{l,\alpha,m}(x)$.

Theorem 3.1. *Let ν be a positive integer. Let $m + \alpha - \nu > 0$. Then*

(a) $w(x) = \exp(-Q_{l,\alpha,m}(x))$ *belongs to* $\mathcal{F}_\nu(C^2+)$.

(b) *If $l \geq 2$ and $\alpha > 0$, then there exists a constant $c_1 > 0$ such that $Q'_{l,\alpha,m}(x)/Q_{l,\alpha,m}(x)$ is quasi-increasing on (c_1, ∞).*

(c) *When $l = 1$, if $\alpha \geq 1$, then there exists a constant $c_2 > 0$ such that $Q'_{l,\alpha,m}(x)/Q_{l,\alpha,m}(x)$ is quasi-increasing on (c_2, ∞) and if $0 < \alpha < 1$, then $Q'_{l,\alpha,m}(x)/Q_{l,\alpha,m}(x)$ is quasi-decreasing on (c_2, ∞).*

(d) *When $l = 1$ and $0 < \alpha < 1$, $Q^{(\nu+1)}_{l,\alpha,m}(x)$ is non-decreasing on a certain positive interval (c_2, ∞).*

Proof. (a) Let $R_l(x) := \exp_l(|x|^\alpha)$ and $x > 0$. Suppose $\prod_{j=1}^{0} R_j(x) := 1$ and $\prod_{j=1}^{-1} R_j(x) := 0$. Then since

$$Q'(x) = mx^{m-1}(R_l(x) - R_l(0)) + \alpha x^{m+\alpha-1} \prod_{j=1}^{l} R_j(x), \tag{3.37}$$

we have

$$\frac{Q'(x)}{Q(x)} = \frac{1}{x}\left(m + \frac{\alpha x^\alpha \prod_{j=1}^{l} R_j(x)}{R_l(x) - R_l(0)}\right).$$

Therefore we have

$$T(x) = \frac{xQ'(x)}{Q(x)} = m + \frac{\alpha x^\alpha \prod_{j=1}^{l} R_j(x)}{R_l(x) - R_l(0)}.$$

Let $S_l(u) = R_l(x)$ for $u = x^\alpha$. Then

$$\left(u \prod_{j=1}^{l} S_j(u) - S_l(u) + S_l(0)\right)' = u \left(\prod_{j=1}^{l} S_j(u)\right)' > 0$$

and

$$\left(u \prod_{j=1}^{l} S_j(u) - S_l(u) + S_l(0) \right) \Bigg|_{u=0} = 0,$$

so we have

$$\frac{u \prod_{j=1}^{l} S_j(u)}{S_l(u) - S_l(0)} = \frac{x^\alpha \prod_{j=1}^{l} R_j(x)}{R_l(x) - R_l(0)} \geq 1. \tag{3.38}$$

Therefore we have

$$T(x) \geq m + \alpha > 1.$$

Here,

$$\frac{x^\alpha \prod_{j=1}^{l} R_j(x)}{R_l(x) - R_l(0)} \to 1 \quad \text{as } x \to 0^+ \tag{3.39}$$

and for sufficiently large $x > 0$

$$\frac{x^\alpha \prod_{j=1}^{l} R_j(x)}{R_l(x) - R_l(0)} \sim x^\alpha \prod_{j=1}^{l-1} R_j(x). \tag{3.40}$$

Then by similar method to the proof of Theorem 1.4 (b) we know that $T(x)$ is quasi-increasing for $x > 0$. Denote for $\alpha \in \mathbb{R}$ and an integer $k \geq 0$,

$${}_\alpha P_k := \alpha(\alpha - 1) \cdots (\alpha - k + 1) \quad \text{and} \quad {}_\alpha P_0 := 1.$$

Then we have the following lemma.

Lemma 3.2. *For $k \geq 1$ and $x > 0$,*

$$\begin{aligned} Q^{(k)}_{l,\alpha,m}(x) &= {}_m P_k \cdot x^{m-k} \cdot (R_l(x) - R_l(0)) \\ &\quad + ({}_{m+\alpha}P_k - {}_m P_k)\, x^{m-k+\alpha} \prod_{j=1}^{l} R_j(x) + E_k(x), \end{aligned} \tag{3.41}$$

where $E_1(x) := 0$ and if we let for $k \geq 2$

$$g_k(x) := ({}_{m+\alpha}P_{k-1} - {}_m P_{k-1}) \cdot x^{m-k+1+\alpha} \left(\prod_{j=1}^{l} R_j(x) \right)', \tag{3.42}$$

then

$$E_k(x) = g_2^{(k-2)}(x) + g_3^{(k-3)}(x) + \cdots + g'_{k-1}(x) + g_k(x). \tag{3.43}$$

Moreover, if $m + \alpha - k + 1 > 0$, then we have $E_k(x) \geq 0$ and

$$Q^{(k)}_{l,\alpha,m}(x) \geq {}_{m+\alpha}P_k \cdot \frac{Q_{l,\alpha,m}(x)}{x^k}. \tag{3.44}$$

Proof. Let $x > 0$. Since

$$xQ'_{l,\alpha,m}(x) = mQ_{l,\alpha,m}(x) + \alpha x^{m+\alpha}\prod_{j=1}^{l} R_j(x), \tag{3.45}$$

we see that (3.41) and (3.43) hold for $k = 1$. Then by the inductive method, we can easily see that (3.41) and (3.43) hold for all $k \geq 1$. First, we know that ${}_{m+\alpha}P_k - {}_mP_k \geq 0$ for $\alpha \geq 0$, $k = 0, 1, \ldots$. For $j = 2, \ldots, k$, $g_j^{(k-j)}(x)$ is finite sum of the terms of $Ax^sF(x)$ form where A is the product of positive numbers or the numbers $> m+\alpha-k+1$, $s \geq m+2\alpha-k$, and $F(x)$ is a function by sum of products of $R_j(x)$. Therefore, if $m+\alpha-k+1 > 0$, then since $g_j^{(k-j)}(x) > 0$, $j = 2, \ldots, k$, we see $E_k(x) \geq 0$. Thus, if $m+\alpha-k+1 > 0$, we have by (3.41) and (3.38)

$$\begin{aligned} Q^{(k)}_{l,\alpha,m}(x) &\geq {}_mP_k \cdot x^{m-k} \cdot (R_l(x) - R_l(0)) + ({}_{m+\alpha}P_k - {}_mP_k)\, x^{m-k+\alpha}\prod_{j=1}^{l} R_j(x) \\ &= {}_{m+\alpha}P_k x^{m-k}(R_l(x) - R_l(0)) = {}_{m+\alpha}P_k \frac{Q_{l,\alpha,m}(x)}{x^k}. \end{aligned}$$

Therefore, we prove the lemma. □

Now, suppose $m + \alpha - \nu > 0$. Then since

$$\lim_{x\to 0+} Q_{l,\alpha,m}(x)/x = 0 = \lim_{x\to 0+} Q'_{l,\alpha,m}(x)$$

and similarly,

$$\lim_{x\to 0-} Q_{l,\alpha,m}(x)/x = 0 = \lim_{x\to 0-} Q'_{l,\alpha,m}(x),$$

we can see from (3.37) that $Q'_{l,\alpha,m}(0) = 0$ and $Q'_{l,\alpha,m}(x)$ is continuous on $\mathbb{R}$. To use the mathematical induction, suppose $Q^{(k-1)}_{l,\alpha,m}(0) = 0$, $1 \leq k \leq \nu$. Then if we show

$$Q^{(k)}_{l,\alpha,m}(0) = \lim_{x\to 0} Q^{(k-1)}_{l,\alpha,m}(x)/x = 0,$$

then $Q^{(k)}_{l,\alpha,m}(0)$ for all $1 \leq k \leq \nu$, because $Q_{l,\alpha,m}(0) = Q'_{l,\alpha,m}(0) = 0$. Then from (3.41),

$$\begin{aligned} \lim_{x\to 0+} Q^{(k-1)}_{l,\alpha,m}(x)/x = \lim_{x\to 0+} \Bigg[&{}_mP_k \cdot x^{m-k} \cdot (R_l(x) - R_l(0)) \\ &+ ({}_{m+\alpha}P_k - {}_mP_k)\, x^{m-k+\alpha}\prod_{j=1}^{l} R_j(x) + \frac{E_{k-1}(x)}{x}\Bigg] = 0. \end{aligned}$$

Similarly, $\lim_{x\to 0-} Q^{(k-1)}_{l,\alpha,m}(x)/x = 0$. Therefore, we know that $Q^{(k)}_{l,\alpha,m}(0) = 0$, $1 \leq k \leq \nu$. Moreover, from the form of (3.41) we have that $Q^{(k)}_{l,\alpha,m}(x)$, $1 \leq k \leq \nu$ are continuous on $\mathbb{R}$. On the other hand, we also know from Lemma 3.2 that

$Q^{(i)}_{l,\alpha,m}(x) > 0$, $i = 0, 1, \ldots, \nu+1$ for $x > 0$. Now, consider for $0 < x < 1$. From the representation of $E_k(x)$, we know that

$$E_k(x) = O(x^{m+2\alpha-k}) \quad \text{and} \quad Q_{l,\alpha,m}(x) \sim x^{m+\alpha}.$$

Then we have from (3.41)

$$\begin{aligned} Q^{(k)}_{l,\alpha,m}(x) &= {}_{m+\alpha}P_k \cdot x^{m-k} \cdot (R_l(x) - R_l(0)) \\ &\quad + ({}_{m+\alpha}P_k - {}_mP_k)\, x^{m-k+\alpha} \left(\prod_{j=1}^{l} R_j(x) - \frac{R_l(x) - R_l(0)}{x^\alpha} \right) + E_k(x) \\ &\le {}_{m+\alpha}P_k \frac{Q_{l,\alpha,m}(x)}{x^k} + O(x^{m+\alpha-k}) + O(x^{m+2\alpha-k}). \end{aligned}$$

Consequently, from (3.44), we have for $0 < x < 1$

$$Q^{(k)}_{l,\alpha,m}(x) \sim x^{m+\alpha-k}. \tag{3.46}$$

Let $x \ge 1$. Then since

$$\left(\prod_{j=1}^{l} R_j(x) \right)^{(i)} \sim x^{i(\alpha-1)} \left(\prod_{j=1}^{l-1} R_j(x) \right)^{i} \prod_{j=1}^{l} R_j(x),$$

we have for $g_k(x)$ defined in Lemma 3.2,

$$g^{(i)}_k(x) \sim x^{m-k+1+\alpha+(i+1)(\alpha-1)} \left(\prod_{j=1}^{l-1} R_j(x) \right)^{i+1} \prod_{j=1}^{l} R_j(x).$$

Therefore, we know that $k \ge 2$

$$g^{(k-2)}_2(x) \sim x^{m+k(\alpha-1)} \left(\prod_{j=1}^{l-1} R_j(x) \right)^{k-1} \prod_{j=1}^{l} R_j(x)$$

and for $3 \le j \le k$

$$g^{(k-j)}_j(x) = o(1) g^{(k-2)}_2(x).$$

Thus, we have for $x \ge 1$ by (3.41)

$$Q^{(k)}_{l,\alpha,m}(x) \sim x^{m+k(\alpha-1)} \left(\prod_{j=1}^{l-1} R_j(x) \right)^{k-1} \prod_{j=1}^{l} R_j(x), \quad k = 1, 2, \ldots, \nu+1, \tag{3.47}$$

since $Q'_{l,\alpha,m}(x) \sim x^{m+\alpha-1} \prod_{j=1}^{l} R_j(x)$. Finally, from (3.46) and (3.47) we have for $0 < x < 1$ and $k = 1, \ldots, \nu$,

$$\frac{Q^{(k+1)}_{l,\alpha,m}(x)}{Q^{(k)}_{l,\alpha,m}(x)} \sim \frac{Q'_{l,\alpha,m}(x)}{Q_{l,\alpha,m}(x)} \sim \frac{1}{x}$$

and for $x \geq 1$ and $k = 1, \ldots, \nu$,

$$\frac{Q^{(k+1)}_{l,\alpha,m}(x)}{Q^{(k)}_{l,\alpha,m}(x)} \sim \frac{Q'_{l,\alpha,m}(x)}{Q_{l,\alpha,m}(x)} \sim x^{\alpha-1}\left(\prod_{j=1}^{l-1} R_j(x)\right). \tag{3.48}$$

Therefore, $w(x) = \exp(-Q_{l,\alpha,m}(x))$ satisfies (ii) of Definition 1.3. Since $m + \alpha - \nu - 1 > -1$, we know from (3.46) that (iii) of Definition 1.3 is satisfied. Thus, $w(x) = \exp(-Q_{l,\alpha,m}(x))$ belongs to $\mathcal{F}_\nu(C^2+)$.

(b) and (c) : Since from (3.48),

$$\frac{Q'_{l,\alpha,m}(x)}{Q_{l,\alpha,m}(x)} \sim x^{\alpha-1}\left(\prod_{j=1}^{l-1} R_j(x)\right),$$

we have the results.

(d) : From (3.47), we can see the result. □

Remark 3.3.

(a) From Theorem 3.1, $w(x) = \exp(-Q_{l,\alpha,m}(x))$ belongs to $\mathcal{F}(C^2+)$.

(b) The weight $w(x) = \exp(-Q_{l,0,m}(x))$, $Q_{l,0,m}(x) = \exp_l(1) \cdot |x|^m$, $m > 1$, is a well known Freud weight satisfying

$$T(a_t) = m, \quad \text{and} \quad a_t \sim t^{\frac{1}{m}}.$$

(c) When $l = 0$, similarly to (b), the weight $w(x) = \exp(-Q_{0,\alpha,m}(x))$, $Q_{0,\alpha,m}(x) = |x|^{m+\alpha}$, $m + \alpha > 1$, is also a well known Freud weight.

(d) From Theorem 1.4 (b), we can see that $w(x) = \exp(-Q_\alpha(x))$, $\alpha > 1$ defined in (3.36), belongs to $\mathcal{F}(C^2+)$.

(e) For $l \geq 1$ and $\alpha > 0$,

$$\lim_{x\to\infty} \frac{Q''_{l,\alpha,m}(x)}{Q'_{l,\alpha,m}(x)} \Big/ \frac{Q'_{l,\alpha,m}(x)}{Q_{l,\alpha,m}(x)} = 1, \qquad \frac{Q''_{l,0,m}(x)}{Q'_{l,0,m}(x)} \Big/ \frac{Q'_{l,0,m}(x)}{Q_{l,0,m}(x)} = 1 - \frac{1}{m}$$

and for $\alpha > 0$,

$$\lim_{x\to\infty} \frac{Q''_\alpha(x)}{Q'_\alpha(x)} \Big/ \frac{Q'_\alpha(x)}{Q_\alpha(x)} = 1.$$

In the following theorem, we consider the special cases with nonnegative even integers m and α.

Theorem 3.4. *Let l be a positive integer. Let m and α be nonnegative even integers with $m + \alpha > 1$. Then $w(x) = \exp(-Q_{l,\alpha,m}(x))$ belongs to $\mathcal{F}_\nu(C^2+)$ for all $\nu \geq 1$.*

Proof. We have already proved for $\nu < m + \alpha$ in Theorem 3.1. Moreover, since m and α are even integers, we know that for any nonnegative integer k, $Q^{(k)}_{l,\alpha,m}(x)$ is continuous on $\mathbb{R}$. Now, we may consider that m and α are nonnegative integers with $\alpha > 0$. Let $x > 0$, $m + \alpha \leq \nu$, and

$$R_{l,\alpha,m}(x) := |x|^m \exp_l(|x|^\alpha).$$

Then we can prove the following representations by the mathematical induction method. For $1 \le k \le m$,

$$R^{(k)}_{l,\alpha,m}(x) = \sum_{s=0}^{k} C_{k,s}(x)x^{m-k+s\alpha}R_l(x).$$

Suppose $m + j\alpha + 1 \le k \le m + (j+1)\alpha$, $j = 0, 1, 2 \ldots$ and let $k_0 := m + (j+1)\alpha - k$. Then

$$R^{(k)}_{l,\alpha,m}(x) = \sum_{s=0}^{k-j} C_{k,s}(x)x^{k_0+s\alpha}R_l(x),$$

where the coefficients functions $C_{k,s}(x)$ are expressed by

$$C_{k,s}(x) = \sum_{\text{finite sum}} \left(\text{positive constant} \cdot \prod_{\text{finite product with } 0 \le p \le l-1} R_p(x) \right).$$

Then for all $k \ge m + \alpha + 1$ we know that

$$Q^{(k)}_{l,\alpha,m}(x) = R^{(k)}_{l,\alpha,m}(x) \ge 0.$$

Consider the case of $m + j\alpha + 1 \le k \le m + (j+1)\alpha$, $j = 0, 1, 2 \ldots$. Then for $0 < x < 1$ we know that

$$R^{(k)}_{l,\alpha,m}(x) \sim x^{k_0}.$$

Therefore, we have for any $k \ge m + \alpha + 1$ and $0 < x < 1$,

$$\frac{Q^{(k+1)}_{l,\alpha,m}(x)}{Q^{(k)}_{l,\alpha,m}(x)} \sim x^{\alpha-1} \text{ or } 1/x \lesssim 1/x \sim \frac{Q'_{l,\alpha,m}(x)}{Q_{l,\alpha,m}(x)}.$$

On the other hand,

$$R^{(k)}_{l,\alpha,m}(x) = C_k x^{m+k(\alpha-1)} \left(\prod_{j=1}^{l-1} R_j(x) \right)^k R_l(x) + \sum_{\text{finite sum}} p(x)F(x)R_l(x),$$

where $C_k > 0$, $p(x)$ is a polynomial with nonnegative coefficients, and

$$F(x) = \prod_{j=1}^{l-1} R_j^{s_j}(x)$$

with $0 \le s_j \le k$ and $0 \le s_1 + s_2 + \cdots + s_{l-1} < (l-1)k$. Then we have for any $k \ge m + \alpha + 1$ and $x \ge 1$

$$Q^{(k)}_{l,\alpha,m}(x) = R^{(k)}_{l,\alpha,m}(x) \sim x^{m+k(\alpha-1)} \left(\prod_{j=1}^{l-1} R_j(x) \right)^k R_l(x).$$

Therefore, we have (3.48). $\square$

Let

$$g(\alpha, x) := \frac{x^\alpha}{1+x}.$$

Then we know that

$$\frac{d^k g(\alpha, x)}{dx^k} = \sum_{j=0}^{k} \binom{k}{j} \frac{{}_\alpha P_{k+1}}{\alpha - j} \frac{x^{\alpha-j}}{(1+x)^{k+1}}$$

and if $\alpha > k$, then $d^k g(\alpha, x)/dx^k > 0$ for $x > 0$.

In the following, we consider the exponential weights with the exponents $Q_\alpha(x)$.

Theorem 3.5. *Let ν be a positive integer and $\alpha > \nu$. Then $w(x) = \exp(-Q_\alpha(x))$ belongs to $\mathcal{F}_\nu(C^2+)$. Moreover, there exists a positive constant $c_2 > 0$ such that $Q'_\alpha(x)/Q_\alpha(x)$ is quasi-increasing on (c_2, ∞).*

Proof. Let $S(x) = (1+|x|)^{x^\alpha}$ and $x > 0$. Then

$$Q'_\alpha(x) = S'(x) = S(x) f(x),$$

where

$$f(x) = \alpha x^{\alpha-1} \ln(1+x) + g(\alpha, x). \tag{3.49}$$

Then for $k = 0, 1, \ldots, \nu$

$$f^{(k)}(x) = {}_\alpha P_{k+1} \cdot x^{\alpha-k-1} \ln(1+x) + \sum_{j=0}^{k} {}_\alpha P_j \cdot \frac{d^{k-j} g(\alpha - j, x)}{dx^{k-j}} > 0.$$

Since $Q'_\alpha(x)$ is continuous on $\mathbb{R}$, $Q'_\alpha(x) = S'(x) = S(x)f(x) > 0$, and for $k = 2, 3, \ldots, \nu+1$

$$Q_\alpha^{(k)}(x) = S^{(k)}(x) = \sum_{j=0}^{k-1} \binom{k}{j} S^{(j)}(x) f^{(k-1-j)}(x), \tag{3.50}$$

we have by the inductive method, $Q_\alpha(x)$ is ν-times continuously differentiable on $\mathbb{R}$ and for $k = 0, 1, \ldots, \nu+1$

$$Q_\alpha^{(k)}(x) > 0.$$

On the other hand, since we have by (3.49) for $0 < x < 1$

$$f^{(k)}(x) \sim x^{\alpha-k}$$

and for $x \geq 1$

$$f^{(k)}(x) \sim x^{\alpha-k-1} \ln x,$$

we have from (3.50)

$$Q_\alpha^{(k)}(x) \sim x^{\alpha-k+1}, \quad \text{as} \quad x \to 0^+ \tag{3.51}$$

and

$$Q_\alpha^{(k)}(x) \sim S(x) \left(x^{\alpha-1} \ln x\right)^k, \quad \text{as} \quad x \to \infty.$$

Therefore, we have for $0 \leq k \leq \nu$

$$Q_\alpha^{(k+1)}(x) \sim Q_\alpha^{(k)}(x)\frac{Q'_\alpha(x)}{Q_\alpha(x)}.$$

Therefore we have (1.2). Since $\alpha - \nu > 0$, we know from (3.51) that (iii) of Definition 1.3 is satisfied. Moreover, since $Q'_\alpha(x)/Q_\alpha(x) \sim f(x)$ for $x \geq 1$, we can see that $Q'_\alpha(x)/Q_\alpha(x)$ is quasi-increasing on a certain interval (c_2, ∞) with $c_2 > 0$. □

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