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CERTAIN TRINOMIAL EQUATIONS AND LACUNARY POLYNOMIALS

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Abstract. We estimate the positive real zeros of certain trinomial equations and then deduce zeros bounds of some lacunary polynomials.

1. Introduction and statement of results

Many of classical inequalities of analysis have been obtained from trinomial equations, and there have been a number of literatures about zero distributions of trinomial equations and lacunary polynomials. See, for example, [1], [2], [3] and [4]. In this paper, we investigate positive real zeros distributions of certain trinomial equations and, using this, we estimate zeros bounds for some lacunary polynomials. While studying these, we will need a new generalized upper bound of the exponential function: for $0 \le x < 1$ and $1 \le n \le 2$ we have

$$e^x \le U(n,x) = 1 - \frac{1}{n} + \frac{1}{n}\left(\frac{1+\left(1-\frac{1}{n}\right)x}{1-\frac{x}{n}}\right)^n \le \frac{1}{1-x}, \tag{1}$$

where $U(1,x) = \frac{1}{1-x}$. For the details about this, see [5]. The first result about trinomial equations follows from the lemma below that will be proved in Section 2.

Lemma 1. *Let n be an integer ≥ 4, and*

$$\frac{1}{2^n} < a \le \frac{1}{4(n-1)}. \tag{2}$$

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Suppose that the polynomial

$$\begin{aligned}P_{n,a}(x) = &\left(a^{\frac{1}{n-1}} - a^{\frac{1}{n}} + 8n - 4\right) x^4 - 4\left(a^{\frac{1}{n-1}} + 4n^2 - 2n - 1\right) x^3 \\ &+ 4\left(a^{\frac{1}{n-1}}(4n^2 - 8n + 5) + 8n^2 - 12n + 4\right) x^2 \\ &- 16\left(1 + 2a^{\frac{1}{n-1}}\right)(n-1)^2 x + 16a^{\frac{1}{n-1}}(n-1)^2\end{aligned}$$

has all its zeros real, namely $d_1 < d_2 < d_3 < d_4$. *Write* $c = a^{-\frac{1}{n}} - 1$,

$$e_1 = \frac{(c+1) - (c+1)^{1-\frac{n}{2}}\sqrt{(c+1)^n - 4(n-1)}}{2(n-1)}$$

and

$$e_2 = \frac{(c+1) + (c+1)^{1-\frac{n}{2}}\sqrt{(c+1)^n - 4(n-1)}}{2(n-1)}.$$

Then

(a) *if*

$$a^{1/n} < x < 1 - \frac{1}{e^{n-1}}$$

and

$$d_1 < x < d_2 \quad or \quad 1 - a^{1/n}e_2 < x < 1 - a^{1/n}e_1,$$

then $x^n - x^{n-1} < -a$, *and*

(b) *if*

$$a^{1/n} < x < 1 - \frac{1}{e^{n-1}}$$

and

$$x > 1 - a \quad or \quad x < \frac{a^{\frac{1}{n}}}{\left(a^{-\frac{1}{n}} - 1\right)^{\frac{1}{n-1}}},$$

then $x^n - x^{n-1} > -a$.

The following theorem is a consequence of Lemma 1.

Theorem 2. *With the same assumptions as in Lemma 1, the polynomial* $u(x) = x^n - x^{n-1} + a$ *has exactly two positive zeros on* $(0, 1)$ *which are not located in*

$$a^{1/n} < x < 1 - \frac{1}{e^{n-1}}, \quad d_1 < x < d_2, \quad 1 - a^{1/n}e_2 < x < 1 - a^{1/n}e_1,$$

and

$$a^{1/n} < x < 1 - \frac{1}{e^{n-1}}, \quad x > 1 - a, \quad x < \frac{a^{\frac{1}{n}}}{\left(a^{-\frac{1}{n}} - 1\right)^{\frac{1}{n-1}}}.$$

Using (a) of Lemma 1, we get zeros bounds for some lacunary polynomials in Theorem 3 below that will be also proved in Section 2. In the proof of Theorem 3, we will consider $u(x)$ in Theorem 2 as a polynomial of degree $n+1$ so that $u(x) = x^{n+1} - x^n + a$.

Theorem 3. *Let, for n positive integer,* $P(z) = a_n z^n + a_{n-k} z^{n-k} + \cdots + a_1 z + a_0 \in \mathbb{C}[z]$*, with* $a_n \neq 0$ *and* $k \geq 1$*, and*

$$H = \max_{0 \leq j \leq n-k} |a_j|.$$

Suppose that the polynomial $P_{n+1,a}(x)$ *(in Lemma 1) has all its zeros real, namely* $d_1' < d_2' < d_3' < d_4'$*. Suppose that* $c = \left(\frac{H}{|a_n|}\right)^{-\frac{1}{n+1}} - 1$,

$$e_1' = \frac{(c+1) - (c+1)^{1-\frac{n+1}{2}} \sqrt{(c+1)^{n+1} - 4n}}{2n},$$
$$e_2' = \frac{(c+1) + (c+1)^{1-\frac{n+1}{2}} \sqrt{(c+1)^{n+1} - 4n}}{2n}.$$

Then if

$$\frac{1}{2^{n+1}} < \left(\frac{H}{|a_n|}\right) \leq \frac{1}{4n},$$

then $P(z)$ *does not have a zero on*

$$d_1' < |z| < d_2',$$

and

$$1 - \left(\frac{H}{|a_n|}\right)^{\frac{1}{n+1}} e_2' < |z| < \min\left\{1, 1 - \left(\frac{H}{|a_n|}\right)^{\frac{1}{n+1}} e_1'\right\}.$$

2. Proofs and examples

In this section, we prove Lemma 1 and Theorem 3, and give some examples for Theorems 2 and 3. We first show Lemma 1.

Proof of Lemma 1. The polynomial $u(x) = x^n - x^{n-1} + a$ has the critical points 0 and $\frac{n-1}{n}$. Since $a \leq \frac{1}{4(n-1)} < \frac{(n-1)^{n-1}}{n^n}$, we have $u\left(\frac{n-1}{n}\right) = a - \frac{(n-1)^{n-1}}{n^n} < 0$. It follows from $u(0) = u(1) = a > 0$ that $u(x)$ has exactly two positive zeros on $(0, 1)$. Also $u(a^{1/n}) = 2a - a^{(n-1)/n} > 0$ since $a > 1/2^n$, and

$$a^{1/n} < \frac{(n-1)^{\frac{n-1}{n}}}{n} < \frac{n-1}{n}.$$

So the positive zeros of $u(x)$ are greater than $a^{1/n}$, and so we assume that

$$a^{1/n} < x < 1.$$

We first want to find x such that

$$x^{n-1} > \frac{a}{1-x}$$

to prove the case (a). This is equivalent to

$$(n-1)\log x > \log a + \log \frac{1}{1-x},$$

and

$$x > a^{\frac{1}{n-1}} \exp\left\{\frac{1}{n-1}\log\frac{1}{1-x}\right\}. \tag{3}$$

By (1),

$$e^x \le U(2,x)\left(\le \frac{1}{1-x}\right) \qquad (0 \le x < 1).$$

Since $\log\frac{1}{1-x} < n-1$, i.e., $(a^{1/n} <) x < 1 - \frac{1}{e^{n-1}}$, the inequality (3) is fulfilled if

$$x > \frac{a^{\frac{1}{n-1}}\left(4(n-1)^2 + \log^2\frac{1}{1-x}\right)}{\left(2n-2-\log\frac{1}{1-x}\right)^2}.$$

Since $2n-2 > \log\frac{1}{1-x}$, the above is again satisfied if

$$x > \frac{a^{\frac{1}{n-1}}\left(4(n-1)^2 + \left(x + \frac{x^2}{2(1-x)}\right)^2\right)}{\left(2n-2-x-\frac{x^2}{2(1-x)}\right)^2} \tag{4}$$

since

$$\log\frac{1}{1-x} < x + \frac{x^2}{2(1-x)}.$$

Multiply both numerator and denominator of the right side of (4) by $4(1-x)^2$ so that we get

$$\begin{aligned} x^5 &- \left(a^{\frac{1}{n-1}} + 8n - 4\right)x^4 - 4\left(a^{\frac{1}{n-1}} + 4n^2 - 2n - 1\right)x^3 \\ &+ 4\left(a^{\frac{1}{n-1}}(4n^2 - 8n + 5) + 8n^2 - 12n + 4\right)x^2 \\ &- 16\left(1 + 2a^{\frac{1}{n-1}}\right)(n-1)^2 x + 16a^{\frac{1}{n-1}}(n-1)^2 > 0. \end{aligned}$$

Since $x > a^{1/n}$, we have $x^5 > x^4 \cdot a^{1/n}$. So the above is fulfilled if

$$P_{n,a}(x) < 0, \tag{5}$$

where $P_{n,a}(x)$ was given in the statement of this lemma. If $P_{n,a}(x)$ has all its zeros real, namely $d_1 < d_2 < d_3 < d_4$, (5) holds when

$$d_1 < x < d_2 \qquad \text{and} \qquad a^{1/n} < x < 1 - \frac{1}{e^{n-1}}.$$

Suppose that $u(x) < 0$ for x real and $a^{1/n} < x < 1 - \frac{1}{e^{n-1}}$, i.e.,

$$x^n - x^{n-1} < -a. \tag{6}$$

Let

$$b = a^{-\frac{1}{n}} \quad \text{and} \quad x = a^{\frac{1}{n}}y.$$

Since $\frac{1}{2^n} < a < \frac{1}{4(n-1)}$, we have

$$(4(n-1))^{1/n} < b < 2.$$

But $1 < \frac{n}{(n-1)^{\frac{n-1}{n}}} \le (4(n-1))^{1/n}$, and so

$$1 < b < 2.$$

Now (6) becomes

$$y^n - by^{n-1} < -1, \tag{7}$$

and since $a^{1/n} < x < 1 - \frac{1}{e^{n-1}}$, we get

$$1 < y < a^{-\frac{1}{n}} \left(1 - \frac{1}{e^{n-1}}\right).$$

Let $b = c + 1$ (and so $0 < c < 1$). Put $y = 1 - z$. Then

$$1 - a^{-1/n} \left(1 - \frac{1}{e^{n-1}}\right) < z < 0,$$

and since $c = a^{-1/n} - 1$,

$$0 < a^{-1/n} \frac{1}{e^{n-1}} = 1 - a^{-1/n} \left(1 - \frac{1}{e^{n-1}}\right) + (a^{-1/n} - 1) < z + c < c.$$

By (7) we have

$$(1 - z)^{n-1}(z + c) > 1. \tag{8}$$

Put $t = z + c > 0$ in (8). Then $0 < t < c$ and

$$t(c + 1 - t)^{n-1} > 1.$$

This is satisfied if

$$t\left((c + 1)^{n-1} - t(c + 1)^{n-2}(n - 1)\right) > 1. \tag{9}$$

In fact, this follows from the inequality

$$\left(1 - \frac{t}{c + 1}\right)^{n-1} > 1 - (n - 1)\frac{t}{c + 1},$$

and so

$$(c + 1 - t)^{n-1} > (c + 1)^{n-1} \left(1 - (n - 1)\frac{t}{c + 1}\right)$$
$$=(c + 1)^{n-1} - t(n - 1)(c + 1)^{n-2}.$$

Solving the inequality (9) in $t = z + c$ gives

$$e_1 < t < e_2,$$

where

$$e_1 = \frac{(c + 1) - (c + 1)^{1-\frac{n}{2}} \sqrt{(c + 1)^n - 4(n - 1)}}{2(n - 1)}$$

and

$$e_2 = \frac{(c + 1) + (c + 1)^{1-\frac{n}{2}} \sqrt{(c + 1)^n - 4(n - 1)}}{2(n - 1)}.$$

The above e_1 and e_2 are real because $(c+1)^n = a^{-1} \geq 4(n-1)$. Hence

$$e_1 - c < z < e_2 - c.$$

The corresponding bounds for $x = a^{1/n}(1-z)$ are

$$1 - a^{1/n}e_2 < x < 1 - a^{1/n}e_1.$$

We now turn to the case (b). Starting with $x^n - x^{n-1} > -a$ we get

$$(1-z)^{n-1}(z+c) < 1 \tag{10}$$

instead of (8) by using same method above. This inequality holds if one of the following two conditions is satisfied:

$$a^{-\frac{n-1}{n}}(z+c) < 1, \text{ i.e., } z < a^{\frac{n-1}{n}} - a^{-\frac{1}{n}} + 1, \tag{11}$$

$$(1-z)^{n-1}c < 1, \text{ i.e., } z > 1 - \frac{1}{\left(a^{-\frac{1}{n}} - 1\right)^{\frac{1}{n-1}}}. \tag{12}$$

In fact, if (11) holds, then $z + c < a^{\frac{n-1}{n}}$ and so

$$(1-z)^{n-1}(z+c) = y^{n-1}(z+c) = a^{-\frac{n-1}{n}}x^{n-1}(z+c) < a^{-\frac{n-1}{n}}a^{\frac{n-1}{n}} = 1.$$

Also if (12) holds, then $(1-z)^{n-1} < 1/c$ and so

$$(1-z)^{n-1}(z+c) < 1/c(z+c) = z/c + 1 < 1.$$

The corresponding bounds for $x = a^{1/n}(1-z)$ are either

$$x > 1 - a \quad \text{or} \quad x < \frac{a^{\frac{1}{n}}}{\left(a^{-\frac{1}{n}} - 1\right)^{\frac{1}{n-1}}},$$

which completes the proof of (b). □

Theorem 2 is immediately obtained from Lemma 1. The following is an example about Theorem 2.

Example 4. It follows from Theorem 2 that $u(x) = x^8 - x^7 + 0.004$ has the positive real zeros on $(0.50191\cdots, 0.51103\cdots) \cup (0.855046\cdots, 0.86126) \cup (0.995881\cdots, 0.999088)$. Now the actual positive real zeros are $0.501982\cdots$ and $0.995883\cdots$.

For the proof of Theorem 3, we will use (a) of Lemma 1.

Proof of Theorem 3. We checked $x^{n+1} - x^n + a = 0$, where $\frac{1}{2^{n+1}} < a \leq \frac{1}{4n}$, has two positive zeros on $(0,1)$ in the proof of Lemma 1 (Here we replace n by

$n+1$). For $|z|<1$,

$$\begin{aligned}|P(z)| &\geq |a_n||z|^n - \left(|a_{n-k}||z|^{n-k} + |a_{n-k-1}||z|^{n-k-1} + \cdots + |a_0|\right) \\ &\geq |a_n||z|^n - H\left(\frac{1-|z|^{n-k+1}}{1-|z|}\right) \\ &> |a_n||z|^n - H\left(\frac{1}{1-|z|}\right) \\ &= |a_n|\frac{-\left(|z|^{n+1} - |z|^n + \frac{H}{|a_n|}\right)}{1-|z|} \\ &> 0\end{aligned}$$

The (a) in Lemma 1 completes the proof. □

We end this paper by giving an example of Theorem 3.

Example 5. For the polynomial $P(z) = z^7 + 0.004z^6 + 0.004z^5 + 0.004z^4 + 0.004z^3 + 0.004z^2 + 0.004z + 0.004$, our result asserts that $P(z)$ does not have a zero on

$$0.51103\cdots \leq |z| \leq 0.85504\cdots,$$
$$0.86126\cdots \leq |z| \leq 0.99588\cdots$$

using computer algebra. Actual zeros of $P(z)$ are located on

$$0.43185\cdots \leq |z| \leq 0.48696\cdots.$$

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