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ON THE COMPUTATIONS OF CONTIGUOUS RELATIONS FOR $_2F_1$ HYPERGEOMETRIC SERIES

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Abstract. Contiguous relations for hypergeometric series contain an enormous amount of hidden information. Applications of contiguous relations range from the evaluation of hypergeometric series to the derivation of summation and transformation formulas for such series. In this paper, a general formula joining three Gauss functions of the form $_2F_1[a_1, a_2; a_3; z]$ with arbitrary integer shifts is presented. Our analysis depends on using shifted operators attached to the three parameters a_1, a_2 and a_3. We also, discussed the existence condition of our formula.

1. Introduction

The theory of generalized hypergeometric function is fundamental in the field of mathematics and mathematical physics. Most of the functions that occur in the analysis are special cases of the hypergeometric functions. Professor John Wallis in his work Arithmetica Infinitorum (1655), first used the term hypergeometric to denote any series which was beyond the ordinary geometric series. In fact, he studied the series

$$1 + a + a(a+1) + a(a+1)(a+2) + \cdots .$$

During the next one hundred and fifty years, many other mathematicians studied similar series, notably Euler, Vandermonde, Hidenberg etc.

In 1812, Gauss defined his famous hypergeometric series as follows

$$\sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n}{(a_3)_n} \frac{z^n}{n!} = 1 + \frac{a_1 a_2}{a_3} \frac{z}{1!} + \frac{a_1(a_1+1)a_2(a_2+1)}{a_3(a_3+1)} \frac{z^2}{2!} + \cdots , \tag{1.1}$$

where

$$(a)_n = a(a+1)\cdots(a+n-1); (a)_0 = 1.$$

The above series is called Gauss series or the ordinary series. It is usually represented by the symbol $_2F_1[a_1, a_2; a_3; z]$, the well known Gauss hypergeometric function. The series given by (1.1) converges when $|z| < 1$ and when

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$z = 1$ provided that $\mathrm{Re}(a_3 - a_1 - a_2) > 0$ and also when $z = -1$ provided that $\mathrm{Re}(a_3 - a_1 - a_2) > -1$.

In the same paper, Gauss [5], derived his famous summation formula

$$ {}_2F_1[a_1, a_2; a_3; 1] = \frac{\Gamma(a_3)\Gamma(a_3 - a_1 - a_2)}{\Gamma(a_3 - a_1)\Gamma(a_3 - a_2)} $$

provided that $\mathrm{Re}(a_3 - a_1 - a_2) > 0$. If in (1.1), we replace z by $\frac{z}{a_2}$ and let $a_2 \to \infty$, then $\frac{(a_2)_n}{a_2^n} z^n \to z^n$, and we arrive at the the well known Kummer series

$$ \sum_{n=0}^{\infty} \frac{(a_1)_n}{(a_3)_n} \frac{z^n}{n!} = 1 + \frac{a_1}{a_3} \frac{z}{1!} + \frac{a_1(a_1+1)}{a_3(a_3+1)} \frac{z^2}{2!} + \cdots . $$

This series is convergent for all values of a_1, a_3 and z (real or complex) excluding $a_3 = 0, -1, -2, \ldots$ and is represented by the symbol ${}_1F_1[a_1; a_3; z]$, the well known confluent hypergeometric functions.

Gauss hypergeometric functions ${}_2F_1$ and its confluent form ${}_1F_1$ constitute the core of special functions and include most of commonly used functions as their special cases. Thus Legendre's function, the incomplete beta function, the complete elliptic functions of the first and second kinds and most of the classical orthogonal polynomials are particular cases of ${}_2F_1$. On the other hand, the confluent hypergeometric function includes, as its special cases, Bessel's functions, parabolic cylinder functions, coulomb wave functions etc. Again, Whittaker functions are also a slightly modified form of the confluent hypergeometric functions.

On account of their usefulness, the functions ${}_2F_1$ and ${}_1F_1$ have already been explored to a considerable extent by a number of eminent scholars notably Gauss, Kummer, Pincherle, Mellin, Barnes, Slater, Luke, Erdélyi, Exton, etc.

Gauss defined two hypergeometric functions to be contiguous if they have the same power series variable and if two of the parameters are pairwise equal and if third pair differs by ± 1. He showed that a hypergeometric function and any two other contiguous to it are linearly related. Since there are six contiguous to a given ${}_2F_1$, one get a total of 15 relations. In fact, only four of the fifteen are really independent as all others may be obtained by elimination and use of the fact that the ${}_2F_1$ is symmetric in a_1 and a_2.

It should be remarked here that whenever hypergeometric functions reduce to gamma functions, the results are very important from an applicative point. Only a few summation theorems are available in the literature.

On the other hand, applications of the contiguous relations range from the evaluation of hypergeometric series to the derivation of the summation and transformation formulas for such series, they can be used to evaluate hypergeometric function which is contiguous to a hypergeometric series. For this, in a series of three research papers, Lavoie, et. al [11–13] have obtained a large number of very interesting results contiguous to Gauss second, Kummer and Bailey theorems for the series ${}_2F_1$ and Watson, Dixon and Whipple theorems for the series $_3F_2$. These results have been obtained, checked and verified with the help of *Mathematica*, a general system of doing mathematics by computer. Very recently Kummer identity has been generalized by Vidúnas [22], by using the contiguous relations. For more details about hypergeometric series and their contiguous relations see [1–4, 6–8, 10, 15–18, 23].

In [23], several properties of coefficients of these general contiguous relations were proved and then used to propose effective ways to compute contiguous relations. Contiguous relations are also used to make a correspondence between Lie algebra and special functions, these correspondence yields formulas of special functions [14].

In [21], contiguous relations were used to establish and prove sharp inequalities between Gaussian hypergeometric function and the power mean. These results extend known inequalities involving the complete elliptic integral and the hypergeometric mean.

Recently, a good progress has been done in the direction of further study of these contiguous relations. In [19], some interesting consequences of the contiguous relations of $_2F_1$ were proved, while in [9], a new method of the shifted operators for computing the contiguous relations of $_2F_1$ are introduced. In [20], a general form of the relation between three gauss function has been established and with the help of the computer algebra system *Mathematica*, two computational examples using the results obtained are presented.

In order to extend the work, our aim, in this paper, is to obtain a general formula joining three Gauss functions of the form $_2F_1[a_1, a_2; a_3; z]$ with arbitrary shifts is presented. Our analysis depends on using shifted operators attached to the three parameters a_1, a_2 and a_3. In the end, we also discussed the existence conditions of our formula.

We devote the rest of our introduction to notations and recall the following helpful definition which introduced in [9].

Definition 1. Let $\mathcal{A}_i^{\alpha_i} : X \to X$, $(i = 1, 2, 3)$, where X is the set of all Gauss functions $_2F_1[a_1, a_2; a_3; z]$ with variable z, and parameters a_1, a_2 and a_3 such that $a_3 \neq 0, -1, -2, \ldots$. Then

$$\mathcal{A}_1^{\alpha_1}\left(C[a_1, a_2, a_3]\ {}_2F_1[a_1, a_2; a_3; z]\right) = C[a_1+\alpha_1, a_2, a_3]\ {}_2F_1[a_1+\alpha_1, a_2; a_3; z],$$

$$\mathcal{A}_2^{\alpha_2}\left(C[a_1, a_2, a_3]\ {}_2F_1[a_1, a_2; a_3; z]\right) = C[a_1, a_2+\alpha_2, a_3]\ {}_2F_1[a_1, a_2+\alpha_2; a_3; z],$$

$$\mathcal{A}_3^{\alpha_3}\left(C[a_1, a_2, a_3]\ {}_2F_1[a_1, a_2; a_3; z]\right) = C[a_1, a_2, a_3+\alpha_3]\ {}_2F_1[a_1, a_2; a_3+\alpha_3; z],$$

where $\alpha_i, i = 1, 2, 3$ are any integers, and $C[a_1, a_2, a_3]$ is an arbitrary constant function of a_1, a_2 and a_3 such that for any such operators

$$\mathcal{A}_i^{\alpha_i}\mathcal{A}_i^{-\alpha_i}\left(C[a_1, a_2, a_3]\ {}_2F_1[a_1, a_2; a_3; z]\right) = \mathcal{I}\left(C[a_1, a_2, a_3]\ {}_2F_1[a_1, a_2; a_3; z]\right).$$

and $\mathcal{I}$ is the identity operator defined on X with

$$\begin{aligned}\mathcal{I}^k\left(C[a_1, a_2, a_3]\ {}_2F_1[a_1, a_2; a_3; z]\right) &= \mathcal{I}\left(C[a_1, a_2, a_3]\ {}_2F_1[a_1, a_2; a_3; z]\right)\\ &= C[a_1, a_2, a_3]\ {}_2F_1[a_1, a_2; a_3; z];\ \forall F \in X.\end{aligned}$$

2. Preliminaries and tools

Theorem (2) in [9] introduced the five shifted operators of the 1^{st} degree $\mathcal{A}_3^{-1}, \mathcal{A}_2^{-1}, \mathcal{A}_1^{-1}, \mathcal{A}_2$ and $\mathcal{A}_3$ in terms of the two operators $\mathcal{A}_1$ and $\mathcal{I}$ as follows:

$$\begin{aligned}
\mathcal{A}_3^{-1} &= \frac{a_1}{a_3-1}\mathcal{A}_1 + \frac{a_3-a_1-1}{a_3-1}\mathcal{I};\ a_3 \neq 1\\
\mathcal{A}_2^{-1} &= \frac{a_1(z-1)}{a_2-a_3}\mathcal{A}_1 + \frac{a_1+a_2-a_3}{a_2-a_3}\mathcal{I};\ a_2 \neq a_3\\
\mathcal{A}_1^{-1} &= \frac{a_1(z-1)}{a_1-a_3}\mathcal{A}_1 + \frac{2a_1+(a_2-a_1)z-a_3}{a_1-a_3}\mathcal{I};\ a_1 \neq a_3\\
\mathcal{A}_2 &= \frac{a_1}{a_2}\mathcal{A}_1 + \frac{a_2-a_1}{a_2}\mathcal{I};\ a_2 \neq 0\\
\mathcal{A}_3 &= \frac{a_1a_3(z-1)}{(a_1-a_3)(a_3-a_2)z}\mathcal{A}_1 - \frac{a_3((a_3-a_2)z-a_1)}{(a_1-a_3)(a_3-a_2)z}\mathcal{I};\\
&\quad a_1 \neq a_3, a_2 \neq a_3 \text{ and } z \neq 0.
\end{aligned}$$

Lemma (6) in [9], defines two sets each of three dimensional vectors $L^{(j)}$ and $M^{(j)}$, $j \in \mathbb{Z}$ as follows:

$$L^{(-1)} = \begin{bmatrix} \frac{a_1(z-1)}{a_1-a_3} \\ \frac{a_1(z-1)}{a_2-a_3} \\ \frac{a_1}{a_3-a_1} \end{bmatrix},\ L^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ L^{(1)} = \begin{bmatrix} 1 \\ \frac{a_1}{a_2} \\ \frac{a_1a_3(z-1)}{(a_2-a_3)(a_3-a_2)z} \end{bmatrix},$$

$$L^{(2)} = \begin{bmatrix} \frac{a_3+(a_1+1-a_2)z-2(a_1+1)}{(a_1+1)(z-1)} \\ \frac{a_1[(a_3-2a_2-2)+(a_2-a_1+1)z]}{a_2(a_2+1)(z-1)} \\ \frac{a_1a_3(z-1)(a_3+1)[a_3+(a_1+a_2-2a_3-1)z]}{(a_1-a_3-1)(a_1-a_3)(a_3-a_2)(a_3-a_2+1)z^2} \end{bmatrix}$$

and

$$M^{(-1)} = \begin{bmatrix} \frac{2a_1-a_3-(a_1-a_2)z}{a_1-a_3} \\ \frac{a_1+a_2-a_3}{a_2-a_3} \\ \frac{a_3-a1-1)}{a_3-1} \end{bmatrix},\ M^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ M^{(1)} = \begin{bmatrix} 0 \\ \frac{a_2-a_1}{a_2} \\ \frac{a_3[a_1-(a_3-a_2)z]}{(a_1-a_3)(a_3-a_2)z} \end{bmatrix},$$

$$M^{(2)} = \begin{bmatrix} \frac{a_1-a_3+1}{(a_1+1)(z-1)} \\ \frac{(a_2-a_1)[(a_3-2a_2-2)+(a_2-a_1+1)z]+a_2(a_2-a_3+1)}{a_2(a_2+1)(z-1)} \\ \frac{a_3(a_3+1)[[a_1-(a_3-a_2)z][a_3-(2a_3-a_1-a_2+1)z]+(a_1-a_3)(a_3-a_2)(z-1)z]}{(a_1-a_3-1)(a_1-a_3)(a_3-a_2)(a_3-a_2+1)z^2} \end{bmatrix}$$

from which we will have in general

$$\left.\begin{aligned} L^{(j)} &= K_{j-1}L^{(j-1)} + T_{j-1}L^{(j-2)}, \\ M^{(j)} &= K_{j-1}M^{(j-1)} + T_{j-1}M^{(j-2)} \end{aligned}\right\},\ j \geq 0 \tag{2.1}$$

or

$$L^{(j)} = T_{j+1}^{-1}\left[L^{(j+2)} - K_{j+1}L^{(j+1)}\right], \quad M^{(j)} = T_{j+1}^{-1}\left[M^{(j+2)} - K_{j+1}M^{(j+1)}\right] \Bigg\}, \ j < 0 \tag{2.2}$$

where

$$K_n = D_n K_0$$
$$= \begin{bmatrix} \frac{a_3-2(a_1+n)+(a_1+n-a_2)z}{(a_1+n)(z-1)} & 0 & 0 \\ 0 & \frac{a_3-2(a_2+n)+(a_2+n-a_1)z}{(a_2+n)(z-1)} & 0 \\ 0 & 0 & \frac{(a_3+n)[(a_3+n-1)+(a_1+a_2-2(a_3+n)+1)z]}{(a_3+n-a_2)(a_1-a_3-n)z} \end{bmatrix},$$

$$T_n = D_n T_0 = \begin{bmatrix} \frac{a_1+n-a_3}{(a_1+n)(z-1)} & 0 & 0 \\ 0 & \frac{a_2+n-a_3}{(a_2+n)(z-1)} & 0 \\ 0 & 0 & \frac{(a_3+n)(a_3+n-1)(z-1)}{(a_3+n-a_2)(a_1-a_3-n)z} \end{bmatrix},$$

where D_n is the n^{th} shifted matrix defined by

$$D_n = \begin{bmatrix} \mathcal{A}_1^n & 0 & 0 \\ 0 & \mathcal{A}_2^n & 0 \\ 0 & 0 & \mathcal{A}_3^n \end{bmatrix}$$

and

$$K_0 = \begin{bmatrix} \frac{a_3-2a_1+(a_1-a_2)z}{a_1(z-1)} & 0 & 0 \\ 0 & \frac{a_3-2a_2+(a_2-a_1)z}{a_2(z-1)} & 0 \\ 0 & 0 & \frac{a_3[(a_3-1)+(a_1+a_2-2a_3+1)z]}{(a_3-a_2)(a_1-a_3)z} \end{bmatrix},$$

$$T_0 = \begin{bmatrix} \frac{a_1-a_3}{a_1(z-1)} & 0 & 0 \\ 0 & \frac{a_2-a_3}{a_2(z-1)} & 0 \\ 0 & 0 & \frac{a_3(a_3-1)(z-1)}{(a_3-a_2)(a_1-a_3)z} \end{bmatrix}.$$

Moreover, Lemma (7) in [9], asserts the possibility of expressing any shifted operator with arbitrary degree $\mathcal{A}_i^j$, $i = 1, 2, 3$ and $j \in \mathbb{Z}$ as a linear relation of $\mathcal{A}_1$ and $\mathcal{I}$ as

$$\mathcal{A}_i^j = l_{ij}\ \mathcal{A}_1 + m_{ij}\ \mathcal{I},\ i = 1, 2, 3;\ j \in \mathbb{Z}, \tag{2.3}$$

where l_{ij} and m_{ij} are the i^{th} row of the two vectors $L^{(j)}$ and $M^{(j)}$ respectively, defined in (2.1) and (2.2) above.

In addition, relation (2.3), can be written in the matrix form

$$X_n = L^{(n)}A_1 + M^{(n)}\mathcal{I},$$

where

$$X_n = D_n \begin{bmatrix} \mathcal{A}_1^0 \\ \mathcal{A}_2^0 \\ \mathcal{A}_3^0 \end{bmatrix} = \begin{bmatrix} \mathcal{A}_1^n \\ \mathcal{A}_2^n \\ \mathcal{A}_3^n \end{bmatrix}.$$

The aim of this paper is to establish a general formula joining three Gauss functions of the form $_2F_1[a_1, a_2; a_3; z]$ with arbitrary integer shifts.

In order to complete the requirements of our desired result, let us introduce our next lemma to obtain a general expression for the mixed shifted operator $\mathcal{A}_1^\alpha \mathcal{A}_2^\beta \mathcal{A}_3^\gamma$ for some integers α, β and γ in terms of the two operators $\mathcal{A}_1$ and $\mathcal{I}$.

Lemma 2. *Let $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ be the linear operators defined in Definition* 1. *Then for any integers α, β and γ we have*

$$\mathcal{A}_1^\alpha \mathcal{A}_2^\beta \mathcal{A}_3^\gamma = G(a_1, a_2, a_3, \alpha, \beta, \gamma, z)\mathcal{A}_1 + H(a_1, a_2, a_3, \alpha, \beta, \gamma, z)\mathcal{I} :=G\mathcal{A}_1 + H\mathcal{I},$$

where

$$\begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} \mathcal{A}_1(l_{3,\gamma}) & l_{3,\gamma} \\ \mathcal{A}_1(m_{3,\gamma}) & m_{3,\gamma} \end{bmatrix} \begin{bmatrix} \mathcal{A}_1(l_{2,\beta}) & l_{2,\beta} \\ \mathcal{A}_1(m_{2,\beta}) & m_{2,\beta} \end{bmatrix} \begin{bmatrix} l_{1,\alpha} \\ m_{1,\alpha} \end{bmatrix}.$$

In addition, for $r = 0, 1, 2, \ldots$

$$\mathcal{A}_1^r \left(l_{ij}\left(a_1, a_2, a_3, z\right)\right) = l_{ij}\left(a_1 + r, a_2, a_3, z\right)$$

and

$$\mathcal{A}_1^r \left(m_{ij}\left(a_1, a_2, a_3, z\right)\right) = m_{ij}\left(a_1 + r, a_2, a_3, z\right),$$

where l_{ij} and m_{ij} are the i^{th} row of the vectors $L^{(j)}$ and $M^{(j)}$ respectively.

Proof. From (2.3), we have $\mathcal{A}_1^\alpha$, $\mathcal{A}_2^\beta$ and $\mathcal{A}_3^\gamma$ as follows

$$\mathcal{A}_1^\alpha = l_{1\alpha}\mathcal{A}_1 + m_{1\alpha}\mathcal{I} = \begin{bmatrix} l_{1\alpha} \\ m_{1\alpha} \end{bmatrix}^T \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{I} \end{bmatrix},$$

$$\mathcal{A}_2^\beta = l_{2\beta}\mathcal{A}_1 + m_{2\beta}\mathcal{I} = \begin{bmatrix} l_{2\beta} \\ m_{2\beta} \end{bmatrix}^T \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{I} \end{bmatrix},$$

and

$$\mathcal{A}_3^\gamma = l_{3\gamma}\mathcal{A}_1 + m_{3\gamma}\mathcal{I} = \begin{bmatrix} l_{3\gamma} \\ m_{3\gamma} \end{bmatrix}^T \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{I} \end{bmatrix},$$

hence, the direct ordered multiplication of $\mathcal{A}_1^\alpha$, $\mathcal{A}_2^\beta$ and $\mathcal{A}_3^\gamma$ can be obtained as

$$\begin{aligned} &\mathcal{A}_1^\alpha \mathcal{A}_2^\beta \mathcal{A}_3^\gamma \\ &= \begin{bmatrix} l_{1\alpha} \\ m_{1\alpha} \end{bmatrix}^T \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{I} \end{bmatrix} \begin{bmatrix} l_{2\beta} \\ m_{2\beta} \end{bmatrix}^T \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{I} \end{bmatrix} \begin{bmatrix} l_{3\gamma} \\ m_{3\gamma} \end{bmatrix}^T \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{I} \end{bmatrix} \\ &= \begin{bmatrix} l_{1\alpha} \\ m_{1\alpha} \end{bmatrix}^T \begin{bmatrix} \mathcal{A}_1(l_{2\beta}) & \mathcal{A}_1(m_{2\beta}) \\ l_{2\beta} & m_{2\beta} \end{bmatrix} \begin{bmatrix} \mathcal{A}_1(l_{3\gamma}) & \mathcal{A}_1(m_{3\gamma}) \\ l_{3\gamma} & m_{3\gamma} \end{bmatrix} \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{I} \end{bmatrix}. \end{aligned} \tag{2.4}$$

If we write

$$\mathcal{A}_1^\alpha \mathcal{A}_2^\beta \mathcal{A}_3^\gamma = G\mathcal{A}_1 + H\mathcal{I} = \begin{bmatrix} G \\ H \end{bmatrix}^T \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{I} \end{bmatrix}, \tag{2.5}$$

then from (2.4) and (2.5), we get

$$\begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} \mathcal{A}_1(l_{3,\gamma}) & l_{3,\gamma} \\ \mathcal{A}_1(m_{3,\gamma}) & m_{3,\gamma} \end{bmatrix} \begin{bmatrix} \mathcal{A}_1(l_{2,\beta}) & l_{2,\beta} \\ \mathcal{A}_1(m_{2,\beta}) & m_{2,\beta} \end{bmatrix} \begin{bmatrix} l_{1,\alpha} \\ m_{1,\alpha} \end{bmatrix}. \tag{2.6}$$

□

It is important to notice that, (2.6) can be generalized as follows

$$\begin{bmatrix} G_i \\ H_i \end{bmatrix} = \begin{bmatrix} \mathcal{A}_1(l_{3,\gamma_i}) & l_{3,\gamma_i} \\ \mathcal{A}_1(m_{3,\gamma_i}) & m_{3,\gamma_i} \end{bmatrix} \begin{bmatrix} \mathcal{A}_1(l_{2,\beta_i}) & l_{2,\beta_i} \\ \mathcal{A}_1(m_{2,\beta_i}) & m_{2,\beta_i} \end{bmatrix} \begin{bmatrix} l_{1,\alpha_i} \\ m_{1,\alpha_i} \end{bmatrix}, i = 1,2,3.$$

3. Main results

The following theorem is the main theorem of the paper, it gives a general formula connecting any three arbitrary shifted Gauss functions.

Theorem 3. *Let $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ be the three linear operators defined in Definition* 1*. Then the following three Gauss functions*

$$(3.1) \qquad {}_2F_1[a_1 + \alpha_i, a_2 + \beta_i; a_3 + \gamma_i; z], \ i = 1,2,3$$

are linearly dependent if they satisfy the following recurrence relation

$$(3.2) \qquad \begin{vmatrix} \mathcal{A}_1^{\alpha_1}\mathcal{A}_2^{\beta_1}\mathcal{A}_3^{\gamma_1} & \mathcal{A}_1^{\alpha_2}\mathcal{A}_2^{\beta_2}\mathcal{A}_3^{\gamma_2} & \mathcal{A}_1^{\alpha_3}\mathcal{A}_2^{\beta_3}\mathcal{A}_3^{\gamma_3} \\ G_1 & G_2 & G_3 \\ H_1 & H_2 & H_3 \end{vmatrix} {}_2F_1[a_1, a_2; a_3; z] = 0$$

for some constants G_i and H_i.

Proof. The idea of the proof is to compute the product $\mathcal{A}_1^{\alpha_i}\mathcal{A}_2^{\beta_i}\mathcal{A}_3^{\gamma_i}$, and then using the result of Lemma 2, replacing α, β and γ by α_i, β_i and γ_i, respectively, to get such product in the form

$$(3.3) \qquad \mathcal{A}_1^{\alpha_i}\mathcal{A}_2^{\beta_i}\mathcal{A}_3^{\gamma_i} = G_i\mathcal{A}_1 + H_i\mathcal{I},$$

where

$$\begin{bmatrix} G_i \\ H_i \end{bmatrix} = \begin{bmatrix} \mathcal{A}_1(l_{3,\gamma_i}) & l_{3,\gamma_i} \\ \mathcal{A}_1(m_{3,\gamma_i}) & m_{3,\gamma_i} \end{bmatrix} \begin{bmatrix} \mathcal{A}_1(l_{2,\beta_i}) & l_{2,\beta_i} \\ \mathcal{A}_1(m_{2,\beta_i}) & m_{2,\beta_i} \end{bmatrix} \begin{bmatrix} l_{1,\alpha_i} \\ m_{1,\alpha_i} \end{bmatrix}$$

for $i = 1, 2, 3$.

Now, assuming that the three Gauss function (3.1) are linearly dependent and hence we have the linear form

$$(3.4) \qquad \sum_{i=1}^{3} c_i \ {}_2F_1[a_1 + \alpha_i, a_2 + \beta_i; a_3 + \gamma_i; z] = 0$$

or in the operators form as

$$(3.5) \qquad \sum_{i=1}^{3} c_i \left(\mathcal{A}_1^{\alpha_i}\mathcal{A}_2^{\beta_i}\mathcal{A}_3^{\gamma_i}\right) {}_2F_1[a_1, a_2; a_3; z] = 0$$

for some non-zero coefficients $c_i, i = 1, 2, 3$.

Substituting (3.3) in (3.5), we get

$$\sum_{i=1}^{3} c_i \left(G_i\mathcal{A}_1 + H_i\mathcal{I}\right) {}_2F_1[a_1, a_2; a_3; z] = 0.$$

Consequently, we will have the systems

$$\sum_{i=1}^{3} c_i G_i = 0 \text{ and } \sum_{i=1}^{3} c_i H_i = 0 \tag{3.6}$$

which have the non-trivial solution

$$c_1 = \begin{vmatrix} G_2 & G_3 \\ H_2 & H_3 \end{vmatrix}, c_2 = \begin{vmatrix} G_3 & G_1 \\ H_3 & H_1 \end{vmatrix} \text{ and } c_3 = \begin{vmatrix} G_1 & G_2 \\ H_1 & H_2 \end{vmatrix} \tag{3.7}$$

substituting in (3.5), we will have

$$\begin{aligned}\Bigg(\begin{vmatrix} G_2 & G_3 \\ H_2 & H_3 \end{vmatrix} \mathcal{A}_1^{\alpha_1}\mathcal{A}_2^{\beta_1}\mathcal{A}_3^{\gamma_1} + \begin{vmatrix} G_3 & G_1 \\ H_3 & H_1 \end{vmatrix} \mathcal{A}_1^{\alpha_2}\mathcal{A}_2^{\beta_2}\mathcal{A}_3^{\gamma_2} \\ + \begin{vmatrix} G_1 & G_2 \\ H_1 & H_2 \end{vmatrix} \mathcal{A}_1^{\alpha_3}\mathcal{A}_2^{\beta_3}\mathcal{A}_3^{\gamma_3}\Bigg)\, {}_2F_1[a_1, a_2; a_3; z] = 0\end{aligned}$$

from which, we will have the desired formula (3.2) and hence the proof of the theorem is completed. □

The existence of the formula (3.2) of Theorem 3 depends on whether the system (3.6) has a non trivial solution. Indeed, the system (3.6) has the non trivial solution (3.7) if and only if the matrix $\left[\begin{smallmatrix} G_1 & G_2 & G_3 \\ H_1 & H_2 & H_3 \end{smallmatrix}\right]$ is of full rank ($= 2$).

Clearly, we can introduce any arbitrary recurrence relation by choosing any arbitrary integer values for the parameter α_i's, β_i's and γ_i's where $i = 1, 2, 3$, provided that the corresponding matrix $\left[\begin{smallmatrix} G_1 & G_2 & G_3 \\ H_1 & H_2 & H_3 \end{smallmatrix}\right]$ is of full rank.

With the help of the computer algebra system "*Mathematica*", we can easily do much computations with the help of our method, to establish and verify such contiguous relations of the form (3.4).

For simplicity, let us write ${}_2F_1\,[a_1, a_2; a_3; z]$ as F also ${}_2F_1\,[a_1 \pm 1, a_2; a_3; z]$ by $F(a_1^+)$ or $F(a_1^-)$, ${}_2F_1\,[a_1, a_2 \pm 1; a_3; z]$ by $F(a_2^+)$ or $F(a_2^-)$ and ${}_2F_1[a_1, a_2; a_3 \pm 1; z]$ by $F(a_3^+)$ or $F(a_3^-)$, that means we shall omit the subscripts. As well as, by means of $F(a_1^{++})$ and $F(a_1^{--})$, we means ${}_2F_1\,[a_1 \pm 2, a_2; a_3; z]$ respectively, an so on.

Table 1, presents the description of the following four Gauss contiguous relations

$$[2a_1 - a_3 + (a_2 - a_1)z]\, F - a_1(1 - z)F(a_1^+) + (a_3 - a_1)F(a_1^-) = 0,$$

$$(a_1 - a_2)F - a_1 F(a_1^+) + a_2 F(a_2^+) = 0,$$

$$(a_1 - a_3 + 1)F - a_1 F(a_1^+) + (a_3 - 1)F(a_3^-) = 0$$

and

$$a_3(a_1 + (a_2 - a_3)z)F - a_1 a_3(1 - z)F(a_1^+) + (a_3 - a_1)(a_3 - a_2)zF(a_3^+) = 0.$$

In such relations, two hypergeometric series differ just in one parameter from the third hypergeometric series, and the difference is 1. It is very important to notice that a contiguous relation between any three contiguous hypergeometric functions can be found by combining linearly a sequence of Gauss contiguous relations.

Shifts	$\mathbf{G}_i$	$\mathbf{H}_i$	$\mathbf{C}_i$
$\alpha_2 = 1$ $\alpha_3 = -1$	$G_1 = 1$ $G_2 = \frac{a_1(z-1)}{a_1-a_3}$ $G_3 = 0$	$H_1 = 0$ $H_2 = -\frac{a_3+a_1(z-2)-a_2 z}{a_1-a_3}$ $H_3 = 1$	$C_1 = \frac{a_1(z-1)}{a_1-a_3}$ $C_2 = -1$ $C_3 = \frac{2a_1-a_3+z(a_2-a_1)}{a_1-a_3}$
$\alpha_2 = 1$ $\beta_3 = 1$	$G_1 = 0$ $G_2 = 1$ $G_3 = \frac{a_1}{a_2}$	$H_1 = 1$ $H_2 = 0$ $H_3 = \frac{a_2-a_1}{a_2}$	$C_1 = \frac{a_2-a_1}{a_2}$ $C_2 = \frac{a_1}{a_2}$ $C_3 = -1$
$\alpha_2 = 1$ $\gamma_3 = -1$	$G_1 = 0$ $G_2 = 1$ $G_3 = \frac{a_1}{a_3-1}$	$H_1 = 1$ $H_2 = 0$ $H_3 = \frac{a_3-a_1-1}{a_3-1}$	$C_1 = \frac{a_3-a_1-1}{a_3-1}$ $C_2 = \frac{a_1}{a_3-1}$ $C_3 = -1$
$\alpha_2 = 1$ $\gamma_3 = 1$	$G_1 = 0$ $G_2 = 1$ $G_3 = \frac{a_1 a_3(z-1)}{(a_1-a_3)(a_3-a_2)z}$	$H_1 = 1$ $H_2 = 0$ $H_3 = \frac{a_3(a_1+(a_2-a_3)z)}{(a_1-a_3)(a_3-a_2)z}$	$C_1 = \frac{a_3(a_1+(a_2-a_3)z)}{(a_1-a_3)(a_3-a_2)z}$ $C_2 = \frac{a_1 a_3(z-1)}{(a_1-a_3)(a_3-a_2)z}$ $C_3 = -1$

TABLE 1. A sample of Gauss contiguous relations

In Table 2, we presents a sample of 3 contiguous relations since their parameters a_1, a_2 and a_3 differ by ± 1

$$\begin{aligned}&(a_1-a_2)(a_1-a_3)F(a_1^-)+a_1(a_3-a_1-a_2)F(a_1^+)\\&\quad+a_2(2a_1-a_3+(a_2-a_1)z)F(a_2^+)=0,\\&a_1(a_3-2a_2+(a_2-a_1)z)F(a_1^+)+(a_1-a_2)(a_2-a_3)F(a_2^-)\\&\quad+a_2(a_1+a_2-a_3)F(a_2^+)=0\end{aligned}$$

and

$$\begin{aligned}&(a_2-a_3+1)(a_2-a_3)F(a_2^-)+a_2(1-a_2+(a_3-a_1-1)z)F(a_2^+)\\&-(a_3-1)(z(a_2-a_1)-2a_2+a_3)F(a_3^-)=0.\end{aligned}$$

In Table 3, we presents the contiguous relations

$$(a_2-a_3+1)F-(2a_2-a_3+2+(a_1-a_2-1)z)F(a_2^+)-(a_2+1)(z-1)F(a_2^{++})=0,$$

$$(a_2-1)(z-1)F-(a_3-2a_2+2+(a_2-a_1-1)z)F(a_2^-)-(a_2-a_3-1)F(a_2^{-\,-})=0$$

and

$$\begin{aligned}&(a_1-a_3+1)(a_2-a_3+1)F-(a_3-1)(2-a_3-(a_1+a_2-2a_3+3)z)F(a_3^-)\\&-(a_3-1)(a_3-2)(1-z)F(a_3^{-\,-})=0\end{aligned}$$

such type of relations called recurrence identities of "consecutive neighbors", for which one parameter of one of the contiguous functions differ by ± 2, [1,8].

Shifts	$\mathbf{G}_i$	$\mathbf{H}_i$	$\mathbf{C}_i$
$\alpha_1 = -1$ $\alpha_2 = 1$ $\beta_3 = 1$	$G_1 = \frac{a_1(z-1)}{a_1-a_3}$ $G_2 = 1$ $G_3 = \frac{a_1}{a_2}$	$H_1 = -\frac{a_3+a_1(z-2)-a_2z}{a_1-a_3}$ $H_2 = 0$ $H_3 = \frac{a_2-a_1}{a_2}$	$C_1 = \frac{a_2-a_1}{a_2}$ $C_2 = \frac{a_1^2+a_1a_2-a_1a_3}{a_2(a_1-a_3)}$ $C_3 = \frac{-2a_1+a_3+a_1z-a_2z}{a_1-a_3}$
$\alpha_1 = 1$ $\beta_2 = -1$ $\beta_3 = 1$	$G_1 = 1$ $G_2 = \frac{a_1(z-1)}{a_2-a_3}$ $G_3 = \frac{a_1}{a_2}$	$H_1 = 0$ $H_2 = \frac{a_1+a_2-a_3}{a_2-a_3}$ $H_3 = \frac{a_2-a_1}{a_2}$	$C_1 = \frac{-2a_1a_2+a_1a_3-a_1^2z+a_1a_2z}{a_2(a_2-a_3)}$ $C_2 = \frac{a_1-a_2}{a_2}$ $C_3 = \frac{a_1+a_2-a_3}{a_2-a_3}$
$\beta_1 = -1$ $\beta_2 = 1$ $\gamma_3 = -1$	$G_1 = \frac{a_1(z-1)}{a_2-a_3}$ $G_2 = \frac{a_1}{a_2}$ $G_3 = \frac{a_1}{a_3-1}$	$H_1 = \frac{a_1+a_2-a_3}{a_2-a_3}$ $H_2 = \frac{a_2-a_1}{a_2}$ $H_3 = \frac{a_3-a_1-1}{a_3-1}$	$C_1 = \frac{-a_1-a_1a_2+a_1a_3}{a_2(a_3-1)}$ $C_2 = \frac{-a_1+a_1a_2+a_1z+a_1^2z-a_1a_3z}{(a_2-a_3)(a_3-1)}$ $C_3 = \frac{-2a_1a_2+a_1a_3-a_1^2z+a_1a_2z}{a_2(a_2-a_3)}$

TABLE 2. A sample of 3 contiguous functions

Shifts	$\mathbf{G}_i$	$\mathbf{H}_i$	$\mathbf{C}_i$
$\beta_2 = 1$ $\beta_3 = 2$	$G_1 = 0$ $G_2 = \frac{a_1}{a_2}$ $G_3 = \frac{a_1(-2+a_3}{a_2(1+a_2)(z-1)}$ $+\frac{a_2(z-2)+z-a_1z)}{a_2(1+a_2)(z-1)}$	$H_1 = 1$ $H_2 = \frac{a_2-a_1}{a_2}$ $H_3 = \frac{a_2(1+a_2)(z-1)+a_1^2z}{a_2(1+a_2)(z-1)}$ $-\frac{a_1(-2+a_3+2a_3(z-1)+z)}{a_2(1+a_2)(z-1)}$	$C_1 = \frac{a_1+a_1a_2-a_1a_3}{a_2(1+a_2)(z-1)}$ $C_2 = -\frac{a_1(2+2a_2-a_3-z}{a_2(1+a_2)(z-1)}$ $+\frac{a_1z-a_2z)}{a_2(1+a_2)(z-1)}$ $C_3 = -\frac{a_1}{a_2}$
$\beta_2 = -1$ $\beta_3 = -2$	$G_1 = 0$ $G_2 = \frac{a_1(z-1)}{a_2-a_3}$ $G_3 = a_1(z-1)$ $\times\frac{(-2-a_3-a_2(z-2)+z+a_1z)}{(a_2-a_3-1)(a_2-a_3)}$	$H_1 = 1$ $H_2 = \frac{a_1+a_2-a_3}{a_2-a_3}$ $H_3 = \frac{-a_2+(a_2-a_3)^2+a_3+a_1^2z}{(a_2-a_3-1)(a_2-a_3)}$ $+\frac{a_1(-2+2a_2+z-a_3(1+z))}{(a_2-a_3-1)(a_2-a_3)}$	$C_1 = \frac{-a_1+a_1a_2+2a_1z}{(a_2-a_3-1)(a_2-a_3)}$ $+\frac{-2a_1a_2z-a_1z^2+a_1a_2z^2}{(a_2-a_3-1)(a_2-a_3)}$ $C_2 = \frac{a_1(z-1)(-2+2a_2-a_3}{-a_2+a_2^2+a_3-2a_2a_3+a_3^2}$ $+\frac{z+a_1z-a_2z)}{-a_2+a_2^2+a_3-2a_2a_3+a_3^2}$ $C_3 = -\frac{a_1(z-1)}{a_2-a_3}$
$\gamma_2 = -1$ $\gamma_3 = -2$	$G_1 = 0$ $G_2 = \frac{a_1}{a_3-1}$ $G_3 = \frac{-a_1(-2+a_3}{(2+(a_3-3)a_3)(z-1)}$ $+\frac{(3+a_1+a_2)z-2a_3z)}{(2+(a_3-3)a_3)(z-1)}$	$H_1 = 1$ $H_2 = \frac{a_3-a_1-1}{a_3-1}$ $H_3 = \frac{(1+a_1-a_3)}{(2+(-3+a_3)a_3)(z-1)}$ $\times\frac{(-2+a_3+(2+a_1)z-a_3z)}{(2+(-3+a_3)a_3)(z-1)}$	$C_1 = \frac{-a_1z(1+a_1+a_2+a_1a_2}{(-2+a_3)(-1+a_3)^2(z-1)}$ $+\frac{2a_3+a_1a_3+a_2a_3-a_3^2)}{(-2+a_3)(-1+a_3)^2(z-1)}$ $C_2 = -\frac{a_1(-2+a_3}{(2-3a_3+a_3^2)(z-1)}$ $+\frac{(3+a_1+a_2-2a_3)z)}{(2-3a_3+a_3^2)(z-1)}$ $C_3 = -\frac{a_1}{a_3-1}$

TABLE 3. A sample of 3 recurrence identities of "consecutive neighbors"

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