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# AN EXTENSION OF FIXED POINT THEOREMS CONCERNING CONE EXPANSION AND COMPRESSION AND ITS APPLICATION

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Abstract. The famous Guo-Krasnosel'skii fixed point theorems concerning cone expansion and compression of norm type and order type are extended, respectively. As an application, the existence of multiple positive solutions for systems of Hammerstein type integral equations is considered.

## 1. Introduction and preliminaries

There are many fixed point theorems. See [16] for an introduction to the study and applications of fixed point theorems. In this paper we will generalize the fixed point theorems concerning cone expansion and compression of norm type and order type.

**Definition 1.1.** Let $E$ be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:
(i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P, -x \in P$ implies $x = \theta$.

Every cone $P \subset E$ induces an ordering in $E$ given by

$$x \leq y \ \text{ if and only if } \ y - x \in P.$$

**Definition 1.2.** An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

**Definition 1.3.** Let $E$ and $E_1$ be two real Banach spaces with cones $P$ and $P_1$ respectively. Then the operator $B : P \to P_1$ is said to be homogeneous on $P$ provided any $t \in \mathbb{R}^+, x \in P$ implies $B(tx) = tBx$.

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**Definition 1.4.** Let $E$ and $E_1$ be two real Banach spaces with cones $P$ and $P_1$ respectively. Then the operator $B : P \to P_1$ is said to be order-preserving on $P$ provided $x_1, x_2 \in P$ with $x_1 \leq x_2$ implies $Bx_1 \leq Bx_2$.

The following theorem, which establishes the existence and uniqueness of fixed point index, is from [5]; an elementary proof can be found in [3]. The proof of the generalization of the fixed point theorems of norm type and order type in the section will invoke the properties of the fixed point index. The proof of the following fixed point index results can be found in [3, 5].

**Lemma 1.1.** *Let $X$ be a retract of a real Banach space $E$. Then, for every bounded relatively open subset $U$ of $X$ and every completely continuous operator $A : \overline{U} \to X$ which has no fixed points on $\partial U$ (relative to $X$), there exists an integer $i(A, U, X)$ satisfying the following conditions*:

($G_1$) **Normality:** $i(A, U, X) = 1$ *if* $Ax \equiv y_0 \in U$ *for any* $x \in \overline{U}$;

($G_2$) **Additivity:** $i(A, U, X) = i(A, U_1, X) + i(A, U_2, X)$ *whenever* $U_1$ *and* $U_2$ *are disjoint open subsets of* $U$ *such that* $A$ *has no fixed points on* $\overline{U} \backslash (U_1 \cup U_2)$;

($G_3$) **Homotopy Invariance:** $i(H(t, \cdot), U, X)$ *is independent of* $t \in [0, 1]$ *whenever* $H : [0, 1] \times \overline{U} \to X$ *is completely continuous and* $H(t, x) \neq x$ *for any* $(t, x) \in [0, 1] \times \partial U$;

($G_4$) **Permanence:** $i(A, U, X) = i(A, U \cap Y, Y)$ *if* $Y$ *is a retract of* $X$ *and* $A(\overline{U}) \subset Y$;

($G_5$) **Excision:** $i(A, U, X) = i(A, U_0, X)$ *whenever* $U_0$ *is an open subset of* $U$ *such that* $A$ *has no fixed points in* $\overline{U} \backslash U_0$;

($G_6$) **Solution:** *If* $i(A, U, X) \neq 0$, *then* $A$ *has at least one fixed point in* $U$.

*Moreover, $i(A, U, X)$ is uniquely defined.*

Now we state the Guo-Krasnosel'skii fixed point theorems concerning cone expansion and compression of norm type and order type as follows (see [3, 5]).

**Theorem 1.1.** *Let $\Omega_1$ and $\Omega_2$ be two bounded open sets in $E$ such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is completely continuous. If one of the two conditions*

($H_1$) $\|Ax\| \leq \|x\|, \forall\ x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \geq \|x\|,\ \forall\ x \in P \cap \partial\Omega_2$

*and*

($H_2$) $\|Ax\| \geq \|x\|, \forall\ x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \leq \|x\|,\ \forall\ x \in P \cap \partial\Omega_2$

*is satisfied, then $A$ has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.*

**Theorem 1.2.** *Let $\Omega_1$ and $\Omega_2$ be two bounded open sets in $E$ such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is completely continuous. If one of the two conditions*

($H_3$) $Ax \ngeq x, \forall\ x \in P \cap \partial\Omega_1$ *and* $Ax \nleq x,\ \forall\ x \in P \cap \partial\Omega_2$

*and*

($H_4$) $Ax \nleq x, \forall\ x \in P \cap \partial\Omega_1$ *and* $Ax \ngeq x,\ \forall\ x \in P \cap \partial\Omega_2$

*is satisfied, then $A$ has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.*

These theorems are extensively applied to many problems of various kinds; see [1, 4, 8, 9, 11, 12, 14], for example. In [2, 17], the authors only dealt with modifications of the Guo-Krasnosel'skii fixed point theorem concerning cone expansion and compression of norm type, respectively. However, fixed point theorem concerning cone expansion and compression of order type to our knowledge has not been generalized or extended. In this paper, the fixed point theorems concerning cone expansion and compression of norm type and order type are both extended.

**Lemma 1.2.** *Let $P$ be a cone in a real Banach space $E$, $\Omega$ a bounded open subset of $E$ with $\theta \in \Omega$, and $A : P \cap \overline{\Omega} \to P$ a completely continuous operator. If*

$$Ax \neq \mu x$$

*for all $x \in P \cap \partial\Omega$ and $\mu \geq 1$, then the fixed point index*

$$i(A, P \cap \Omega, P) = 1.$$

**Lemma 1.3.** *Let $P$ be a cone in a real Banach space $E$, $\Omega$ a bounded open subset of $E$, and $A : P \cap \overline{\Omega} \to P$ a completely continuous operator. If*

(i) $\inf\limits_{x \in P \cap \partial\Omega} \|Ax\| > 0$ *and*

(ii) $Ax \neq \mu x$ *for all $x \in P \cap \partial\Omega$ and $\mu \in (0, 1]$, then the fixed point index*

$$i(A, P \cap \Omega, P) = 0.$$

**Lemma 1.4.** *Let $P$ be a cone in a real Banach space $E$, $\Omega$ a bounded open subset of $E$, and $A : P \cap \overline{\Omega} \to P$ a completely continuous operator. Assume that there exists a $u_0 \in P, u_0 \neq \theta$ such that*

$$x - Ax \neq \mu u_0$$

*for all $x \in P \cap \partial\Omega$ and $\mu \geq 0$, then the fixed point index*

$$i(A, P \cap \Omega, P) = 0.$$

## 2. Main results

In this section, we present the main results of this paper.

**Theorem 2.1.** *Let $P$ be a cone in a real Banach space $E$, $\Omega$ a bounded open subset of $E$ with $\theta \in \Omega$, and $A : P \cap \overline{\Omega} \to P$ a completely continuous operator. Assume that there exists another cone $P_1$ in another real Banach space $E_1$ and a homogeneous operator $B : P \to P_1$ with $\{Bx \mid x \in P \cap \partial\Omega\} \subset P_1 \backslash \{\theta\}$, such that*

$$BAx \leq Bx, \quad \forall\, x \in P \cap \partial\Omega, \tag{2.1}$$

*this partial order is induced by the cone $P_1$ in $E_1$, then the fixed point index*

$$i(A, P \cap \Omega, P) = 1.$$

*Proof.* If there exist $x_0 \in P \cap \partial\Omega$ and $\lambda_0 \geq 1$ such that $Ax_0 = \lambda_0 x_0$, then $\lambda_0 > 1$. Therefore

$$B(Ax_0) = B(\lambda_0 x_0) = \lambda_0 Bx_0 > Bx_0,$$

which contradicts (2.1). Hence the proof is finished by Lemma 1.2. □

**Theorem 2.2.** *Let $P$ be a cone in a real Banach space $E$, $\Omega$ a bounded open subset of $E$, and $A : P \cap \overline{\Omega} \to P$ a completely continuous operator. If*

(i) $\inf\limits_{x \in P \cap \partial\Omega} \|Ax\| > 0$ *and*

(ii) *there exists another cone $P_1$ in another real Banach space $E_1$ and a homogeneous operator $B : P \to P_1$ with $\{Bx \mid x \in P \cap \partial\Omega\} \subset P_1 \backslash \{\theta\}$, such that*

$$BAx \geq Bx, \quad \forall\, x \in P \cap \partial\Omega, \tag{2.2}$$

*this partial order is induced by the cone $P_1$ in $E_1$, then the fixed point index*

$$i(A, P \cap \Omega, P) = 0.$$

*Proof.* If there exist $x_0 \in P \cap \partial\Omega$ and $0 < \lambda_0 \leq 1$ such that $Ax_0 = \lambda_0 x_0$, then $0 < \lambda_0 < 1$. Therefore

$$B(Ax_0) = B(\lambda_0 x_0) = \lambda_0 Bx_0 < Bx_0,$$

which contradicts (2.2). Hence the proof is finished by Lemma 1.3. □

**Theorem 2.3.** *Let $\Omega_1$ and $\Omega_2$ be two bounded open sets in $E$ such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is completely continuous. Assume that there exists two cones $P_1$ and $P_2$ in the Banach spaces $E_1$ and $E_2$ respectively, and homogeneous operators $B_1 : P \to P_1$ with $\{B_1 x \mid x \in P \cap \partial\Omega_1\} \subset P_1 \backslash \{\theta\}$ and $B_2 : P \to P_2$ with $\{B_2 x \mid x \in P \cap \partial\Omega_2\} \subset P_2 \backslash \{\theta\}$. If one of the two conditions*:

($H_1^*$) $B_1 Ax \leq B_1 x, \forall\, x \in P \cap \partial\Omega_1$ *and* $\inf_{x \in P \cap \partial\Omega_2} \|Ax\| > 0,\ B_2 Ax \geq B_2 x$, $\forall\, x \in P \cap \partial\Omega_2$;

($H_2^*$) $\inf_{x \in P \cap \partial\Omega_1} \|Ax\| > 0, B_1 Ax \geq B_1 x, \forall\, x \in P \cap \partial\Omega_1$ *and* $B_2 Ax \leq B_2 x$, $\forall\, x \in P \cap \partial\Omega_2$;

*is satisfied, then $A$ has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.*

The proof is easy by Theorems 2.1 and 2.2, and hence we omit it.

*Remark* 2.1. We claim that Theorem 2.3 is the extension of the fixed point theorem of cone expansion and compression of norm type. Indeed, if we take $B_1 x = B_2 x = \|x\|, \forall\, x \in P$, then $B_1(B_2) : P \to \mathbb{R}^+$ is a homogeneous operator and $\|x\| \neq 0, \forall\, x \in P \cap \partial\Omega_1$ or $x \in P \cap \partial\Omega_2$. Moreover $\inf_{x \in P \cap \partial\Omega_1} \|Ax\| > 0$ and $\inf_{x \in P \cap \partial\Omega_2} \|Ax\| > 0$ is satisfied naturally by $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$.

**Theorem 2.4.** *Let $P$ be a cone in a real Banach space $E$, $\Omega$ a bounded open subset of $E$ with $\theta \in \Omega$, and $A : P \cap \overline{\Omega} \to P$ a completely continuous operator.*

*Assume that there exists another cone $P_1$ in another real Banach space $E_1$ and an order-preserving operator $B : P \to P_1$, such that*

$$BAx \ngeq Bx, \quad \forall\, x \in P \cap \partial\Omega, \tag{2.3}$$

*this partial order is induced by the cone $P_1$ in $E_1$, then the fixed point index*

$$i(A, P \cap \Omega, P) = 1.$$

*Proof.* If there exist $x_0 \in P \cap \partial\Omega$ and $\lambda_0 \geq 1$ such that $Ax_0 = \lambda_0 x_0$, then $Ax_0 \geq x_0$. Therefore

$$BAx_0 \geq Bx_0,$$

which contradicts (2.3). Hence the proof is finished by Lemma 1.2. □

**Theorem 2.5.** *Let $P$ be a cone in a real Banach space $E$, $\Omega$ a bounded open subset of $E$, and $A : P \cap \overline{\Omega} \to P$ a completely continuous operator. Assumed that there exists another cone $P_1$ in another real Banach space $E_1$ and an order-preserving operator $B : P \to P_1$, such that*

$$BAx \nleq Bx, \quad \forall\, x \in P \cap \partial\Omega, \tag{2.4}$$

*this partial order is induced by the cone $P_1$ in $E_1$, then the fixed point index*

$$i(A, P \cap \Omega, P) = 0.$$

*Proof.* If there exist $u_0 \in P, u_0 \neq \theta$, $x_0 \in P \cap \partial\Omega$ and $\lambda_0 \geq 0$, such that $x_0 - Ax_0 = \lambda_0 u_0$, then $Ax_0 \leq x_0$. Therefore

$$BAx_0 \leq Bx_0,$$

which contradicts (2.4). Hence the proof is finished by Lemma 1.4. □

**Theorem 2.6.** *Let $\Omega_1$ and $\Omega_2$ be two bounded open sets in $E$ such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is completely continuous. Assume that there exists two cones $P_1$ and $P_2$ in the Banach spaces $E_1$ and $E_2$ respectively, and order-preserving operators $B_1 : P \to P_1,\ B_2 : P \to P_2$. If one of the two conditions*:

($H_3^*$) *$B_1Ax \ngeq B_1x, \forall\, x \in P \cap \partial\Omega_1$ and $B_2Ax \nleq B_2x, \forall\, x \in P \cap \partial\Omega_2$;*
($H_4^*$) *$B_1Ax \nleq B_1x, \forall\, x \in P \cap \partial\Omega_1$ and $B_2Ax \ngeq B_2x, \forall\, x \in P \cap \partial\Omega_2$;*

*is satisfied, then $A$ has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.*

The proof is easy by Theorems 2.4 and 2.5, and hence we omit it.

*Remark* 2.2. While using Guo-Krasnosel'skii fixed point theorems concerning cone expansion and compression of order type, we find that the partial order induced by the cone $P$ is difficult to check in a Banach space $E$. So we introduce another cone $P_1$ in another real Banach space $E_1$, the partial order induced by $P_1$ is easily satisfied. Obviously, if we take $B_1 \equiv B_2 \equiv \mathbf{I}$ (the identical mapping), then $B_1(B_2) : P \to P$ is an order-preserving operator. So far, we realize that the fixed point theorems concerning cone expansion and compression of order type is a special case of Theorem 2.6, namely it is improved.

## 3. Applications

In this section, we apply the results in Section 2 to the existence of multiple positive solutions for system of Hammerstein type integral equations given by

$$\begin{cases} \varphi(x) = \displaystyle\int_G k_1(x,y) f_1(y, \varphi(y), \psi(y))dy, \\ \psi(x) = \displaystyle\int_G k_2(x,y) f_2(y, \varphi(y), \psi(y))dy, \end{cases} \tag{3.1}$$

where $G$ is a bounded closed domain in $\mathbb{R}^n$, $k_i(x,y) : G \times G \to \mathbb{R}^+$ is a nonnegative continuous function, and $f_i(x,u,v) : G \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function $(i = 1, 2)$. If we let

$$A_i(\varphi, \psi)(x) = \int_G k_i(x,y) f_i(y, \varphi(y), \psi(y))dy, \quad i = 1, 2,$$

and $A(\varphi, \psi) = (A_1(\varphi, \psi), A_2(\varphi, \psi))$, then (3.1) is equivalent to the fixed point of operator $A$.

Let the operator $K_i$ be defined as

$$(K_i\varphi)(x) = \int_G k_i(x,y)\varphi(y)dy (x \in G), \quad i = 1, 2,$$

and the spectral radiuses $r(K_i) > 0$ $(i = 1, 2)$. $f_i(x,u,v) : G \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ a continuous function and $k_i$ a nonnegative function imply that linear operator $K_i$ is complete continuous $(i = 1, 2)$. Evidently, operator $A_i$ is also complete continuous $(i = 1, 2)$. Then by the well-known Krein-Rutman Theorem [7], there exist continuous functions $g_i(x) \geq 0$, $g_i(x) \not\equiv 0$ such that

$$\int_G k_i(y,x) g_i(y)dy = r(K_i) g_i(x), \quad \forall\, x \in G, \ i = 1, 2. \tag{3.2}$$

We make the following assumptions:

$(C_1)$ there exist continuous nonnegative function $a_i(x)$ $(i = 1, 2)$, such that

$$k_i(x,y) \geq a_i(x) k_i(\tau, y), \ \forall\, x, y, \tau \in G, \tag{3.3}$$

$$\delta_i = \int_G a_i(x) g_i(x)dx > 0, \tag{3.4}$$

where $g_i(x)$ are determined by (3.2);

$(C_2)$ If one of the two conditions

$$\lim_{u \to 0^+} \frac{f_1(x,u,v)}{u} > r^{-1}(K_1) \tag{3.5}$$

hold uniformly for $x \in G, v \in \mathbb{R}^+$ and

$$\lim_{v \to 0^+} \frac{f_2(x,u,v)}{v} > r^{-1}(K_2) \tag{3.6}$$

hold uniformly for $x \in G, u \in \mathbb{R}^+$ is satisfied;

$(C_3)$ Let

$$\lim_{u\to+\infty}\frac{f_1(x,u,v)}{u} > r^{-1}(K_1) \tag{3.7}$$

hold uniformly for $x \in G, v \in \mathbb{R}^+$ and

$$\lim_{v\to+\infty}\frac{f_2(x,u,v)}{v} > r^{-1}(K_2) \tag{3.8}$$

hold uniformly for $x \in G, u \in \mathbb{R}^+$.

$(C_4)$ There exist a $r_1 > 0$ with $0 < u + v < r_1$, such that

$$f_i(x,u,v) \le \lambda r_1 \ (\ \forall\ x \in G), \tag{3.9}$$

where $0 < \lambda \le (\|h_1\| + \|h_2\|)^{-1},\ h_i(x) = \int_G k_i(x,y)dy,\ \|h_i\| = \max_{x\in G}|h_i(x)|$, $i = 1, 2$.

**Theorem 3.1.** *Suppose that conditions* $(C_1)$-$(C_4)$ *hold. Then problem* (3.1) *has at least two positive continuous solutions* $(\varphi_1, \psi_1)$ *and* $(\varphi_2, \psi_2)$ *satisfying*

$$0 < \|\varphi_1\| + \|\psi_1\| < r_1 < \|\varphi_2\| + \|\psi_2\|.$$

*Proof.* Let $C(G) = \Big\{\varphi \mid \varphi(x) \text{ is continuous on G}\Big\}$, X$=C(G)\times C(G)$. The norm in $X$ is defined as $\|(\varphi,\psi)\|_X = \|\varphi\| + \|\psi\|$, and obviously $X$ is a Banach space. Let

$$P_i = \Big\{\varphi \in C(G) \mid \varphi(x) \ge 0, \int_G g_i(x)\varphi(x)dx \ge \delta_i\|\varphi\|\Big\},$$

where $\delta_i$ are determined by (3.4)$(i = 1, 2)$. It is easy to see that $P = P_1 \times P_2$ is a cone in $X$. Let

$$T(\varphi,\psi) = (r^{-1}(K_1)K_1\varphi, r^{-1}(K_2)K_2\psi).$$

Now we first show that the operator $A$ maps $P$ into $P$. In fact, for any $(\varphi,\psi) \in P$, by virtue of (3.3) we get

$$\begin{aligned}\int_G g_1(x)A_1(\varphi,\psi)(x)dx &= \int_G g_1(x)dx\int_G k_1(x,y)f_1(y,\varphi(y),\psi(y))dy \\ &\ge \int_G g_1(x)dx\int_G a_1(x)k_1(\tau,y)f_1(y,\varphi(y),\psi(y))dy \\ &\ge \delta_1 A_1(\varphi,\psi)(\tau),\ \forall\ \tau\in G,\end{aligned}$$

which implies $A_1(\varphi,\psi) \in P_1$. Similarly $A_2(\varphi,\psi) \in P_2$ and hence $A(\varphi,\psi) \in P$, namely, $A(P) \subset P$. Similarly $T(P) \subset P$. We take

$$B(\varphi,\psi) = \int_G g_1(x)\varphi(x)dx,$$

then $B$ maps $P$ into another cone $P_1 = [0, +\infty)$ in a real Banach space $E_1 = \mathbb{R}$ and evidently $B$ is an order-preserving and homogeneous operator. Let

$$\Omega_r = \Big\{(\varphi, \psi) \in X \mid \|\varphi\| + \|\psi\| < r\Big\},$$

then $\theta \in \Omega_r$.

Condition (3.5) implies that we can find a number $r_0$ with $0 < r_0 < r_1$ such that

$$f_1(x, u, v) \geq r^{-1}(K_1)u, \quad \forall\ 0 < u \leq r_0. \tag{3.10}$$

Without loss of generality, we may assume that (3.5) hold in $(C_2)$ and $A$ has no fixed point on $\partial\Omega_{r_0}$. In virtue of (3.2) and (3.10), for any $(\varphi, \psi) \in \partial\Omega_{r_0} \cap P$, we get

$$\begin{aligned}
&BA(\varphi, \psi) - B(\varphi, \psi)\\
&= \int_G g_1(x)A_1(\varphi(x), \psi(x))dx - B(\varphi, \psi)\\
&= \int_G g_1(x)dx \int_G k_1(x, y)f_1(y, \varphi(y), \psi(y))dy - \int_G g_1(x)\varphi(x)dx\\
&= r(K_1)\int_G g_1(y)f_1(y, \varphi(y), \psi(y))dy - \int_G g_1(x)\varphi(x)dx\\
&\geq 0.
\end{aligned}$$

It is clear that $BA(\varphi, \psi) - B(\varphi, \psi) \neq 0$, thus $BA(\varphi, \psi) - B(\varphi, \psi) > 0$. Hence for any $(\varphi, \psi) \in \partial\Omega_{r_0} \cap P$, such that

$$BA(\varphi, \psi) \nleq B(\varphi, \psi).$$

It follows from Theorem 2.5 that the fixed point index

$$i(A, \Omega_{r_0} \cap P, P) = 0. \tag{3.11}$$

From (3.7) in $(C_3)$, then there exist positive numbers $\varepsilon$ and $b_1$, such that

$$f_1(x, u, v) \geq (r^{-1}(K_1) + \varepsilon)u - b_1, \quad \forall\ x \in G,\ u \geq 0,\ v \geq 0. \tag{3.12}$$

We put

$$R > \max\Big\{r_1, \frac{2b_1\displaystyle\int_G g_1(y)dy}{\varepsilon r(K_1)\delta_1}\Big\}$$

and let

$$\Omega_R = \Big\{(\varphi, \psi) \in X \mid \|\varphi\| + \|\psi\| < R\Big\}.$$

Without loss of generality, we may assume that $A$ has no fixed point on $\partial\Omega_R$. For any $(\varphi, \psi) \in \partial\Omega_R \cap P$, we have $\|\varphi\| + \|\psi\| = R$. Suppose $\|\varphi\| \geq \frac{R}{2}$, because the proof is similar when $\|\psi\| \geq \frac{R}{2}$. In virtue of (3.12) and (3.2) we get

$$
\begin{aligned}
& BA(\varphi,\psi) - B(\varphi,\psi) \\
= & \int_G g_1(x)A_1(\varphi(x),\psi(x))dx - B(\varphi,\psi) \\
= & \int_G g_1(x)dx \int_G k_1(x,y)f_1(y,\varphi(y),\psi(y))dy - \int_G g_1(x)\varphi(x)dx \\
= & \, r(K_1)\int_G g_1(y)f_1(y,\varphi(y),\psi(y))dy - \int_G g_1(x)\varphi(x)dx \\
\geq & \, r(K_1)(r^{-1}(K_1)+\varepsilon)\int_G g_1(y)\varphi(y)dy - b_1\int_G g_1(y)dy - \int_G g_1(x)\varphi(x)dx \\
= & \, \varepsilon r(K_1)\int_G g_1(y)\varphi(y)dy - b_1\int_G g_1(y)dy \\
\geq & \, \varepsilon r(K_1)\delta_1\|\varphi\| - b_1\int_G g_1(y)dy > 0.
\end{aligned}
$$

Hence for any $(\varphi,\psi) \in \partial\Omega_R \cap P$, such that

$$BA(\varphi,\psi) \nleq B(\varphi,\psi).$$

It follows from Theorem 2.5 that the fixed point index

$$i(A,\Omega_R \cap P, P) = 0. \tag{3.13}$$

Without loss of generality, we may assume that $A$ has no fixed point on $\partial\Omega_{r_1}$. For any $(\varphi,\psi) \in \partial\Omega_{r_1} \cap P$, then $\|\varphi\| + \|\psi\| = r_1$. By (3.9), we get

$$A_i(\varphi,\psi)(x) \leq \lambda r_1 \int_G k_i(x,y)dy = \lambda r_1 h_i(x), \quad (i=1,2),$$

then $\|A_i(\varphi,\psi)\| \leq \lambda r_1\|h_i\|$. Therefore,

$$\|A(\varphi,\psi)\|_X \leq \lambda r_1(\|h_1\| + \|h_2\|) \leq r_1 = \|(\varphi,\psi)\|_X.$$

By taking $Bx = \|x\|_X$, $\forall\, x \in P$ in Theorem 2.1, we see that the fixed point index

$$i(A,\Omega_{r_1} \cap P, P) = 1. \tag{3.14}$$

It is clear that $\overline{\Omega}_{r_0} \subset \Omega_{r_1}$, $\overline{\Omega}_{r_1} \subset \Omega_R$. We see that (3.11), (3.13) and (3.14) imply by virtue of $(G_2)$ in Lemma 1.1 the fixed point index $i(A,(\Omega_R\backslash\Omega_{r_1}) \cap P, P) = -1, i(A,(\Omega_{r_1}\backslash\Omega_{r_0}) \cap P, P) = 1$. Hence, $(G_6)$ in Lemma 1.1 implies that $A$ has at least two positive continuous solutions $(\varphi_1,\psi_1)$ and $(\varphi_2,\psi_2)$ satisfying

$$0 < \|\varphi_1\| + \|\psi_1\| < r_1 < \|\varphi_2\| + \|\psi_2\|.$$

This completes the proof. □

*Remark* 3.1. In [13, 14], the authors only obtained the existence of positive solutions to systems of nonlinear Hammerstein type integral equations. However, the existence of multiple positive solutions is obtained here and the main method used in the proof is essentially different from the literature [13, 14].

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