

## CONVERGENCE THEOREMS FOR SET-VALUED DENJOY-PETTIS INTEGRABLE MAPPINGS

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ABSTRACT. In this paper, we introduce the Denjoy-Pettis integral of set-valued mappings and investigate some properties of the set-valued Denjoy-Pettis integral. Finally we obtain the Dominated Convergence Theorem and Monotone Convergence Theorem for set-valued Denjoy-Pettis integrable mappings.

### 1. Introduction

Saks [11] introduced the Denjoy integral of real-valued functions which is a natural extension of the Lebesgue integral. Gordon [8] introduced the Denjoy-Pettis integral of Banach-valued functions in terms of the Denjoy integral which is the Denjoy extension of Pettis integral. The notion of integral of set-valued mappings is very useful in many branches of mathematics like mathematical economics, control theory, convex analysis, etc. It has been introduced by many authors and in different ways. Several types of integrals of set-valued mappings were introduced and studied by Aumann [1], Cascales and Rodriguez [2], Debreu [3], Di Piazza and Musial [4, 5], El Amri and Hess [6], Papageoriou [10] and others.

In this paper, we introduce the Denjoy-Pettis integral of set-valued mappings and investigate some properties of the set-valued Denjoy-Pettis integral. Finally we obtain two convergence theorems for set-valued Denjoy-Pettis integrable mappings.

### 2. Preliminaries

Throughout this paper,  $\mathcal{L}$  denotes the family of all Lebesgue measurable subsets of  $[a, b]$  and  $X$  a Banach space with dual  $X^*$ . The closed unit ball of  $X^*$  is denoted by  $B_{X^*}$ .  $CL(X)$  denotes the family of all nonempty closed subsets of  $X$ ,  $C(X)$  the family of all nonempty closed convex subsets of  $X$ ,  $CB(X)$  the family of all nonempty closed bounded convex subsets of  $X$ ,  $CWK(X)$  the family of all nonempty convex weakly compact subsets of  $X$  and  $CK(X)$

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the family of all nonempty convex compact subsets of  $X$ . Note that if  $X$  is reflexive, then  $CWK(X) = CB(X)$ .

For  $A \subseteq X$  and  $x^* \in X^*$ , let  $s(x^*, A) = \sup\{x^*(x) : x \in A\}$ , the support function of  $A$ .

For  $A, B \in CL(X)$ , let  $H(A, B)$  denote the Hausdorff metric of  $A$  and  $B$  defined by

$$H(A, B) = \max \left( \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right),$$

where  $d(a, B) = \inf_{b \in B} \|a - b\|$  and  $d(b, A) = \inf_{a \in A} \|a - b\|$ . Especially,

$$H(A, B) = \sup_{\|x^*\| \leq 1} |s(x^*, A) - s(x^*, B)|$$

whenever  $A, B$  are convex sets. The number  $\|A\|$  is defined by  $\|A\| = H(A, \{0\}) = \sup_{x \in A} \|x\|$ . If  $A \in CB(X)$  and  $x_1^*, x_2^* \in X^*$ , then

$$|s(x_1^*, A) - s(x_2^*, A)| \leq \|x_1^* - x_2^*\| \|A\|.$$

Note that  $(CWK(X), H)$  is a complete metric space.

The mapping  $F : [a, b] \rightarrow CL(X)$  is called a *set-valued mapping*.  $F$  is said to be *scalarly measurable* if for every  $x^* \in X^*$ , the real-valued function  $s(x^*, F(\cdot))$  is measurable.  $F$  is said to be *measurable* if  $F^{-1}(A) = \{t \in [a, b] : F(t) \cap A \neq \emptyset\} \in \mathcal{L}$  for every  $A \in CL(X)$ .

Let  $F : [a, b] \rightarrow CL(X)$ . Then the following statements are equivalent [10]:

- (1)  $F : [a, b] \rightarrow CL(X)$  is measurable;
- (2)  $F^{-1}(U) = \{t \in [a, b] : F(t) \cap U \neq \emptyset\} \in \mathcal{L}$  for every open subset  $U$  of  $X$ ;
- (3) (Castaing representation) there exists a sequence  $\{f_n\}$  of measurable functions  $f_n : [a, b] \rightarrow X$  such that  $F(t) = cl\{f_n(t)\}$  for all  $t \in [a, b]$ .

If  $F : [a, b] \rightarrow CL(X)$  is measurable then  $F : [a, b] \rightarrow CL(X)$  is scalarly measurable. Let  $X$  be a separable Banach space. Then  $F : [a, b] \rightarrow CWK(X)$  is measurable if and only if  $F : [a, b] \rightarrow CWK(X)$  is scalarly measurable [6].

$f : [a, b] \rightarrow X$  is called a *selection* of  $F : [a, b] \rightarrow CL(X)$  if  $f(t) \in F(t)$  for all  $t \in [a, b]$ .

**Definition 2.1** ([8]). Let  $F : [a, b] \rightarrow X$  and let  $E \subseteq [a, b]$ .

(a) The function  $F$  is said to be *absolutely continuous* on  $E$  ( $F$  is AC on  $E$ ) if for each  $\epsilon > 0$  there exists  $\delta > 0$  such that  $\sum_{i=1}^n \|F(d_i) - F(c_i)\| < \epsilon$  whenever  $\{[c_i, d_i] : 1 \leq i \leq n\}$  is a finite collection of non-overlapping intervals that have endpoints in  $E$  and satisfy  $\sum_{i=1}^n (d_i - c_i) < \delta$ .

(b) The function  $F$  is said to be *generalized absolutely continuous* on  $E$  ( $F$  is ACG on  $E$ ) if  $F$  is continuous on  $E$  and  $E$  can be expressed as a countable union of sets on each of which  $F$  is AC.

**Definition 2.2** ([8]). Let  $F : [a, b] \rightarrow X$  and let  $t \in (a, b)$ . A vector  $z$  in  $X$  is called the *approximate derivative* of  $F$  at  $t$  if there exists a measurable set  $E \subseteq [a, b]$  that has  $t$  as a point of density such that  $\lim_{\substack{s \rightarrow t \\ s \in E}} \frac{F(s) - F(t)}{s - t} = z$ . We will write  $F'_{ap}(t) = z$ .

A function  $f : [a, b] \rightarrow \mathbb{R}$  is said to be *Denjoy integrable* on  $[a, b]$  if there exists an ACG function  $F : [a, b] \rightarrow \mathbb{R}$  such that  $F'_{ap} = f$  almost everywhere on  $[a, b]$ . In this case, we write  $\int_a^b f(t)dt = F(b) - F(a)$ .

The function  $f$  is Denjoy integrable on a set  $A \subseteq [a, b]$  if  $f\chi_A$  is Denjoy integrable on  $[a, b]$ . In this case, we write  $\int_A f(t)dt = \int_a^b f\chi_A(t)dt$ .

**Definition 2.3** ([8]). A function  $f : [a, b] \rightarrow X$  is said to be *Denjoy-Pettis integrable* on  $[a, b]$  if for each  $x^* \in X^*$  the function  $x^*f$  is Denjoy integrable on  $[a, b]$  and if for every subinterval  $[c, d]$  of  $[a, b]$  there exists a vector  $x_{[c,d]} \in X$  such that  $x^*(x_{[c,d]}) = \int_c^d x^*f(t)dt$  for all  $x^* \in X^*$ . In this case, the vector  $x_{[a,b]}$  is called the *Denjoy-Pettis integral* of  $f$  on  $[a, b]$  and is denoted by  $(DP) \int_a^b f(t)dt$ .

The function  $f : [a, b] \rightarrow X$  is Denjoy-Pettis integrable on a set  $A \subseteq [a, b]$  if the function  $f\chi_A$  is Denjoy-Pettis integrable on  $[a, b]$ . In this case, we write  $(DP) \int_A f(t)dt = (DP) \int_a^b f\chi_A(t)dt$ .

**Theorem 2.4** ([13]). *Let  $A \in C(X)$ . Then the support function  $s(\cdot, A)$*

- (1) *is positively homogeneous, i.e.,  $s(\lambda x^*, A) = \lambda s(x^*, A)$  for all  $\lambda \geq 0$  and  $x^* \in X^*$ ;*
- (2) *is a convex function on  $X^*$ ;*
- (3) *is weak\* lower semi-continuous on  $X^*$ .*

*Conversely, if a function  $\varphi : X^* \rightarrow [-\infty, +\infty]$  satisfies the conditions (1)-(3), then there exists an  $A \in C(X)$  such that  $\varphi(x^*) = s(x^*, A)$  for each  $x^* \in X^*$ . The set  $A$  is unique and given by  $A = \{x \in X : x^*(x) \leq \varphi(x^*) \text{ for all } x^* \in X^*\}$ .*

**Theorem 2.5** ([13]). *If  $A_n \in CWK(X)$  for each  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} s(x^*, A_n)$  exists for each  $x^* \in X^*$ , then there exists an  $M > 0$  such that  $\sup_{n \in \mathbb{N}} \|A_n\| \leq M$ .*

### 3. Results

In this section, we introduce the Denjoy-Pettis integral of set-valued mappings and obtain some properties of the Denjoy-Pettis integral.

**Definition 3.1.** A set-valued mapping  $F : [a, b] \rightarrow CWK(X)$  is said to be *Denjoy-Pettis integrable* on  $[a, b]$  if for each  $x^* \in X^*$   $s(x^*, F(\cdot))$  is Denjoy integrable on  $[a, b]$  and for every subinterval  $[c, d]$  of  $[a, b]$  there exists  $W_{[c,d]} \in CWK(X)$  such that

$$(1) \quad s(x^*, W_{[c,d]}) = \int_c^d s(x^*, F(t))dt$$

for each  $x^* \in X^*$ . We write  $W_{[c,d]} = (DP) \int_c^d F(t)dt$ .

Note that when a set-valued mapping is a function  $f : [a, b] \rightarrow X$ , then the set  $W_{[c,d]}$  is reduced to vector in  $X$  and the equality (1) turns into

$$x^*(W_{[c,d]}) = \int_c^d x^* f(t) dt$$

for each  $x^* \in X^*$ . In this case, the function  $f : [a, b] \rightarrow X$  is Denjoy-Pettis integrable on  $[a, b]$ .

The following two theorems follow easily from the properties of the Denjoy integral and the support function.

**Theorem 3.2.** *Let  $F : [a, b] \rightarrow CWK(X)$  and  $c \in (a, b)$ . Then*

- (1) *if  $F$  is Denjoy-Pettis integrable on  $[a, b]$ , then  $F$  is Denjoy-Pettis integrable on every subinterval of  $[a, b]$ ;*
- (2) *if  $F$  is Denjoy-Pettis integrable on  $[a, c]$  and  $[c, b]$ , then  $F$  is Denjoy-Pettis integrable on  $[a, b]$  and*

$$(DP) \int_a^b F(t) dt = (DP) \int_a^c F(t) dt + (DP) \int_c^b F(t) dt.$$

**Theorem 3.3.** *Let  $F : [a, b] \rightarrow CWK(X)$  and  $G : [a, b] \rightarrow CWK(X)$  be Denjoy-Pettis integrable on  $[a, b]$  and  $\lambda \geq 0$ . Then*

- (1)  *$F + G$  is Denjoy-Pettis integrable on  $[a, b]$  and*

$$(DP) \int_a^b \{F(t) + G(t)\} dt = (DP) \int_a^b F(t) dt + (DP) \int_a^b G(t) dt;$$

- (2)  *$\lambda F$  is Denjoy-Pettis integrable on  $[a, b]$  and*

$$(DP) \int_a^b \lambda F(t) dt = \lambda (DP) \int_a^b F(t) dt.$$

**Theorem 3.4.** *Let  $f : [a, b] \rightarrow X$  be Denjoy-Pettis integrable on  $[a, b]$  and  $F : [a, b] \rightarrow CWK(X)$  and  $G : [a, b] \rightarrow CWK(X)$  be Denjoy-Pettis integrable on  $[a, b]$ . Then*

- (1) *if  $f(t) \in F(t)$  a.e., then*

$$(DP) \int_a^b f(t) dt \in (DP) \int_a^b F(t) dt;$$

- (2) *if  $F(t) \subseteq G(t)$  a.e., then  $(DP) \int_a^b F(t) dt \subseteq (DP) \int_a^b G(t) dt$ ;*
- (3) *if  $F(t) = G(t)$  a.e., then  $(DP) \int_a^b F(t) dt = (DP) \int_a^b G(t) dt$ .*

*Proof.* (1) Since  $f : [a, b] \rightarrow X$  is Denjoy-Pettis integrable on  $[a, b]$ , for each  $x^* \in X^*$   $x^* f$  is Denjoy integrable on  $[a, b]$  and

$$\int_a^b x^* f(t) dt = x^* ((DP) \int_a^b f(t) dt).$$

If  $f(t) \in F(t)$  a.e., then for each  $x^* \in X^*$

$$\begin{aligned} x^*((DP) \int_a^b f(t)dt) &= \int_a^b x^* f(t)dt \\ &\leq \int_a^b s(x^*, F(t))dt \\ &= s(x^*, (DP) \int_a^b F(t)dt). \end{aligned}$$

Since  $(DP) \int_a^b F(t)dt \in CWK(X)$ , by the separation theorem

$$(DP) \int_a^b f(t)dt \in (DP) \int_a^b F(t)dt.$$

(2) If  $F(t) \subseteq G(t)$  a.e., then for each  $x^* \in X^*$

$$\int_a^b s(x^*, F(t))dt \leq \int_a^b s(x^*, G(t))dt$$

and so

$$s(x^*, (DP) \int_a^b F(t)dt) \leq s(x^*, (DP) \int_a^b G(t)dt).$$

Since  $(DP) \int_a^b F(t)dt, (DP) \int_a^b G(t)dt \in CWK(X)$ , by the separation theorem

$$(DP) \int_a^b F(t)dt \subseteq (DP) \int_a^b G(t)dt.$$

(3) The proof is similar to (1). □

**Theorem 3.5.** *Let  $X$  be a separable Banach space. If  $F : [a, b] \rightarrow CWK(X)$  is Denjoy-Pettis integrable on  $[a, b]$ , then  $F : [a, b] \rightarrow CWK(X)$  is measurable on  $[a, b]$ .*

*Proof.* If  $F : [a, b] \rightarrow CWK(X)$  is Denjoy-Pettis integrable on  $[a, b]$ , then  $s(x^*, F(\cdot))$  is Denjoy integrable on  $[a, b]$  and so  $s(x^*, F(\cdot))$  is measurable on  $[a, b]$  for each  $x^* \in X^*$  by [9, Theorem 15.8]. Thus  $F : [a, b] \rightarrow CWK(X)$  is scalarly measurable on  $[a, b]$ . Hence  $F : [a, b] \rightarrow CWK(X)$  is measurable on  $[a, b]$ . □

$F : [a, b] \rightarrow CL(X)$  is said to be *Denjoy integrably bounded* on  $[a, b]$  if there exists a Denjoy integrable real-valued function  $h$  on  $[a, b]$  such that for each  $t \in [a, b]$ ,  $\|x\| \leq h(t)$  for all  $x \in F(t)$ .

**Theorem 3.6.** *Let  $X$  be a separable Banach space. If  $F : [a, b] \rightarrow CWK(X)$  and  $G : [a, b] \rightarrow CWK(X)$  are Denjoy integrably bounded and Denjoy-Pettis integrable on  $[a, b]$ , then  $H(F, G)$  is Denjoy integrable on  $[a, b]$  and*

$$H \left( (DP) \int_a^b F(t)dt, (DP) \int_a^b G(t)dt \right) \leq \int_a^b H(F(t), G(t))dt.$$

*Proof.* Since  $F$  and  $G$  are measurable, there exist Castaing representations  $\{f_n\}$  and  $\{g_n\}$  for  $F$  and  $G$ . Since  $f_n$  and  $g_n$  are measurable for all  $n \in \mathbb{N}$ ,

$$H(F(t), G(t)) = \max \left( \sup_{n \geq 1} \inf_{k \geq 1} \|f_n(t) - g_k(t)\|, \sup_{n \geq 1} \inf_{k \geq 1} \|g_n(t) - f_k(t)\| \right)$$

is measurable. Since  $F$  and  $G$  are Denjoy integrably bounded on  $[a, b]$ , there exist Denjoy integrable real-valued functions  $h_1$  and  $h_2$  such that for each  $t \in [a, b]$ ,  $\|x\| \leq h_1(t)$  for all  $x \in F(t)$  and  $\|x\| \leq h_2(t)$  for all  $x \in G(t)$ . Since  $h_1$  and  $h_2$  are nonnegative and Denjoy integrable on  $[a, b]$ ,  $h_1$  and  $h_2$  are Lebesgue integrable on  $[a, b]$ . Hence we have

$$H(F(t), G(t)) \leq H(F(t), \{0\}) + H(G(t), \{0\}) \leq h_1(t) + h_2(t)$$

for each  $t \in [a, b]$ . Therefore  $H(F, G)$  is Lebesgue integrable on  $[a, b]$  and so  $H(F, G)$  is Denjoy integrable on  $[a, b]$ . Hence we have

$$\begin{aligned} & H \left( (DP) \int_a^b F(t) dt, (DP) \int_a^b G(t) dt \right) \\ &= \sup_{\|x^*\| \leq 1} \left| s(x^*, (DP) \int_a^b F(t) dt) - s(x^*, (DP) \int_a^b G(t) dt) \right| \\ &= \sup_{\|x^*\| \leq 1} \left| \int_a^b s(x^*, F(t)) dt - \int_a^b s(x^*, G(t)) dt \right| \\ &\leq \sup_{\|x^*\| \leq 1} \int_a^b |s(x^*, F(t)) - s(x^*, G(t))| dt \\ &\leq \int_a^b \sup_{\|x^*\| \leq 1} |s(x^*, F(t)) - s(x^*, G(t))| dt \\ &= \int_a^b H(F(t), G(t)) dt. \end{aligned} \quad \square$$

**Theorem 3.7.** *Let  $X$  be a reflexive and separable Banach space and let  $F_n : [a, b] \rightarrow CWK(X)$  be a Denjoy-Pettis integrable set-valued mapping for each  $n \in \mathbb{N}$  and let  $F : [a, b] \rightarrow CWK(X)$  be a set-valued mapping such that  $\lim_{n \rightarrow \infty} H(F_n(t), F(t)) = 0$  on  $[a, b]$ . If there exists a Denjoy integrable function  $h$  on  $[a, b]$  such that  $\|F_n(t)\| \leq h(t)$  on  $[a, b]$  for each  $n \in \mathbb{N}$ , then  $F : [a, b] \rightarrow CWK(X)$  is Denjoy-Pettis integrable on  $[a, b]$  and*

$$\lim_{n \rightarrow \infty} H \left( (DP) \int_a^b F_n(t) dt, (DP) \int_a^b F(t) dt \right) = 0.$$

*Proof.* Since  $F_n : [a, b] \rightarrow CWK(X)$  is Denjoy-Pettis integrable on  $[a, b]$  for each  $n \in \mathbb{N}$ , for each  $n \in \mathbb{N}$ ,  $s(x^*, F_n(\cdot))$  is Denjoy integrable on  $[a, b]$  for each

$x^* \in X^*$  and for every subinterval  $[c, d]$  of  $[a, b]$  there exists  $(DP) \int_c^d F_n(t)dt \in CWK(X)$  such that

$$s(x^*, (DP) \int_c^d F_n(t)dt) = \int_c^d s(x^*, F_n(t))dt$$

for each  $x^* \in X^*$ . Since  $\lim_{n \rightarrow \infty} H(F_n(t), F(t)) = \lim_{n \rightarrow \infty} \sup_{\|x^*\| \leq 1} |s(x^*, F_n(t)) - s(x^*, F(t))| = 0$  on  $[a, b]$ ,  $\lim_{n \rightarrow \infty} |s(x^*, F_n(t)) - s(x^*, F(t))| = 0$  on  $[a, b]$  for each  $x^* \in B_{X^*}$  and so  $x^* \in X^*$ . Thus  $\lim_{n \rightarrow \infty} s(x^*, F_n(t)) = s(x^*, F(t))$  on  $[a, b]$  for each  $x^* \in X^*$ . For each  $n \in \mathbb{N}$ ,  $|s(x^*, F_n(t))| \leq \sup_{\|x^*\| \leq 1} |s(x^*, F_n(t))| = H(F_n(t), \{0\}) = \|F_n(t)\| \leq h(t)$  on  $[a, b]$  for each  $x^* \in B_{X^*}$ . Hence  $|s(x^*, F_n(t))| \leq \|x^*\|h(t)$  on  $[a, b]$  for each  $x^* \in X^*$ . Since  $h$  is Denjoy integrable on  $[a, b]$ ,  $\|x^*\|h$  is Denjoy integrable on  $[a, b]$  for each  $x^* \in X^*$ . By the Dominated Convergence Theorem for the Denjoy integral, for each  $x^* \in X^*$ ,  $s(x^*, F(\cdot))$  is Denjoy integrable on  $[a, b]$  and for every subinterval  $[c, d]$  of  $[a, b]$

$$\lim_{n \rightarrow \infty} \int_c^d s(x^*, F_n(t))dt = \int_c^d s(x^*, F(t))dt.$$

Let  $[c, d]$  be any subinterval of  $[a, b]$ . We define a function  $\varphi : X^* \rightarrow \mathbb{R}$  by  $\varphi(x^*) = \int_c^d s(x^*, F(t))dt$ . Then  $\varphi$  is positively homogeneous and convex. Now we prove that  $\varphi$  is continuous on  $X^*$ .  $(DP) \int_c^d F_n(t)dt \in CWK(X)$  for each  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} s(x^*, (DP) \int_c^d F_n(t)dt) = \lim_{n \rightarrow \infty} \int_c^d s(x^*, F_n(t))dt = \int_c^d s(x^*, F(t))dt$  exists for each  $x^* \in X^*$ . By Theorem 2.5 there exists an  $M > 0$  such that  $\sup_{n \in \mathbb{N}} \left\| (DP) \int_c^d F_n(t)dt \right\| \leq M$ . For each  $\epsilon > 0$  let  $\delta = \epsilon/M$ . If  $x_1^*, x_2^* \in X^*$ ,  $\|x_1^* - x_2^*\| < \delta$ , then

$$\begin{aligned} |\varphi(x_1^*) - \varphi(x_2^*)| &= \left| \int_c^d s(x_1^*, F(t))dt - \int_c^d s(x_2^*, F(t))dt \right| \\ &= \lim_{n \rightarrow \infty} \left| \int_c^d s(x_1^*, F_n(t))dt - \int_c^d s(x_2^*, F_n(t))dt \right| \\ &= \lim_{n \rightarrow \infty} \left| s(x_1^*, (DP) \int_c^d F_n(t)dt) - s(x_2^*, (DP) \int_c^d F_n(t)dt) \right| \\ &\leq \lim_{n \rightarrow \infty} \|x_1^* - x_2^*\| \left\| (DP) \int_c^d F_n(t)dt \right\| \\ &\leq M \|x_1^* - x_2^*\| \\ &< M\delta = \epsilon. \end{aligned}$$

Thus  $\varphi$  is continuous on  $X^*$ . By Theorem 2.4 there exists  $W_{[c,d]} \in C(X)$  such that  $\varphi(x^*) = s(x^*, W_{[c,d]})$  for each  $x^* \in X^*$ . Since  $|\varphi(x^*)| = \left| \int_c^d s(x^*, F(t))dt \right| < \infty$  for each  $x^* \in X^*$ ,  $W_{[c,d]} \in CB(X)$  by the Resonance Theorem. Since

$X$  is reflexive,  $W_{[c,d]} \in CWK(X)$  and  $s(x^*, W_{[c,d]}) = \varphi(x^*) = \int_c^d s(x^*, F(t))dt$  for each  $x^* \in X^*$ . Hence  $F : [a, b] \rightarrow CWK(X)$  is Denjoy-Pettis integrable on  $[a, b]$  and  $W_{[c,d]} = (DP) \int_c^d F(t)dt$ .

Since  $\lim_{n \rightarrow \infty} H(F_n(t), F(t)) = 0$  on  $[a, b]$ , for each  $\epsilon > 0$  and  $t \in [a, b]$  there exists  $N \in \mathbb{N}$  such that  $n \geq N \Rightarrow H(F_n(t), F(t)) < \epsilon$ . For some  $n \in \mathbb{N}$  with  $n \geq N$ ,

$$\begin{aligned} \|F(t)\| &= H(F(t), \{0\}) \leq H(F(t), F_n(t)) + H(F_n(t), \{0\}) \\ &< \|F_n(t)\| + \epsilon \leq h(t) + \epsilon. \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary,  $\|F(t)\| \leq h(t)$  on  $[a, b]$ , i.e.,  $F : [a, b] \rightarrow CWK(X)$  is Denjoy integrably bounded on  $[a, b]$ . By Theorem 3.6  $H(F_n, F)$  is Denjoy integrable on  $[a, b]$  and

$$H\left((DP) \int_a^b F_n(t)dt, (DP) \int_a^b F(t)dt\right) \leq \int_a^b H(F_n(t), F(t))dt$$

for each  $n \in \mathbb{N}$ . By the Dominated Convergence Theorem for the Denjoy integral we have

$$\lim_{n \rightarrow \infty} H\left((DP) \int_a^b F_n(t)dt, (DP) \int_a^b F(t)dt\right) \leq \lim_{n \rightarrow \infty} \int_a^b H(F_n(t), F(t))dt = 0.$$

Thus

$$\lim_{n \rightarrow \infty} H\left((DP) \int_a^b F_n(t)dt, (DP) \int_a^b F(t)dt\right) = 0. \quad \square$$

Let  $F_n : [a, b] \rightarrow CWK(X)$  be a set-valued mapping for each  $n \in \mathbb{N}$ . The sequence  $\{F_n\}$  is said to be *monotone increasing* (resp. *monotone decreasing*) if for each  $n \in \mathbb{N}$   $F_n(t) \subseteq F_{n+1}(t)$  (resp.  $F_n(t) \supseteq F_{n+1}(t)$ ) for all  $t \in [a, b]$ . The sequence  $\{F_n\}$  is said to be *monotone* if it is monotone increasing or monotone decreasing.

**Theorem 3.8.** *Let  $X$  be a reflexive and separable Banach space and let  $\{F_n\}$  be a monotone sequence of Denjoy-Pettis integrable  $CWK(X)$ -valued mappings on  $[a, b]$  and let  $F : [a, b] \rightarrow CWK(X)$  be a set-valued mapping such that  $H(F_1, F)$  is bounded and  $\lim_{n \rightarrow \infty} H(F_n(t), F(t)) = 0$  on  $[a, b]$ . If  $\lim_{n \rightarrow \infty} (DP) \int_a^b F_n(t)dt \in CWK(X)$ , then  $F : [a, b] \rightarrow CWK(X)$  is Denjoy-Pettis integrable on  $[a, b]$  and*

$$\lim_{n \rightarrow \infty} H\left((DP) \int_a^b F_n(t)dt, (DP) \int_a^b F(t)dt\right) = 0.$$

*Proof.* Assume that  $\{F_n\}$  is a monotone increasing sequence of Denjoy-Pettis integrable  $CWK(X)$ -valued mappings on  $[a, b]$ . The proof is similar when  $\{F_n\}$  is a monotone decreasing sequence of Denjoy-Pettis integrable  $CWK(X)$ -valued mappings on  $[a, b]$ . Since  $F_n : [a, b] \rightarrow CWK(X)$  is Denjoy-Pettis integrable on  $[a, b]$  for each  $n \in \mathbb{N}$ , for each  $n \in \mathbb{N}$ ,  $s(x^*, F_n(\cdot))$  is Denjoy



integrable on  $[a, b]$  for each  $x^* \in X^*$  and for every subinterval  $[c, d]$  of  $[a, b]$  there exists  $(DP) \int_c^d F_n(t)dt \in CWK(X)$  such that

$$s(x^*, (DP) \int_c^d F_n(t)dt) = \int_c^d s(x^*, F_n(t))dt$$

for each  $x^* \in X^*$ . Since  $\lim_{n \rightarrow \infty} H(F_n(t), F(t)) = \lim_{n \rightarrow \infty} \sup_{\|x^*\| \leq 1} |s(x^*, F_n(t)) - s(x^*, F(t))| = 0$  on  $[a, b]$ ,  $\lim_{n \rightarrow \infty} |s(x^*, F_n(t)) - s(x^*, F(t))| = 0$  on  $[a, b]$  for each  $x^* \in B_{X^*}$  and so  $x^* \in X^*$ . Thus  $\lim_{n \rightarrow \infty} s(x^*, F_n(t)) = s(x^*, F(t))$  on  $[a, b]$  for each  $x^* \in X^*$ . Since  $\{F_n\}$  is a monotone increasing sequence of Denjoy-Pettis integrable  $CWK(X)$ -valued mappings on  $[a, b]$ ,  $\{s(x^*, F_n(\cdot))\}$  is a monotone increasing sequence of Denjoy integrable functions on  $[a, b]$  for each  $x^* \in X^*$ . Let  $\lim_{n \rightarrow \infty} (DP) \int_a^b F_n(t)dt = W$ , where  $W \in CWK(X)$ . Then

$$\begin{aligned} & \lim_{n \rightarrow \infty} H \left( (DP) \int_a^b F_n(t)dt, W \right) \\ &= \lim_{n \rightarrow \infty} \sup_{\|x^*\| \leq 1} \left| s(x^*, (DP) \int_a^b F_n(t)dt) - s(x^*, W) \right| \\ &= \lim_{n \rightarrow \infty} \sup_{\|x^*\| \leq 1} \left| \int_a^b s(x^*, F_n(t))dt - s(x^*, W) \right| = 0. \end{aligned}$$

Hence  $\lim_{n \rightarrow \infty} \int_a^b s(x^*, F_n(t))dt = s(x^*, W)$  for each  $x^* \in B_{X^*}$  and so  $x^* \in X^*$ . Since  $W \in CWK(X)$ ,  $s(x^*, W)$  is finite for each  $x^* \in X^*$ . Thus  $\lim_{n \rightarrow \infty} \int_a^b s(x^*, F_n(t))dt$  is finite for each  $x^* \in X^*$ . Since  $\{F_n\}$  is monotone increasing on  $[a, b]$ ,  $\{s(x^*, F_n(\cdot)) - s(x^*, F_1(\cdot))\}$  is a monotone increasing sequence of nonnegative Denjoy integrable functions on  $[a, b]$  for each  $x^* \in X^*$ . Hence for each subinterval  $[c, d]$  of  $[a, b]$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_c^d \{s(x^*, F_n(t)) - s(x^*, F_1(t))\}dt \\ & \leq \lim_{n \rightarrow \infty} \int_a^b \{s(x^*, F_n(t)) - s(x^*, F_1(t))\}dt \\ & = \lim_{n \rightarrow \infty} \int_a^b s(x^*, F_n(t))dt - \int_a^b s(x^*, F_1(t))dt \end{aligned}$$

for each  $x^* \in X^*$ . Since  $\lim_{n \rightarrow \infty} \int_a^b s(x^*, F_n(t))dt$  is finite for each  $x^* \in X^*$ , for each subinterval  $[c, d]$  of  $[a, b]$ ,  $\lim_{n \rightarrow \infty} \int_c^d s(x^*, F_n(t))dt$  is also finite for each  $x^* \in X^*$ . By the Monotone Convergence Theorem for the Denjoy integral, for each  $x^* \in X^*$ ,  $s(x^*, F(\cdot))$  is Denjoy integrable on  $[a, b]$  and for every subinterval  $[c, d]$  of  $[a, b]$

$$\lim_{n \rightarrow \infty} \int_c^d s(x^*, F_n(t))dt = \int_c^d s(x^*, F(t))dt.$$

Using the same method as in the proof of Theorem 3.7, we obtain that  $F : [a, b] \rightarrow CWK(X)$  is Denjoy-Pettis integrable on  $[a, b]$ . Since  $F_n$  and  $F$  are measurable on  $[a, b]$ ,  $H(F_n, F)$  is measurable on  $[a, b]$  for each  $n \in \mathbb{N}$ . Since  $\{F_n\}$  is monotone increasing on  $[a, b]$  and  $\lim_{n \rightarrow \infty} s(x^*, F_n(t)) = s(x^*, F(t))$  on  $[a, b]$  for each  $x^* \in X^*$ ,

$$\begin{aligned} H(F_n(t), F(t)) &= \sup_{\|x^*\| \leq 1} |s(x^*, F_n(t)) - s(x^*, F(t))| \\ &\geq \sup_{\|x^*\| \leq 1} |s(x^*, F_{n+1}(t)) - s(x^*, F(t))| \\ &= H(F_{n+1}(t), F(t)) \end{aligned}$$

on  $[a, b]$  for each  $n \in \mathbb{N}$ . In particular,  $H(F_n(t), F(t)) \leq H(F_1(t), F(t))$  on  $[a, b]$  for each  $n \in \mathbb{N}$ . Since  $H(F_1, F)$  is bounded on  $[a, b]$ ,  $H(F_n, F)$  is Lebesgue integrable and so Denjoy integrable on  $[a, b]$  for each  $n \in \mathbb{N}$ . Hence we have

$$\begin{aligned} &H \left( (DP) \int_a^b F_n(t) dt, (DP) \int_a^b F(t) dt \right) \\ &= \sup_{\|x^*\| \leq 1} \left| s(x^*, (DP) \int_a^b F_n(t) dt) - s(x^*, (DP) \int_a^b F(t) dt) \right| \\ &= \sup_{\|x^*\| \leq 1} \left| \int_a^b s(x^*, F_n(t)) dt - \int_a^b s(x^*, F(t)) dt \right| \\ &\leq \sup_{\|x^*\| \leq 1} \int_a^b |s(x^*, F_n(t)) - s(x^*, F(t))| dt \\ &\leq \int_a^b \sup_{\|x^*\| \leq 1} |s(x^*, F_n(t)) - s(x^*, F(t))| dt \\ &= \int_a^b H(F_n(t), F(t)) dt \end{aligned}$$

for each  $n \in \mathbb{N}$ . By the Monotone Convergence Theorem for the Denjoy integral we have

$$\lim_{n \rightarrow \infty} H \left( (DP) \int_a^b F_n(t) dt, (DP) \int_a^b F(t) dt \right) \leq \lim_{n \rightarrow \infty} \int_a^b H(F_n(t), F(t)) dt = 0.$$

Thus

$$\lim_{n \rightarrow \infty} H \left( (DP) \int_a^b F_n(t) dt, (DP) \int_a^b F(t) dt \right) = 0. \quad \square$$

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