FIXED POINT THEORY FOR VARIOUS CLASSES OF PERMISSIBLE MAPS VIA INDEX THEORY

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ABSTRACT. In this paper we use degree and index theory to present new applicable fixed point theory for permissible maps.

1. Introduction

In [16] we presented new fixed point theory for Urysohn type maps using an argument based on constructing multivalued maps $\mathcal{K}_n$. In this paper we remove the maps $\mathcal{K}_n$ and replace them with a more natural sequentially compact condition. In particular we use degree and index theory to obtain applicable fixed point theorems in Fréchet spaces. The proof relies on fixed point theory in Banach spaces and viewing a Fréchet space as the projective limit of a sequence of Banach spaces. Our final goal is to discuss permissible maps and to help us achieve this we first discuss various subclasses of this class, namely the J maps and more generally the admissible maps. One of the advantages of J maps is that no knowledge of homology theory is needed to construct the index. Our theory is partly motivated by the papers [1, 14, 15, 16].

Existence in Section 2 will rely on degree and index theory so we begin by discussing the maps we will consider in this paper. In this paper we consider maps with nonempty closed values. Let A be a compact subset of a metric space X. A is called ∞–proximally connected in X if for every $\epsilon > 0$ there is a $\delta > 0$ such that for any $n = 1, 2, \ldots$, and any map $g : \partial \triangle^n \to N_\delta(A)$ there exists a map $g' : \triangle^n \to N_\epsilon(A)$ such that $g(x) = g'(x)$ for $x \in \partial \triangle^n$; here $\triangle^n$ is the n–dimensional standard simplex and $N_\epsilon(A) = \{x \in X : \operatorname{dist}(x, A) < \epsilon\}$. Let X and Y be two metric spaces and $F : X \to 2^Y$. We say $F \in J(X, Y)$ if F is upper semicontinuous with nonempty, compact, ∞–proximally connected values. If Z is another metric space and $F \in J(X, Y)$ with $r : Z \to X$ continuous, then it is well known [11] that $F \circ r \in J(Z, Y)$. In this paper we will also discuss a special subclass of J maps, namely the Kututani maps. Let $F : X \to CK(Y)$;

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here $CK(Y)$ denotes the family of nonempty compact convex subsets of Y. We say $F : X \to CK(Y)$ is Kakutani if F is upper semicontinuous.

Let Ω be a bounded open subset of a Banach space E and assume $T : \overline{\Omega} \to 2^E$ is a Kakutani countably condensing map with $0 \notin (I - T)(\partial\Omega)$. Then [17, Chapter 2, 3, 11] guarantees that $\deg(I - T, \Omega, 0)$ is well defined and has the usual properties.

Next let Ω be an open subset of a Banach space E and assume $T \in J(\overline{\Omega}, E)$ is a compact map with $0 \notin (I - T)(\partial\Omega)$. Then [6, p. 4868] guarantees that $\deg(I - T, \Omega, 0)$ is well defined and has the usual properties. It is possible to extend the degree for countably condensing J maps (see [3]). Let E be a Banach space and Ω an open bounded subset of E. Also let $T \in J(\overline{\Omega}, E)$ be a countably condensing map with $0 \notin (I - T)(\partial\Omega)$. Let

$$A_1 = \overline{co}\,(T\,(\overline{\Omega})), \quad A_n = \overline{co}\,(T\,(\overline{\Omega} \cap A_{n-1}))$$

for $n = 2, 3, \ldots$, and

$$A_\infty = \bigcap_{n=1}^{\infty} A_n.$$

Fix a retraction $R : E \to A_\infty$. If $\Omega \cap A_\infty = \emptyset$, we let the degree of $I - T$ on Ω with respect to 0, denoted $\deg(I - T, \Omega, 0)$, be zero. If $\Omega \cap A_\infty \neq \emptyset$ we let

$$\deg(I - T, U, \Omega, 0) = \deg(I - T \circ R, R^{-1}(\Omega), 0),$$

where the right hand side is the Andres, Gabor, Gorniewicz degree.

Let C be a closed convex subset of a Banach space E and U an open bounded subset of E. Assume $T : \overline{W} \to 2^C$ is a Kakutani countably condensing map with $x \notin T\,x$ for $x \in \partial W$; here $W = U \cap C$ and in this situation $\overline{W}$ (respectively ∂W) denotes the closure of W in C (respectively the boundary of W in C). Then [3, 9, 18] guarantees that $\text{ind}(T, C, W)$ is well defined and has the usual properties.

It is possible to extend the index for countably condensing J maps (see [3]). Let C be a closed convex subset of a Banach space E and U an open bounded subset of E. Assume $T \in J(\overline{W}, C)$ is countably condensing map with $x \notin T\,x$ for $x \in \partial W$ where $W = U \cap C$. Let

$$A_1 = \overline{co}\,(T\,(\overline{W})), \quad A_n = \overline{co}\,(T\,(\overline{W} \cap A_{n-1}))$$

for $n = 2, 3, \ldots$, and

$$A_\infty = \bigcap_{n=1}^{\infty} A_n.$$

Fix a retraction $R : E \to A_\infty$. If $W \cap A_\infty = \emptyset$, we let $\text{ind}(T, C, W) = 0$. If $W \cap A_\infty \neq \emptyset$ we let

$$\text{ind}(T, C, W) = \deg(I - T \circ R, R^{-1}(U), 0),$$

where the right hand side is the Andres, Gabor, Gorniewicz degree (see [6]).

Also in Section 2 we discuss maps which arise from homology theory [9]. Let X and Y be metric spaces. A continuous single valued map $p : Y \to X$ is called a Vietoris map [10] if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic

(ii) p is a proper map, i.e., for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

Let $D(X, Y)$ be the set of all pairs $X \xleftarrow{p} Z \xrightarrow{q} Y$, where p is a Vietoris map, q is continuous and Z is a metric space.

Definition 1.1. A multifunction $\phi : X \to C(Y)$ is admissible, and we write $\phi \in Ad(X, Y)$, if $\phi : X \to C(Y)$ is upper semicontinuous, and if there exists a metric space Z and two continuous maps $p : Z \to X$ and $q : Z \to Y$ such that

(i) p is a Vietoris map, and

(ii) $\phi(x) = q(p^{-1}(x))$ for any $x \in X$;

here $C(Y)$ denotes the family of nonempty, compact subsets of Y.

Remark 1.1. (i) It should be noted that ϕ upper semicontinuous is redundant in Definition 1.1.

(ii) (p, q) is called a selected pair of ϕ and we write $(p, q) \subset \phi$.

Let E be a normed space. Let A and C be two subsets of E. A pair $A \xleftarrow{p} Z \xrightarrow{q} C$ is called a countably condensing pair from A to C if $\alpha \left(q \left(p^{-1}(\Omega) \right) \right) < \alpha(\Omega)$ for all countably bounded subsets Ω of A with $\alpha(\Omega) \neq 0$ (here α denotes the Kuratowski measure of noncompactness).

A pair (p, q) is called compact if q is compact. Let U be an open subset of a normed space E. By $\mathcal{K}(\overline{U}, E)$ we mean the family of compact pairs (p, q) from $\overline{U}$ to E for which $\text{Fix}(p, q) \cap \partial U = \emptyset$ (recall that a pair (p, q) is from $\overline{U}$ to E if there exists a metric space Z for which $\overline{U} \xleftarrow{p} Z \xrightarrow{q} E$); here $\text{Fix}(p, q) = \{x \in \overline{U} : x \in q(p^{-1}(x))\}$. In 1976 Kucharski [13], using the coincidence index in $\mathbb{R}^n$ and Schauder projections, defined the coincidence index on $\mathcal{K}(\overline{U}, E)$ and established the following result.

Theorem 1.1. *There exists a map* $I : \mathcal{K}(\overline{U}, E) \to \mathbb{Q}$ (*called the coincidence index* (*degree*)) *which satisfies the following properties*:

(I) *if* $I(p, q) \neq 0$, *then* $\text{Fix}(p, q) \neq \emptyset$;

(II) *if* $h : Z \times [0, 1] \to E$ *is a compact map such that* $\text{Fix}(p, h) \cap \partial U = \emptyset$, *then* $I(p, h_0) = I(p, h_1)$; *here* $h_0(y) = h(y, 0)$, $h_1(y) = h(y, 1)$ *and* $\text{Fix}(p, h) = \{x \in \overline{U} : x \in h(p^{-1}(x) \times \{t\})$ *for some* $t \in [0, 1]\}$.

Let E be a Banach space and U an open subset of E.

Definition 1.2. A pair $\overline{U} \xleftarrow{p} Z \xrightarrow{q} E$ is called a countably condensing pair from $\overline{U}$ to E if $\alpha \left(q \left(p^{-1}(\Omega) \right) \right) < \alpha(\Omega)$ for all countably bounded subsets Ω of $\overline{U}$ with $\alpha(\Omega) \neq 0$.

Definition 1.3. $(p, q) \in \mathcal{M}(\overline{U}, E)$ if $\overline{U}$ is bounded and (p, q) is a countably condensing pair from $\overline{U}$ to E with no fixed points on ∂U (i.e., $\text{Fix}(p, q) \cap \partial U = \emptyset$).

Now let $(p, q) \in \mathcal{M}(\overline{U}, E)$. We showed in [2, 14] that we can associate with each pair (p, q) a compact pair $(p, q^\star)$ with

$$\text{Fix}(p, q) = \text{Fix}(p, q^\star)$$

(here of course $\text{Fix}(p, q) = \{x \in \overline{U} : x \in q(p^{-1}(x))\}$) and that we could define (a well defined) coincidence index (degree) $I(p, q)$ as

$$I(p, q) = I(p, q^\star).$$

Theorem 1.2. *If* $(p, q) \in \mathcal{M}(\overline{U}, E)$ *and* $I(p, q) \neq 0$, *then* $\text{Fix}(p, q) \neq \emptyset$.

Remark 1.2. If the pair in Definition 1.3 was condensing (see [4]) instead of countably condensing then $\overline{U}$ bounded is not needed in the above argument (see [4]).

Let E be a Banach space, U an open bounded subset of E and let $\phi \in \text{Ad}(\overline{U}, E)$ be countably condensing with $\text{Fix}\phi \cap \partial U = \emptyset$; here $\text{Fix}\phi = \{x \in U : x \in \phi(x)\}$ and ϕ is called countably condensing if there exists a selected pair (p, q) of ϕ which is countably condensing. We define the coincidence index (degree) $I(\phi, U)$ by putting

$$I(\phi, U) = \{I(p, q) : (p, q) \subset \phi \text{ such that } (p, q) \text{ is } P\text{-concentrative}\} ;$$

note $\text{Fix}\phi = \text{Fix}(p, q)$.

If $I(\phi, U) \neq \{0\}$, then $\text{Fix}\phi \neq \emptyset$. To see this note if $I(\phi, U) \neq \{0\}$, then there exists a selected pair (p, q) of ϕ which is countably condensing with $I(p, q) \neq 0$. Then Theorem 1.5 guarantees that $\text{Fix}(p, q) \neq \emptyset$ and so $\text{Fix}\phi \neq \emptyset$.

Finally in Section 2 we discuss permissible maps. Let X and Y be Hausdorff topological spaces. We say $F : X \to 2^Y$ (here 2^Y denotes the family of nonempty subsets of Y) is locally compact if for every $x \in X$ there exists a neighborhood U of x such that the restriction $F|_U : U \to 2^Y$ is compact. Now if $F : X \to 2^X$ we let $F^n(x) = F(F^{n-1}(x))$.

Definition 1.4. Let $F : X \to 2^X$ be upper semicontinuous, $x \in X$ and $A \subseteq X$. We say A attracts x if for each neighborhood U of A there is a $n \in \{1, 2, \ldots\}$ with $F^n(x) \subseteq U$. Also we say A is an attractor for F if it attracts all points in X. Now we say the map F is of compact attraction if it has a compact attractor and is locally compact.

Definition 1.5. A multivalued map $F : X \to 2^Y$ is in the class $\mathcal{A}_m(X, Y)$ if (i) F is continuous, and (ii) for each $x \in X$ the set $F(x)$ consists of one or m acyclic components; here m is a positive integer. We say F is of class $\mathcal{A}_0(X, Y)$ if F is upper semicontinuous and for each $x \in X$ the set $F(x)$ is acyclic.

Definition 1.6. A decomposition $(F_1, \ldots, F_n)$ of a multivalued map $F : X \to 2^Y$ is a sequence of maps

$$X = X_0 \xrightarrow{F_1} X_1 \xrightarrow{F_2} X_2 \xrightarrow{F_3} \cdots \xrightarrow{F_{n-1}} X_{n-1} \xrightarrow{F_n} X_n = Y,$$

where $F_i \in \mathcal{A}_{m_i}(X_{i-1}, X_i)$, $F = F_n \circ \cdots \circ F_1$. One can say that the map F is determined by the decomposition $(F_1, \ldots, F_n)$. The number n is said to be the length of the decomposition $(F_1, \ldots, F_n)$. We will denote the class of decompositions by $\mathcal{D}(X, Y)$.

Definition 1.7. An upper semicontinuous map $F : X \to 2^Y$ is permissible provided it admits a selector $G : X \to 2^Y$ which is determined by a decomposition $(G_1, \ldots, G_n) \in \mathcal{D}(X, Y)$. We denote the class of permissible maps from X into Y by $\mathcal{P}(X, Y)$.

Let X be a closed convex subset of a normed space E and let $F : X \to 2^X$ be a permissible map which is of compact attraction. Let U be an open subset of X with $\mathrm{Fix} F \cap \partial U = \emptyset$. Then the index $i(X, F, U)$ is well defined (see [8, p. 42] or see [10, Sections 50–53]) and has the usual properties ([8, p. 43]).

Now let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha \in I}$ be a family of locally convex spaces. For each $\alpha \in I, \beta \in I$ for which $\alpha \leq \beta$ let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha \in I} E_\alpha : x_\alpha = \pi_{\alpha,\beta}(x_\beta) \forall \alpha, \beta \in I, \alpha \leq \beta \right\}$$

is a closed subset of $\prod_{\alpha \in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha \in I}$ and is denoted by $\lim_\leftarrow E_\alpha$ (or $\lim_\leftarrow \{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection $\cap_{\alpha \in I} E_\alpha$ [12, p. 439].)

2. Fixed point theory in Fréchet spaces

Let $E = (E, \{|\cdot|_n\}_{n \in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$; here $N = \{1, 2, \ldots\}$. We assume that the family of seminorms satisfies

$$(2.1) \qquad |x|_1 \leq |x|_2 \leq |x|_3 \leq \cdots \text{ for every } x \in E.$$

A subset X of E is bounded if for every $n \in N$ there exists $r_n > 0$ such that $|x|_n \leq r_n$ for all $x \in X$. For $r > 0$ and $x \in E$ we denote $B(x, r) = \{y \in E : |x - y|_n \leq r \forall n \in N\}$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$(2.2) \qquad x \sim_n y \text{ if and only if } |x - y|_n = 0.$$

We denote by $\mathbf{E}^n = (E/\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (2.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \geq n$ (again this seminorm is denoted by $|\cdot|_n$). Also (2.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m/\sim_n$ can be regarded as a subset

of $\mathbf{E}_n$. Now $\mu_{n,m}\mu_{m,k} = \mu_{n,k}$ if $n \le m \le k$ and $\mu_n = \mu_{n,m}\mu_m$ if $n \le m$. We now assume the following condition holds:

(2.3) $\begin{cases} \text{for each } n \in N, \text{ there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \to E_n. \end{cases}$

Remark 2.1. (i) For convenience the norm on E_n is denoted by $|\cdot|_n$.

(ii) In our applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.

(iii) Note if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$), then $x \in E$. However if $x \in E_n$, then x is not necessarily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example if $E = C[0, \infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0, n]$ and $E_n = C[0, n]$.

Finally we assume

(2.4) $\begin{cases} E_1 \supseteq E_2 \supseteq \cdots \text{ and for each } n \in N, \\ |j_n\mu_{n,n+1}j_{n+1}^{-1}x|_n \le |x|_{n+1} \forall x \in E_{n+1} \end{cases}$

(here we use the notation from [12], i.e., decreasing in the generalized sense). Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$, where $\cap_1^\infty$ is the generalized intersection [12]) denote the projective limit of $\{E_n\}_{n \in N}$ (note $\pi_{n,m} = j_n\mu_{n,m}j_m^{-1} : E_m \to E_n$ for $m \ge n$) and note $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$ we set $X_n = j_n\mu_n(X)$, and we let $\overline{X_n}$, $\mathrm{int}X_n$ and ∂X_n denote respectively the closure, the interior and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by

$$\mathrm{pseudo-int}(X) = \{x \in X : j_n\mu_n(x) \in \overline{X_n}\backslash\partial X_n \text{ for every } n \in N\}.$$

The set X is pseudo-open if $X = \mathrm{pseudo-int}(X)$. For $r > 0$ and $x \in E_n$ we denote $B_n(x, r) = \{y \in E_n : |x - y|_n \le r\}$.

We now show how easily one can extend fixed point theory in Banach spaces to applicable fixed point theory in Fréchet spaces. In this case the map F_n will be related to F by the closure property (2.10).

Theorem 2.1. *Let E and E_n be as described above, X a bounded subset of E and $F : Y \to 2^E$ where $\overline{\mathrm{int}X_n} \subseteq Y_n$ for each $n \in N$. Also for each $n \in N$ assume there exists $F_n : \overline{\mathrm{int}X_n} \to 2^{E_n}$ and suppose the following conditions are satisfied:*

(2.5) $\begin{cases} \text{for each } n \in N, F_n : \overline{\mathrm{int}X_n} \to CK(E_n) \text{ is a} \\ \text{upper semicontinuous countably condensing map,} \end{cases}$

(2.6)   *for each* $n \in N, 0 \notin (I - F_n)(\partial \mathrm{int}X_n)$,

(2.7)   *for each* $n \in N, \deg(I - F_n, \mathrm{int}X_n, 0) \ne 0$,

(2.8) $\begin{cases} \text{for each } n \in \{2, 3, \dots\} \text{ if } y \in \mathrm{int}X_n \text{ solves } y \in F_ny \text{ in } E_n, \\ \text{then } j_k\mu_{k,n}j_n^{-1}(y) \in \mathrm{int}X_k \text{ for } k \in \{1, \dots, n-1\}, \end{cases}$

(2.9)
$$\begin{cases} \textit{for any sequence } \{y_n\}_{n \in N} \textit{ with } y_n \in \mathrm{int} X_n \\ \textit{and } y_n \in F_n y_n \textit{ in } E_n \textit{ for } n \in N \textit{ and} \\ \textit{for every } k \in N \textit{ there exists a subsequence} \\ N_k \subseteq \{k+1, k+2, \ldots\}, N_k \subseteq N_{k-1} \textit{ for} \\ k \in \{1, 2, \ldots\}, \ N_0 = N, \textit{ and a } z_k \in \overline{\mathrm{int} X_k} \textit{ with} \\ j_k \mu_{k,n} j_n^{-1}(y_n) \to z_k \textit{ in } E_k \textit{ as } n \to \infty \textit{ in } N_k, \end{cases}$$

and

(2.10)
$$\begin{cases} \textit{if there exists a } w \in Y \textit{ and a sequence } \{y_n\}_{n \in N} \\ \textit{with } y_n \in \mathrm{int} X_n \textit{ and } y_n \in F_n y_n \textit{ in } E_n \textit{ such that} \\ \textit{for every } k \in N \textit{ there exists a subsequence } S \subseteq \\ \{k+1, k+2, \ldots\} \textit{ of } N \textit{ with } j_k \mu_{k,n} j_n^{-1}(y_n) \to j_k \mu_k(w) \\ \textit{in } E_k \textit{ as } n \to \infty \textit{ in } S, \textit{ then } w \in Fw \textit{ in } E. \end{cases}$$

Then F has a fixed point in E.

Remark 2.2. Notice to check (2.9) we need to show that for each $k \in N$ the sequence $\{j_k \mu_{k,n} j_n^{-1}(y_n)\}_{n \in N_{k-1}} \subseteq \overline{\mathrm{int} X_k}$ is sequentially compact.

Proof. For each $n \in N$ (note X_n is bounded since X is bounded for if $y \in X_n$, then there exists a $x \in X$ with $y = j_n \mu_n(x)$) there exists $y_n \in \mathrm{int} X_n$ with $y_n \in F_n y_n$ in E_n. Lets look at $\{y_n\}_{n \in N}$. Notice $y_1 \in \mathrm{int} X_1$ and $j_1 \mu_{1,k} j_k^{-1}(y_k) \in \mathrm{int} X_1$ for $k \in \{2, 3, \ldots\}$. Now (2.9) with $k = 1$ guarantees that there exists a subsequence $N_1 \subseteq \{2, 3, \ldots\}$ and a $z_1 \in \overline{\mathrm{int} X_1}$ with $j_1 \mu_{1,n} j_n^{-1}(y_n) \to z_1$ in E_1 as $n \to \infty$ in N_1. Look at $\{y_n\}_{n \in N_1}$. Now $j_2 \mu_{2,n} j_n^{-1}(y_n) \in \mathrm{int} X_2$ for $k \in N_1$. Now (2.9) with $k = 2$ guarantees that there exists a subsequence $N_2 \subseteq \{3, 4, \ldots\}$ of N_1 and a $z_2 \in \overline{\mathrm{int} X_2}$ with $j_2 \mu_{2,n} j_n^{-1}(y_n) \to z_2$ in E_2 as $n \to \infty$ in N_2. Note from (2.4) and the uniqueness of limits that $j_1 \mu_{1,2} j_2^{-1} z_2 = z_1$ in E_1 since $N_2 \subseteq N_1$ (note $j_1 \mu_{1,n} j_n^{-1}(y_n) = j_1 \mu_{1,2} j_2^{-1} j_2 \mu_{2,n} j_n^{-1}(y_n)$ for $n \in N_2$). Proceed inductively to obtain subsequences of integers

$$N_1 \supseteq N_2 \supseteq \cdots, N_k \subseteq \{k+1, k+2, \ldots\}$$

and $z_k \in \overline{\mathrm{int} X_k}$ with $j_k \mu_{k,n} j_n^{-1}(y_n) \to z_k$ in E_k as $n \to \infty$ in N_k. Note $j_k \mu_{k,k+1} j_{k+1}^{-1} z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \ldots\}$.

Fix $k \in N$. Note

$$\begin{aligned} z_k &= j_k \mu_{k,k+1} j_{k+1}^{-1} z_{k+1} = j_k \mu_{k,k+1} j_{k+1}^{-1} j_{k+1} \mu_{k+1,k+2} j_{k+2}^{-1} z_{k+2} \\ &= j_k \mu_{k,k+2} j_{k+2}^{-1} z_{k+2} = \cdots = j_k \mu_{k,m} j_m^{-1} z_m = \pi_{k,m} z_m \end{aligned}$$

for every $m \geq k$. We can do this for each $k \in N$. As a result $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $y \in Y$ since $z_k \in \overline{\mathrm{int} X_k} \subseteq Y_k$ for each $k \in N$. Also since $y_n \in F_n y_n$ in E_n for $n \in N_k$ and $j_k \mu_{k,n} j_n^{-1}(y_n) \to z_k = y$ in E_k as $n \to \infty$ in N_k we have from (2.10) that $y \in Fy$ in E. $\square$

Remark 2.3. From the proof we see that condition (2.8) can be removed from the statement of Theorem 2.1. We include it only to explain condition (2.9) (see Remark 2.2).

Remark 2.4. Note we could replace $\overline{\mathrm{int}X_n} \subseteq Y_n$ above with $\overline{\mathrm{int}X_n}$ a subset of the closure of Y_n in E_n if Y is a closed subset of E (so in this case we can take $Y = X$ if X is a closed subset of E). To see this note $z_k \in \overline{\mathrm{int}X_k}$, $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and $\pi_{k,m}(y_m) \to z_k$ in E_k as $m \to \infty$ and we can conclude that $y \in \overline{Y} = Y$ (note $q \in \overline{Y}$ if and only if for every $k \in N$ there exists $(x_{k,m}) \in Y$, $x_{k,m} = \pi_{k,n}(x_{n,m})$ for $n \geq k$ with $x_{k,m} \to j_k\mu_k(q)$ in E_k as $m \to \infty$).

Remark 2.5. Suppose in Theorem 2.1 we replace (2.9) with

$(2.9)^\star$
$$\begin{cases} \text{for any sequence } \{y_n\}_{n \in N} \text{ with } y_n \in \mathrm{int}X_n \\ \text{and } y_n \in F_n y_n \text{ in } E_n \text{ for } n \in N \text{ and} \\ \text{for every } k \in N \text{ there exists a subsequence} \\ N_k \subseteq \{k+1, k+2, \ldots\}, N_k \subseteq N_{k-1} \text{ for} \\ k \in \{1, 2, \ldots\}, N_0 = N, \text{ and a } z_k \in \mathrm{int}X_k \text{ with} \\ j_k\mu_{k,n}j_n^{-1}(y_n) \to z_k \text{ in } E_k \text{ as } n \to \infty \text{ in } N_k. \end{cases}$$

In addition we assume $F : Y \to 2^E$ with $\overline{\mathrm{int}X_n} \subseteq Y_n$ for each $n \in N$ is replaced by $F : X \to 2^E$ and suppose (2.10) is true with $w \in Y$ replaced by $w \in X$. Then the result in Theorem 2.1 is again true.

The proof follows the reasoning in Theorem 2.1 except in this case $z_k \in \mathrm{int}X_k$ and $y \in X$.

Remark 2.6. In fact we could replace (in fact we can remove it as mentioned in Remark 2.3) (2.8) in Theorem 2.1 with

$(2.8)^\star$
$$\begin{cases} \text{for each } n \in \{2, 3, \ldots\} \text{ if } y \in \mathrm{int}X_n \text{ solves } y \in F_n y \text{ in } E_n, \\ \text{then } j_k\mu_{k,n}j_n^{-1}(y) \in \overline{\mathrm{int}X_k} \text{ for } k \in \{1, \ldots, n-1\} \end{cases}$$

and the result above is again true.

Remark 2.7. Usually in our applications we have $\partial X_n = \partial \mathrm{int}X_n$ (so $\overline{X_n} = \overline{\mathrm{int}X_n}$). If X is a pseudo-open subset of E, then for each $n \in N$ we have X_n is a open subset of E_n so $\mathrm{int}X_n = X_n$. To see this note $X_n \subseteq \overline{X_n} \backslash \partial X_n$ since if $y \in X_n$, then there exists $x \in X$ with $y = j_n\mu_n(x)$ and this together with $X = \mathrm{pseudo-int}X$ yields $j_n\mu_n(x) \in \overline{X_n} \backslash \partial X_n$, i.e., $y \in \overline{X_n} \backslash \partial X_n$. In addition notice
$$\overline{X_n} \backslash \partial X_n = (\mathrm{int}X_n \cup \partial X_n) \backslash \partial X_n = \mathrm{int}X_n \backslash \partial X_n = \mathrm{int}X_n$$
since $\mathrm{int}X_n \cap \partial X_n = \emptyset$. Consequently
$$X_n \subseteq \overline{X_n} \backslash \partial X_n = \mathrm{int}X_n, \text{ so } X_n = \mathrm{int}X_n.$$

Note also if F_n is compact (or condensing) in (2.5), then the assumption that X is bounded can be removed in Theorem 2.1.

The result in Theorem 2.1 clearly extends to countably condensing J maps.

Theorem 2.2. *Let E and E_n be as described above, X a bounded subset of E and $F : Y \to 2^E$ where $\overline{\mathrm{int}X_n} \subseteq Y_n$ for each $n \in N$. Also for each $n \in N$*

assume there exists $F_n : \overline{\mathrm{int}\,X_n} \to 2^{E_n}$ *and suppose the following condition is satisfied*:

(2.11) $\qquad \begin{cases} \textit{for each } n \in N, F_n \in J(\overline{\mathrm{int}\,X_n}, E_n) \\ \textit{is a countably condensing map}. \end{cases}$

Also assume (2.6), (2.7), (2.8), (2.9) *and* (2.10) *hold. Then* F *has a fixed point in* E.

Next we present some results using fixed point index.

Theorem 2.3. *Let* E *and* E_n *be as described in the beginning of Section* 2, C *a convex subset in* E, U *a pseudo-open bounded subset of* E *and* $F : Y \to 2^E$ *with* $Y \subseteq E$, *and* $\overline{W_n} = \overline{U_n \cap \overline{C_n}} \subseteq Y_n$ *for each* $n \in N$ (*here* $W_n = U_n \cap \overline{C_n}$). *Also for each* $n \in N$ *assume there exists* $F_n : \overline{W_n} \to 2^{E_n}$ *and suppose the following conditions are satisfied*:

(2.12) $\qquad \begin{cases} \textit{for each } n \in \{2, 3, \ldots\} \textit{ if } y \in W_n \textit{ solves } y \in F_n y \textit{ in } E_n, \\ \textit{then } j_k \mu_{k,n} j_n^{-1}(y) \in W_k \textit{ for } k \in \{1, \ldots, n-1\}, \end{cases}$

(2.13) $\qquad \begin{cases} \textit{for each } n \in N, F_n \in J(\overline{W_n}, \overline{C_n}) \\ \textit{is a countably condensing map}, \end{cases}$

(2.14) $\qquad \begin{cases} \textit{for each } n \in N, x \notin F_n x \textit{ for } x \in \partial W_n \\ (\textit{here } \partial W_n \textit{ denotes the boundary of } W_n \textit{ in } \overline{C_n}), \end{cases}$

(2.15) $\qquad\qquad \textit{for each } n \in N, \mathrm{ind}(F_n, \overline{C_n}, W_n) \neq 0,$

(2.16) $\qquad \begin{cases} \textit{for any sequence } \{y_n\}_{n \in N} \textit{ with } y_n \in W_n \\ \textit{and } y_n \in F_n y_n \textit{ in } E_n \textit{ for } n \in N \textit{ and} \\ \textit{for every } k \in N \textit{ there exists a subsequence} \\ N_k \subseteq \{k+1, k+2, \ldots\}, N_k \subseteq N_{k-1} \textit{ for} \\ k \in \{1, 2, \ldots\}, N_0 = N, \textit{ and a } z_k \in \overline{W_k} \textit{ with} \\ j_k \mu_{k,n} j_n^{-1}(y_n) \to z_k \textit{ in } E_k \textit{ as } n \to \infty \textit{ in } N_k, \end{cases}$

and

(2.17) $\qquad \begin{cases} \textit{if there exists a } w \in Y \textit{ and a sequence } \{y_n\}_{n \in N} \\ \textit{with } y_n \in W_n \textit{ and } y_n \in F_n y_n \textit{ in } E_n \textit{ such that} \\ \textit{for every } k \in N \textit{ there exists a subsequence } S \subseteq \\ \{k+1, k+2, \ldots\} \textit{ of } N \textit{ with } j_k \mu_{k,n} j_n^{-1}(y_n) \to j_k \mu_k(w) \\ \textit{in } E_k \textit{ as } n \to \infty \textit{ in } S, \textit{ then } w \in Fw \textit{ in } E. \end{cases}$

Then F *has a fixed point in* E.

Proof. Fix $n \in N$. Note from Remark 2.7 that $U_n = \mathrm{int}\,U_n$. We now show

(2.18) $\qquad\qquad\qquad \overline{C_n}$ is convex.

To see this let $\hat{x}, \hat{y} \in \mu_n(C)$ and $\lambda \in [0,1]$. Then for every $x \in \mu_n^{-1}(\hat{x})$ and $y \in \mu_n^{-1}(\hat{y})$ we have $\lambda x + (1-\lambda)y \in C$ since C is convex and so $\lambda \hat{x} + (1-\lambda)\hat{y} =$

$\lambda\mu_n(x) + (1-\lambda)\mu_n(y)$. It is easy to check that $\lambda\mu_n(x) + (1-\lambda)\mu_n(y) = \mu_n(\lambda x + (1-\lambda)y)$ so as a result

$$\lambda\hat{x} + (1-\lambda)\hat{y} = \mu_n(\lambda x + (1-\lambda)y) \in \mu_n(C),$$

and so $\mu_n(C)$ is convex. Now since j_n is linear we have $C_n = j_n(\mu_n(C))$ is convex and as a result $\overline{C_n}$ is convex, so (2.18) holds.

Now there exists $y_n \in U_n \cap \overline{C_n}$ with $y_n \in F_n y_n$ in E_n. Essentially the same reasoning as in Theorem 2.1 establishes the result. $\qquad\square$

Remark 2.8. Condition (2.12) can be removed from the statement of Theorem 2.3.

Remark 2.9. In Theorem 2.3 it is possible to replace $\overline{\overline{C_n \cap U_n}} \subseteq Y_n$ with $\overline{C_n \cap U_n}$ a subset of the closure of Y_n in E_n provided Y is a closed subset of E so in this case we could have $Y = C \cap \overline{U}$ if $\overline{C_n \cap U_n}$ is a subset of the closure of $j_n\mu_n(C \cap \overline{U})$ in E_n and if C is closed.

Remark 2.10. Suppose in Theorem 2.3 we replace (2.16) with

$(2.16)^\star$ $\quad \begin{cases} \text{for any sequence } \{y_n\}_{n\in N} \text{ with } y_n \in W_n \\ \text{and } y_n \in F_n y_n \text{ in } E_n \text{ for } n \in N \text{ and} \\ \text{for every } k \in N \text{ there exists a subsequence} \\ N_k \subseteq \{k+1, k+2, \ldots\}, N_k \subseteq N_{k-1} \text{ for} \\ k \in \{1, 2\ldots\}, N_0 = N, \text{ and a } z_k \in W_k \text{ with} \\ j_k\mu_{k,n}j_n^{-1}(y_n) \to z_k \text{ in } E_k \text{ as } n \to \infty \text{ in } N_k. \end{cases}$

In addition we assume $F : Y \to 2^E$ with $\overline{W_n} \subseteq Y_n$ for each $n \in N$ is replaced by $F : Y \to 2^E$ with $W_n \subseteq Y_n$ for each $n \in N$. Then the result in Theorem 2.3 is again true.

We now use the index theory for admissible maps in Section 1 to obtain applicable fixed point theory in Fréchet spaces.

Theorem 2.4. *Let E and E_n be as described above, X a bounded subset of E and $F : Y \to 2^E$ where $\overline{\mathrm{int}X_n} \subseteq Y_n$ for each $n \in N$. Also for each $n \in N$ assume there exists $F_n : \overline{\mathrm{int}X_n} \to 2^{E_n}$ and suppose (2.8) and the following conditions are satisfied:*

(2.19) $\qquad$ *for each $n \in N, F_n \in \mathrm{Ad}(\overline{\mathrm{int}X_n}, E_n)$ is countably condensing,*

(2.20) $\qquad\qquad$ *for each $n \in N, \mathrm{Fix}F_n \cap \partial\mathrm{int}X_n = \emptyset,$*

and

(2.21) $\qquad\qquad$ *for each $n \in N, I(F_n, \mathrm{int}X_n) \neq \{0\}.$*

Also assume (2.9) and (2.10) hold. Then F has a fixed point in E.

Proof. For each $n \in N$ there exists (see Theorem 1.2) $y_n \in \mathrm{int}X_n$ with $y_n \in F_n y_n$. Now essentially the same reasoning as in Theorem 2.1 guarantees the result. $\qquad\square$

Remark 2.11. Note Remark 2.3, Remark 2.4, Remark 2.5 and Remark 2.6 hold in this situation also.

Finally we discuss permissible maps.

Theorem 2.5. *Let E and E_n be as described in the beginning of Section 2, C a convex subset in E, U a pseudo-open bounded subset of E and $F : Y \to 2^E$ with $Y \subseteq E$, and $\overline{C_n} \subseteq Y_n$ (or $\overline{W_n} = \overline{U_n \cap \overline{C_n}} \subseteq Y_n$) for each $n \in N$ (here $W_n = U_n \cap \overline{C_n}$). Also for each $n \in N$ assume there exists $F_n : \overline{C_n} \to 2^{E_n}$ and suppose (2.12) and the following conditions are satisfied:*

$$(2.22) \qquad \begin{cases} \text{for each } n \in N, F_n \in \mathcal{P}(\overline{C_n}, \overline{C_n}) \\ \text{is of compact attraction,} \end{cases}$$

$$(2.23) \qquad \begin{cases} \text{for each } n \in N, x \notin F_n x \text{ for } x \in \partial W_n \\ \text{(here } \partial W_n \text{ denotes the boundary of } W_n \text{ in } \overline{C_n}), \end{cases}$$

and

$$(2.24) \qquad \text{for each } n \in N, \operatorname{ind}(F_n, \overline{C_n}, W_n) \neq \{0\}$$

Also assume (2.16) and (2.17) hold. Then F has a fixed point in E.

Proof. Fix $n \in N$. Note from Remark 2.7 that $U_n = \operatorname{int} U_n$ and from Theorem 2.3 we know $\overline{C_n}$ is convex. Now there exists $y_n \in U_n \cap \overline{C_n}$ with $y_n \in F_n y_n$ in E_n. Essentially the same reasoning as in Theorem 2.1 establishes the result. □

Remark 2.12. Condition (2.12) can be removed from the statement of Theorem 2.5.

Remark 2.13. In Theorem 2.5 it is possible to replace $\overline{C_n} \subseteq Y_n$ (or $\overline{W_n} \subseteq Y_n$) with $\overline{C_n}$ a subset of the closure of Y_n in E_n (or $\overline{W_n}$ a subset of the closure of Y_n in E_n) provided Y is a closed subset of E.

Remark 2.14. Suppose in Theorem 2.5 we replace (2.16) with (2.16)*. In addition we assume $F : Y \to 2^E$ with $\overline{C_n} \subseteq Y_n$ (or $\overline{W_n} \subseteq Y_n$) for each $n \in N$ is replaced by $F : Y \to 2^E$ with $C_n \subseteq Y_n$ (or $W_n \subseteq Y_n$) for each $n \in N$. Then the result in Theorem 2.5 is again true.

Remark 2.15. One could write the result in Theorem 2.5 with $\mathcal{P}$ replaced by $J(\mathcal{A})^c$ using the index theory from [7] (this has the advantage that no knowledge of homology theory is needed to construct the index).

As an application of our fixed point results we conclude the paper by applying Theorem 2.1 to the integral inclusion

$$(2.25) \qquad y(t) \in \int_0^\infty K(t,s) F(s, y(s)) ds \text{ for } t \in [0, \infty).$$

Here $K : [0, \infty) \times [0, \infty) \to \mathbb{R}$ and $F : [0, \infty) \times \mathbb{R} \to CK(\mathbb{R})$ with $CK(\mathbb{R})$ denoting the family of nonempty, convex, compact subsets of $\mathbb{R}$.

Theorem 2.6. *Let $1 \leq p < \infty$ be a constant and $1 < q \leq \infty$ the conjugate to p. Suppose the following conditions are satisfied:*

(2.26)   *for each $t \in [0, \infty)$, the map $s \mapsto K(t, s)$ is measurable,*

(2.27) $$\sup_{t \in [0,\infty)} \left(\int_0^\infty |K(t,s)|^q ds \right)^{\frac{1}{q}} < \infty,$$

(2.28) $\begin{cases} \int_0^\infty |K(t',s) - K(t,s)|^q ds \to 0 \; as \; t \to t' \\ for \; each \; t' \in [0, \infty), \end{cases}$

(2.29) $\begin{cases} F : [0, \infty) \times \mathbb{R} \to CK(\mathbb{R}) \; is \; a \; L^p\text{-}Carathéodory \; function: \\ by \; this \; we \; mean \\ (a) \; for \; each \; measurable \; u : [0, \infty) \to \mathbb{R}, \; the \; map \\ x \mapsto F(x, u(x)) \; has \; measurable \; single \; valued \; selections \\ (b) \; for \; a.e. \; x \in [0, \infty), \; the \; map \; u \mapsto F(x, u) \; is \\ upper \; semicontinuous \\ (c) \; for \; each \; r > 0 \; there \; exists \; h_r \in L^p[0, \infty) \; with \\ |F(x, y)| \leq h_r(x) \; for \; a.e. \; x \in [0, \infty) \; and \; all \; y \in \mathbb{R} \\ with \; |y| \leq r; \; here \; |F(x, y)| = \sup\{|v| : v \in F(x, y)\}, \end{cases}$

(2.30) $\begin{cases} there \; exists \; a \; function \; \psi : [0, \infty) \to [0, \infty) \; continuous \\ and \; nonincreasing \; and \; a \; \phi \in L^p[0, \infty) \; with \\ |F(s, y)| \leq \phi(s) \psi(|y|) \; for \; all \; y \in \mathbb{R} \; and \\ a.e. \; s \in [0, \infty), \end{cases}$

and

(2.31) $\begin{cases} \exists r > 0 \; with \; r > K_1 \psi(r) \; where \\ K_1 = \sup_{t \in [0,\infty)} \int_0^\infty |K(t,s)| \phi(s) ds. \end{cases}$

Then (2.26) has at least one solution in $C[0, \infty)$.

Remark 2.16. Note $(2.29)(a)$ could be replaced by: the map $x \mapsto F(x, u)$ is measurable for all $u \in \mathbb{R}$.

Proof. Here $E = C[0, \infty)$, $\mathbf{E}^k$ consists of the class of functions in E which coincide on the interval $[0, k]$, $E_k = C[0, k]$ with of course $\pi_{n,m} = j_n \mu_{n,m} j_m^{-1} :$ $E_m \to E_n$ defined by $\pi_{n,m}(x) = x|_{[0,n]}$. We will apply Theorem 2.1 with

$$X = \{u \in C[0, \infty) : |u|_n \leq r \text{ for each } n \in N\} \,;$$

here $|u|_n = \sup_{t \in [0,n]} |u(t)|$. Fix $n \in N$ and note

$$X_n = \overline{X_n} = \{u \in C[0, n] : |u|_n \leq r\}$$

with

$$\text{int} X_n = \{u \in C[0, n] : |u|_n < r\}.$$

Let

$$F_n y(t) = \int_0^n K(t, s) F(s, y(s)) ds \text{ for } t \in [0, n]$$

and

$$Fy(t) = \int_0^\infty K(t,s)F(s,y(s))ds \text{ for } t \in [0,\infty).$$

Now let $K_n : L^p[0,n] \to C[0,n]$ and $\mathcal{F}_n : C[0,n] \to L^p[0,n]$ be given by

$$K_n y(t) = \int_0^n K(t,s)y(s)ds$$

and

$$\mathcal{F}_n(y) = \{u \in L^p[0,n] : u(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,n]\}.$$

Also let $Y = X$ (we will use Remark 2.4). Clearly (2.8) holds and a standard argument in the literature guarantees that

$$F_n = K_n \circ \mathcal{F}_n : \overline{\text{int}X_n} \to CK(E_n) \text{ is upper semicontinuous and compact,}$$

so (2.5) holds. To show (2.6) and (2.7) we fix $n \in N$ and show

(2.32) $$y \notin \lambda F_n y \text{ for } \lambda \in [0,1] \text{ and } y \in \partial \text{int} X_n.$$

If (2.32) holds, then clearly (2.6) is satisfied and (2.7) follows from the homotopy property of the degree. Now suppose there exists $y \in \partial \text{int} X_n$ (i.e., $|y|_n = r$) and $\lambda \in [0,1]$ with $y \in \lambda F_n y$. Since it is well known in the literature that $\mathcal{F}_n(y) \neq \emptyset$ there exists $v \in \mathcal{F}_n(y)$ with

$$y(t) = \lambda \int_0^n K(t,s)v(s)ds \text{ for } t \in [0,n].$$

In addition (2.30) guarantees that $|v(s)| \le \phi(s)\psi(|y(s)|)$ for a.e. $s \in [0,n]$, and so for $t \in [0,n]$ we have

$$|y(t)| \le \psi(|y|_n) \int_0^n |K(t,s)|\phi(s)ds \le \psi(|y|_n) K_1.$$

Consequently

$$|y|_n \le \psi(|y|_n) K_1,$$

so $r \le \psi(r)K_1$, a contradiction. To show (2.9) consider a sequence $\{y_n\}_{n \in N}$ with $y_n \in C[0,n]$, $y_n \in F_n y_n$ on $[0,n]$ and $|y_n|_n < r$. Now to show (2.9) we will show for a fixed $k \in N$ that $\{j_k \mu_{k,n} j_n^{-1}(y_n)\}_{n \in S} \subseteq \overline{\text{int}X_k}$ is sequentially compact for any subsequence S of $\{k, k+1, \dots\}$. Note for $n \in S$ that $j_k \mu_{k,n} j_n^{-1}(y_n) = y_n|_{[0,k]}$ so $\{j_k \mu_{k,n} j_n^{-1}(y_n)\}_{n \in S}$ is uniformly bounded since $|y_n|_n \le r$ for $n \in S$ implies $|y_n|_k \le r$ for $n \in S$. Also $\{j_k \mu_{k,n} j_n^{-1}(y_n)\}_{n \in S}$ is equicontinuous on $[0,k]$ since for $n \in S$ and $t, x \in [0,k]$ (note there exists $h_r \in L^p[0,\infty)$ with $|F(s,y_n(s))| \le h_r(s)$ for a.e. $s \in [0,n]$) we have

$$|j_k \mu_{k,n} j_n^{-1}(y_n(t)) - j_k \mu_{k,n} j_n^{-1}(y_n(x))|$$
$$\le \int_0^n |K(t,s) - K(x,s)|h_r(s)ds$$
$$\le \left(\int_0^\infty [h_r(s)]^p ds\right)^{\frac{1}{p}} \left(\int_0^\infty |K(t,s) - K(x,s)|^q ds\right)^{\frac{1}{q}}.$$

The Arzela-Ascoli theorem guarantees that $\{j_k \mu_{k,n} j_n^{-1}(y_n)\}_{n \in S} \subseteq \overline{\text{int} X_k}$ is sequentially compact, so (2.9) holds. Finally we show (2.10). Suppose there exists $w \in C[0, \infty)$ and a sequence $\{y_n\}_{n \in N}$ with $y_n \in \text{int} X_n$ and $y_n \in F_n y_n$ in $C[0, n]$ such that for every $k \in N$ there exists a subsequence $S \subseteq \{k+1, k+2, \ldots\}$ of N with $y_n \to w$ in $C[0, k]$ as $n \to \infty$ in S. If we show

$$w(t) \in \int_0^\infty K(t, s) F(s, w(s)) ds \text{ for } t \in [0, \infty),$$

then (2.10) will hold. The argument presented below follows closely that in [5]. Fix $t \in [0, \infty)$. Consider $k \geq t$ and $n \in S$ (as described above). Then $y_n(t) \in F_n y_n(t)$, $t \in [0, n]$, for $n \in S$. Now there exists $v_n \in \mathcal{F}_n(y_n)$ with

$$y_n(x) = \int_0^n K(x, s) v_n(s) ds \text{ for } x \in [0, n],$$

and so

$$(2.33) \qquad y_n(t) - \int_0^k K(t, s) v_n(s) ds = \int_k^n K(t, s) v_n(s) ds.$$

Now (2.29) guarantees that there exists a $h_r \in L^p[0, \infty)$ with $|v_n(s)| \leq h_r(s)$ for a.e. $s \in [0, n]$. Then

$$\left| y_n(t) - \int_0^k K(t, s) v_n(s) ds \right| \leq \int_k^n h_r(s) |K(t, s)| ds$$

and so

$$(2.34) \qquad \left| y_n(t) - \int_0^k K(t, s) v_n(s) ds \right| \leq \int_k^\infty h_r(s) |K(t, s)| ds.$$

Consider $\{v_n\}_{n \in S}$. A standard result from the literature guarantees that $\mathcal{F}_k : C[0, k] \to L^p[0, k]$ is upper semicontinuous with respect to the weak topology (w–u.s.c.) and also weakly completely continuous. Now since $v_n \in \mathcal{F}_k(y_n)$ for $n \in S$, there exists a $u_k \in L^p[0, k]$ and a subsequence of S (without loss of generality assume its S) with v_n converging weakly to u_k (i.e., $v_n \rightharpoonup u_k$ in $L^p[0, k]$) as $n \to \infty$ in S. Now $y_n \to w$ in $C[0, k]$ and $v_n \rightharpoonup u_k$ in $L^p[0, k]$ as $n \to \infty$ in S together with $v_n \in \mathcal{F}_k(y_n)$ for $n \in S$ and the fact that $\mathcal{F}_k : C[0, k] \to L^p[0, k]$ is w–u.s.c. guarantees

$$(2.35) \qquad\qquad\qquad u_k \in \mathcal{F}_k(w).$$

Note as well that $|w|_k \leq r$ since $|y_n|_k \leq r$ for $n \in S$, and also we have $|u_k(x)| \leq h_r(x)$ for a.e. $x \in [0, k]$. Let $n \to \infty$ through S in (2.34) to obtain

$$(2.36) \qquad \left| w(t) - \int_0^k K(t, s) u_k(s) ds \right| \leq \int_k^\infty h_r(s) |K(t, s)| ds.$$

Similarly we can show that there exists $u_{k+1} \in L^p[0, k+1]$ and a subsequence of S, say S_1, with $v_n \rightharpoonup u_{k+1}$ in $L^p[0, k+1]$ as $n \to \infty$ in S_1 and with $u_{k+1} \in \mathcal{F}_{k+1}(w)$. Of course this implies $v_n \rightharpoonup u_{k+1}$ in $L^p[0, k]$ as $n \to \infty$ in S_1

so $u_{k+1}(x) = u_k(x)$ for a.e. $x \in [0, k]$. In addition note $|u_{k+1}(x)| \leq h_r(x)$ for a.e. $x \in [0, k+1]$. Continue and construct $u_{k+2}, u_{k+3}, \ldots$. For $l \in \{k, k+1, \ldots\} \equiv P$ let $u_l^\star(x)$ be any extension to $[0, \infty)$ of u_l with $|u_l^\star(x)| \leq h_r(x)$ for a.e. $x \in (l, \infty)$. Also let

$$\mathcal{F}_l^\star(w) = \{v \in L^p[0, \infty) : v(x) \in F(x, w(x)) \text{ for a.e. } x \in [0, l],$$
$$|v(x)| \leq h_r(x) \text{ for a.e. } x \in [0, \infty)\}.$$

Now $\{u_l^\star\}_{l \in P}$ is a weakly compact sequence in $L^p[0, \infty)$ so there exists a subsequence which converges weakly to a function $u \in L^p[0, \infty)$. Note $u(x) = u_k(x)$ for a.e. $x \in [0, k]$ since $u_{k+m}(x) = u_k(x)$ for a.e. $x \in [0, k]$, here $m \in N_0$. This together with (2.36) yields

$$(2.37) \qquad \left| w(t) - \int_0^k K(t, s) u(s) ds \right| \leq \int_k^\infty h_r(s) |K(t, s)| ds.$$

Let

$$\mathcal{F}(w) = \{v \in L^p[0, \infty) : v(x) \in F(x, w(x)) \text{ for a.e. } x \in [0, \infty)\}$$

(note $|w|_k \leq r$ for each $k \in N_0$ so $w \in BC[0, \infty)$). We next <u>claim</u> that $\mathcal{F}(w) = \cap_{l \in N_0} \mathcal{F}_l^\star(w)$ (and $\mathcal{F}(w)$ is nonempty, closed and convex). Note first $\sup_{t \in [0, \infty)} |w(t)| \leq r$ so $|F(x, w(x))| \leq h_r(x)$ for a.e. $x \in [0, \infty)$. Let w_k be the restriction to the interval $[0, k]$, $k \in N_0$, of w. Note that

$$\mathcal{F}_k(w_k) = \{v \in L^p[0, k] : v(x) \in F(x, w_k(x)) \text{ for a.e. } x \in [0, k]\}$$

is closed in $L^p[0, k]$ for all $k \in N_0$. Let

$$\mathcal{F}_k^\star(w_k) = \{v \in L^p[0, \infty) : v \in \mathcal{F}_k(w_k) \text{ for } x \in [0, k]$$
$$\text{and } v(x) = 0 \text{ for } x > k\}.$$

It is immediate that $\mathcal{F}_k^\star(w_k)$ is a closed set in $L^p[0, \infty)$ for each $k \in N_0$. Let

$$R_k = \{v \in L^p[0, \infty) : v(x) = 0 \text{ for } x \in [0, k],$$
$$|v(x)| \leq h_r(x) \text{ for a.e. } x \in (k, \infty)\}$$

and notice it is clear that

$$\mathcal{F}_k^\star(w) = \mathcal{F}_k^\star(w_k) \oplus R_k.$$

It is immediate that $\mathcal{F}_k^\star(w)$ is a closed set in $L^p[0, \infty)$. Also for each $k \in N_0$ we have $\mathcal{F}(w) \subseteq \mathcal{F}_k^\star(w)$ and so

$$\mathcal{F}(w) \subseteq \bigcap_{l \in N_0} \mathcal{F}_l^\star(w).$$

On the other hand if $v \in \mathcal{F}_l^\star(w)$ for each $l \in N_0$ then $v(x) \in F(x, w(x))$ for a.e. $x \in [0, \infty)$ and so

$$\bigcap_{l \in N_0} \mathcal{F}_l^\star(w) \subseteq \mathcal{F}(w).$$

Thus $\mathcal{F}(w) = \cap_{l \in N_0} \mathcal{F}_l^\star(w)$ and also $\mathcal{F}(w)$ is a closed subset of $L^p[0, \infty)$. Thus our claim is established. Now since u belongs to $\cap_{l \in N_0} \mathcal{F}_l^\star(w)$ (note for each $l \in N_0$, $u \in \mathcal{F}_l^\star(w)$) we have $u \in \mathcal{F}(w)$. Let $k \to \infty$ in (2.37) to obtain

$$w(t) = \int_0^\infty K(t, s) u(s) ds$$

and so

$$w(t) \in \int_0^\infty K(t, s) F(s, w(s)) ds.$$

Thus (2.10) holds. Our result now follows from Theorem 2.1 (with Remark 2.4). $\qquad\square$

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