

## COMPATIBLE MAPPINGS OF TYPE (I) AND (II) ON INTUITIONISTIC FUZZY METRIC SPACES IN CONSIDERATION OF COMMON FIXED POINT

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ABSTRACT. In this paper, we formulate the definition of compatible mappings of type (I) and (II) in intuitionistic fuzzy metric spaces and prove a common fixed point theorem by using the conditions of compatible mappings of type (I) and (II) in complete intuitionistic fuzzy metric spaces. Our results intuitionistically fuzzify the result of Cho, Sedghi, and Shobe [4].

### 1. Introduction

Motivated by the potential applicability of fuzzy topology to quantum par-tical physics particularly in connection with both string and  $e^{(\infty)}$  theory devel-oped by El Naschie [7], [8], Park introduced and discussed in [22] a notion of intuitionistic fuzzy metric spaces which is based both on the idea of intuition-istic fuzzy set due to Atanassov [2] and the concept of fuzzy metric space given by George and Veeramani [11]. Actually, Park's notion is useful in modelling some phenomena where it is necessary to study relationship between two prob-ability functions. It has direct physics motivation in the context of the two slit experiment as foundation of E-infinity of high energy physics, recently studied by El Naschie [9], [10].

Alaca et. al [1] using the idea of intuitionistic fuzzy sets defined the notion of intuitionistic fuzzy metric space as Park [22] with the help of continuous  $t$ -norms and continuous  $t$ -conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [19]. Further they introduced the notion of Cauchy sequences in intuitionistic fuzzy metric spaces and proved the well known fixed point theorem of Banach [3] and Edelstein [6] extended for intuitionistic fuzzy metric spaces with the help of Grabiec [12]. Turkoglu et. al [30], introduced the concept of compatible maps and compatible maps of type  $(\alpha)$  and type  $(\beta)$  in

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intuitionistic fuzzy metric spaces and gave some relation between the concept of compatible maps of type  $(\alpha)$  and type  $(\beta)$ .

Turkoglu et al. [30] gave generalization of Jungck's common fixed point theorem [15] to intuitionistic fuzzy metric spaces.

Gregory et al. [13], Sadati and Park [26] studied the concept of intuitionistic fuzzy metric spaces and its applications. Sharma and Deshpande [29] proved common fixed point theorems for finite number of mappings without continuity and compatibility on intuitionistic fuzzy metric spaces.

On the other hand Cho, Sedghi, and Shobe [4] introduced definitions of compatible mappings of type (I) and (II) in fuzzy metric spaces and proved some common fixed point theorems for four mappings under the condition of compatible mappings of type (I) and (II) in complete fuzzy metric spaces. They extended, generalized and improved the corresponding results given by many authors.

In this paper, we formulate the definition of compatible mappings of type (I) and (II) in intuitionistic fuzzy metric spaces and prove a common fixed point theorem under the condition of compatible mappings of type (I) and (II) in intuitionistic fuzzy metric spaces. We also give an example to validate our main theorem. Our results intuitionistically fuzzify the results of Cho Sedghi and Shobe [4].

## 2. Preliminaries

**Definition 1** ([27]). A binary operation  $*$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is continuous  $t$ -norm if  $*$  is satisfying the following conditions:

- (i)  $*$  is commutative and associative,
- (ii)  $*$  is continuous,
- (iii)  $a * 1 = a$  for all  $a \in [0, 1]$ ,
- (iv)  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$  for all  $a, b, c, d \in [0, 1]$ .

**Definition 2** ([27]). A binary operation  $\diamond$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is continuous  $t$ -conorm if  $\diamond$  is satisfying the following conditions:

- (i)  $\diamond$  is commutative and associative,
- (ii)  $\diamond$  is continuous,
- (iii)  $a \diamond 0 = a$  for all  $a \in [0, 1]$ ,
- (iv)  $a \diamond b \leq c \diamond d$  whenever  $a \leq c$  and  $b \leq d$  for all  $a, b, c, d \in [0, 1]$ .

*Remark 1.* The concept of triangular norms ( $t$ -norms) and triangular conorms ( $t$ -conorms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and unions respectively. These concepts were originally introduced by Menger [21] in his study of statistical metric spaces. Several examples for these concepts were proposed by many authors ([5], [17], [18], [31]).

**Definition 3** ([1]). A 5-tuple  $(X, M, N, *, \diamond)$  is said to be an intuitionistic fuzzy metric space if  $X$  is an arbitrary set  $*$  is a continuous  $t$ -norm,  $\diamond$  is continuous  $t$ -conorm and  $M, N$  are fuzzy sets on  $X^2 \times [0, \infty)$  satisfying the following conditions:

- (i)  $M(x, y, t) + N(x, y, t) \leq 1$  for all  $x, y \in X$  and  $t > 0$ ,
- (ii)  $M(x, y, 0) = 0$  for all  $x, y \in X$ ,
- (iii)  $M(x, y, t) = 1$  for all  $x, y \in X$  and  $t > 0$  if and only if  $x = y$ ,
- (iv)  $M(x, y, t) = M(y, x, t)$  for all  $x, y \in X, t > 0$ ,
- (v)  $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$  for all  $x, y, z \in X$  and  $s, t > 0$ ,
- (vi) for all  $x, y \in X, M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$  is left continuous,
- (vii)  $\lim_{t \rightarrow \infty} M(x, y, t) = 1$  for all  $x, y \in X$  and  $t > 0$ ,
- (viii)  $N(x, y, 0) = 1$  for all  $x, y \in X$ ,
- (ix)  $N(x, y, t) = 0$  for all  $x, y \in X$  and  $t > 0$  if and only if  $x = y$ ,
- (x)  $N(x, y, t) = N(y, x, t)$  for all  $x, y \in X$  and  $t > 0$ ,
- (xi)  $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$  for all for all  $x, y, z \in X$  and  $s, t > 0$ ,
- (xii) for all  $x, y \in X, N(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$  is right continuous,
- (xiii)  $\lim_{t \rightarrow \infty} N(x, y, t) = 0$  for all  $x, y \in X$ .

Then  $(M, N)$  is called an intuitionistic fuzzy metric on  $X$ . The function  $M(x, y, t)$  and  $N(x, y, t)$  denote the degree of nearness and the degree of non nearness between  $x$  and  $y$  with respect to  $t$  respectively.

*Remark 2.* Every fuzzy metric space  $(X, M, *)$  is an intuitionistic fuzzy metric space of the form  $(X, M, 1 - M, *, \diamond)$  such that  $t$ -norm  $*$  and  $t$ -conorm  $\diamond$  are associated (Lowen [20]), i.e.,  $x \diamond y = 1 - ((1 - x) * (1 - y))$  for all  $x, y \in X$ .

**Example 1.** Let  $(X, d)$  be a metric space. Define  $t$ -norm  $a * b = \min\{a, b\}$  and  $t$ -conorm  $a \diamond b = \max\{a, b\}$  and for all  $x, y \in X$  and  $t > 0$ ,

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

Then  $(X, M, N, *, \diamond)$  is an intuitionistic fuzzy metric space. We call this intuitionistic fuzzy metric  $(M, N)$  induced by the metric  $d$  the standard intuitionistic fuzzy metric.

*Remark 3.* In intuitionistic fuzzy metric space  $(X, M, N, *, \diamond)$ ,  $M(x, y, \cdot)$  is non-decreasing and  $N(x, y, \cdot)$  is non increasing for all  $x, y \in X$ .

**Lemma 1** ([25]). *Let  $(X, M, N, *, \diamond)$  be an intuitionistic fuzzy metric space. Then  $M$  and  $N$  are continuous functions on  $X \times X \rightarrow (0, +\infty)$ .*

**Definition 4.** Let  $A$  and  $S$  be mappings from an intuitionistic fuzzy metric space  $(X, M, N, *, \diamond)$  into itself. Then the pair  $(A, S)$  is said to be compatible of type (I) if for all  $t > 0$ ,  $\lim_{n \rightarrow \infty} M(ASx_n, x, t) \leq M(Sx, x, t)$  and  $\lim_{n \rightarrow \infty} N(ASx_n, x, t) \geq N(Sx, x, t)$  whenever  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = x \in X$ .

**Definition 5.** Let  $A$  and  $S$  be mappings from an intuitionistic fuzzy metric space  $(X, M, N, *, \diamond)$  into itself. Then the pair  $(A, S)$  is said to be compatible of type (II) if and only if  $(S, A)$  is compatible of type (I).

*Remark 4.* In [15], [16], [23], [24], we can find the equivalent formulation of above definitions and their examples in metric spaces. Such mappings are independent of each other and more general than commuting and weakly commuting mappings (see [15], [28]).

**Proposition 1.** Let  $A$  and  $S$  be mappings from an intuitionistic fuzzy metric space  $(X, M, N, *, \diamond)$  into itself. Suppose that the pair  $(A, S)$  is compatible of type (I) (respectively II) and  $Az = Sz$  for some  $z \in X$ . Then for all  $t > 0$ ,  $M(Az, SSz, t) \geq M(Az, ASz, t)$  and  $N(Az, SSz, t) \leq N(Az, ASz, t)$  (respectively  $M(Sz, AAz, t) \geq M(Sz, SAz, t)$  and  $N(Sz, AAz, t) \leq N(Sz, SAz, t)$ ).

*Proof.* See Proposition 2.16 of [4]. □

**Lemma 2.** Let  $(X, M, N, *, \diamond)$  be an intuitionistic fuzzy metric space. If we define  $E_{\lambda, M} : X^2 \rightarrow R^+ \cup \{0\}$  by  $E_{\lambda, M}(x, y) = \inf\{t > 0 : M(x, y, t) > 1 - \lambda\}$  for all  $\lambda \in (0, 1)$  and  $x, y \in X$  and  $E_{\lambda, N} : X^2 \rightarrow R^+ \cup \{0\}$  by  $E_{\lambda, N}(x, y) = \sup\{t > 0 : N(x, y, t) < \lambda\}$  for all  $\lambda \in (0, 1)$  and  $x, y \in X$ . Then we have

(i) For all  $\mu \in (0, 1)$  there exists  $\lambda \in (0, 1)$  such that

$$E_{\mu, M}(x_1, x_n) \leq E_{\lambda, M}(x_1, x_2) + E_{\lambda, M}(x_2, x_3) + \cdots + E_{\lambda, M}(x_{n-1}, x_n),$$

$$E_{\mu, N}(x_1, x_n) \geq E_{\lambda, N}(x_1, x_2) + E_{\lambda, N}(x_2, x_3) + \cdots + E_{\lambda, N}(x_{n-1}, x_n)$$

for all  $x_1, x_2, \dots, x_n \in X$ .

(ii) The sequence  $\{x_n\}_{n \in \mathbb{N}}$  is convergent in intuitionistic fuzzy metric space  $(X, M, N, *, \diamond)$  if and only if  $E_{\lambda, M}(x_n, x) \rightarrow 0$ ,  $E_{\lambda, N}(x_n, x) \rightarrow 0$ . Also the sequence  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy sequence if and only if it is a Cauchy sequence with  $E_{\lambda, M}$  and  $E_{\lambda, N}$ .

*Proof.* (i) For any  $\mu \in (0, 1)$  we can find a  $\lambda \in (0, 1)$  such that

$$\frac{(1 - \lambda) * (1 - \lambda) * \cdots * (1 - \lambda)}{n} \geq 1 - \mu \quad \text{and} \quad \frac{\lambda \diamond \lambda \diamond \cdots \diamond \lambda}{n} \leq \mu.$$

So by triangle inequality, we have

$$\begin{aligned} & M(x_1, x_n, E_{\lambda, M}(x_1, x_2) + E_{\lambda, M}(x_2, x_3) + \cdots + E_{\lambda, M}(x_{n-1}, x_n) + n\delta) \\ & \geq M(x_1, x_2, E_{\lambda, M}(x_1, x_2) + \delta) + \cdots + M(x_{n-1}, x_n, E_{\lambda, M}(x_{n-1}, x_n) + \delta) \\ & \geq \frac{(1 - \lambda) * (1 - \lambda) * \cdots * (1 - \lambda)}{n} \geq 1 - \mu \end{aligned}$$

and

$$\begin{aligned} & N(x_1, x_n, E_{\lambda, N}(x_1, x_2) + E_{\lambda, N}(x_2, x_3) + \cdots + E_{\lambda, N}(x_{n-1}, x_n) - n\delta) \\ & \leq N(x_1, x_2, E_{\lambda, N}(x_1, x_2) - \delta) + \cdots + M(x_{n-1}, x_n, E_{\lambda, N}(x_{n-1}, x_n) - \delta) \end{aligned}$$

$$\leq \frac{\lambda \diamond \lambda \cdots \diamond \lambda}{n} \leq \mu$$

for all  $\delta > 0$ , which implies that

$$E_{\mu,M}(x_1, x_n) \leq E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n) + n\delta$$

and

$$E_{\mu,N}(x_1, x_n) \geq E_{\mu,N}(x_1, x_2) + E_{\mu,N}(x_2, x_3) + \cdots + E_{\mu,N}(x_{n-1}, x_n) - n\delta.$$

Since  $\delta > 0$  is arbitrary, we have

$$E_{\mu,M}(x_1, x_n) \leq E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n)$$

and

$$E_{\mu,N}(x_1, x_n) \geq E_{\mu,N}(x_1, x_2) + E_{\mu,N}(x_2, x_3) + \cdots + E_{\mu,N}(x_{n-1}, x_n).$$

For (ii), since  $M$  and  $N$  are continuous in its third place therefore

$$E_{\lambda,M}(x, y) = \inf\{t > 0 : M(x, y, t) > 1 - \lambda\},$$

and

$$E_{\lambda,N}(x, y) = \sup\{t > 0 : N(x, y, t) < \lambda\}.$$

Thus we have

$$M(x_n, x, \eta) > 1 - \lambda \Leftrightarrow E_{\lambda,M}(x_n, x) < \eta$$

and

$$N(x_n, x, \eta) < \lambda \Leftrightarrow E_{\lambda,N}(x_n, x) > \eta$$

for all  $\eta > 0$ . □

**Lemma 3.** *Let  $(X, M, N, *, \diamond)$  be an intuitionistic fuzzy metric space. If a sequence  $\{x_n\}$  in  $X$  is such that for any  $n \in \mathbb{N}$ ,  $M(x_n, x_{n+1}, t) \geq M(x_0, x_1, k^n t)$  and  $N(x_n, x_{n+1}, t) \leq N(x_0, x_1, k^n t)$  for all  $k > 1$ , then the sequence  $\{x_n\}$  is a Cauchy sequence.*

*Proof.* For all  $\lambda \in (0, 1)$  and  $x_n, x_{n+1} \in X$ , we have

$$\begin{aligned} E_{\lambda,M}(x_{n+1}, x_n) &= \inf\{t > 0 : M(x_{n+1}, x_n, t) > 1 - \lambda\} \\ &\leq \{t > 0 : M(x_0, x_1, k^n t) > 1 - \lambda\} \\ &= \inf\left\{\frac{t}{k^n} : M(x_0, x_1, t) > 1 - \lambda\right\} \\ &= \frac{1}{k^n} \inf\{t > 0 : M(x_0, x_1, t) > 1 - \lambda\} \\ &= \frac{1}{k^n} E_{\lambda,M}(x_0, x_1) \end{aligned}$$

and

$$\begin{aligned} E_{\lambda,N}(x_{n+1}, x_n) &= \sup\{t > 0 : N(x_{n+1}, x_n, t) < \lambda\} \\ &\geq \{t > 0 : N(x_0, x_1, k^n t) < \lambda\} \\ &= \sup\left\{\frac{t}{k^n} : N(x_0, x_1, t) < \lambda\right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{k^n} \sup\{t > 0 : M(x_0, x_1, t) < \lambda\} \\
&= \frac{1}{k^n} E_{\lambda, N}(x_0, x_1).
\end{aligned}$$

By Lemma 2, for all  $\mu \in (0, 1)$  there exists  $\lambda \in (0, 1)$  such that

$$E_{\mu, M}(x_n, x_m) \leq E_{\lambda, M}(x_n, x_{n+1}) + E_{\lambda, M}(x_{n+1}, x_{n+2}) + \cdots + E_{\lambda, M}(x_{m-1}, x_m),$$

$$E_{\mu, N}(x_n, x_m) \geq E_{\lambda, N}(x_n, x_{n+1}) + E_{\lambda, N}(x_{n+1}, x_{n+2}) + \cdots + E_{\lambda, N}(x_{m-1}, x_m).$$

Therefore,

$$\begin{aligned}
E_{\mu, M}(x_n, x_m) &\leq \frac{1}{k^n} E_{\lambda, M}(x_0, x_1) + \frac{1}{k^{n+1}} E_{\lambda, M}(x_0, x_1) + \cdots + \frac{1}{k^{m-1}} E_{\lambda, M}(x_0, x_1) \\
&= E_{\lambda, M}(x_0, x_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \rightarrow 0,
\end{aligned}$$

$$\begin{aligned}
E_{\mu, N}(x_n, x_m) &\geq \frac{1}{k^n} E_{\lambda, N}(x_0, x_1) + \frac{1}{k^{n+1}} E_{\lambda, N}(x_0, x_1) + \cdots + \frac{1}{k^{m-1}} E_{\lambda, N}(x_0, x_1) \\
&= E_{\lambda, N}(x_0, x_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \rightarrow 0.
\end{aligned}$$

Hence the sequence  $\{x_n\}$  is a Cauchy sequence.  $\square$

### 3. Main results

Let  $\Phi$  be the set of all continuous and increasing functions  $\phi : [0, 1]^5 \rightarrow [0, 1]$  in any coordinate and  $\phi(t, t, t, t, t) > t$  for all  $t \in [0, 1)$ . Also let  $\Psi$  be the set of all continuous and increasing functions  $\psi : [0, 1]^5 \rightarrow [0, 1]$  in any coordinate and  $\psi(t, t, t, t, t) < t$  for all  $t \in [0, 1)$ .

**Example 2.** Consider the function  $\phi : [0, 1]^5 \rightarrow [0, 1]$  defined as follows:

- (i)  $\phi(x_1, x_2, x_3, x_4, x_5) = (\min\{x_i\})^h$  for some  $0 < h < 1$ .
- (ii)  $\phi(x_1, x_2, x_3, x_4, x_5) = x_i^h$  for some  $0 < h < 1$ .
- (iii)  $\phi(x_1, x_2, x_3, x_4, x_5) = \max\{x_1^{\alpha_1}, x_2^{\alpha_2}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_5^{\alpha_5}\}$ , where  $0 < \alpha_i < 1$  for  $i = 1, 2, 3, 4, 5$ .

Consider the function  $\psi : [0, 1]^5 \rightarrow [0, 1]$  defined as follows:

- (i)  $\psi(x_1, x_2, x_3, x_4, x_5) = (\max\{x_i\})^h$  for some  $h > 1$ .
- (ii)  $\psi(x_1, x_2, x_3, x_4, x_5) = x_i^h$  for some  $h > 1$ .
- (iii)  $\psi(x_1, x_2, x_3, x_4, x_5) = \min\{x_1^{\alpha_1}, x_2^{\alpha_2}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_5^{\alpha_5}\}$ , where  $\alpha_i > 1$  for  $i = 1, 2, 3, 4, 5$ .

Throughout this paper  $(X, M, N, *, \diamond)$  will denote intuitionistic fuzzy metric space with continuous  $t$ -norm  $*$  and continuous  $t$ -conorm  $\diamond$  defined by  $t * t = t$  and  $(1 - t) \diamond (1 - t) = (1 - t)$  for all  $t \in [0, 1]$ .

**Theorem 1.** *Let  $(X, M, N, *, \diamond)$  be a complete intuitionistic fuzzy metric space. Let  $A, B, S$  and  $T$  be mappings from  $X$  into itself such that*

$$(1.1) \quad A(X) \sqsubseteq T(X), B(X) \sqsubseteq S(X),$$

(1.2) *there exists a constant  $k \in (0, \frac{1}{2})$  such that*

$$M(Ax, By, kt) \geq \phi \left( \begin{array}{cc} M(Sx, Ty, t), & M(Ax, Sx, t), \\ M(By, Ty, t), & M(Ax, Ty, \alpha t), \\ M(By, Sx, (2-\alpha)t) & \end{array} \right)$$

and

$$N(Ax, By, kt) \leq \psi \left( \begin{array}{cc} N(Sx, Ty, t), & N(Ax, Sx, t), \\ N(By, Ty, t), & N(Ax, Ty, \alpha t), \\ N(By, Sx, (2-\alpha)t) & \end{array} \right)$$

for all  $x, y \in X, \alpha \in (0, 2), t > 0$  and  $\phi \in \Phi, \psi \in \Psi$ .

If the mappings  $A, B, S$  and  $T$  satisfy any one of the following conditions:

(1.3) *the pairs  $(A, S)$  and  $(B, T)$  are compatible of type (II) and  $A$  or  $B$  is continuous,*

(1.4) *the pairs  $(A, S)$  and  $(B, T)$  are compatible of type (I) and  $S$  or  $T$  is continuous. Then  $A, B, S$  and  $T$  have a unique common fixed point in  $X$ .*

*Proof.* Let  $x_0 \in X$  be an arbitrary point. Since  $A(X) \sqsubseteq T(X), B(X) \sqsubseteq S(X)$ , there exist  $x_1, x_2 \in X$  such that  $Ax_0 = Tx_1, Bx_1 = Sx_2$ . Inductively, construct the sequences  $\{y_n\}$  and  $\{x_n\}$  in  $X$  such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

for  $n = 1, 2, 3, \dots$ . Then by  $\alpha = 1 - q$  and  $q \in (\frac{1}{2}, 1)$  if we set  $d_m(t) = M(y_m, y_{m+1}, t)$  for all  $t > 0$  and  $d_m^*(t) = N(y_m, y_{m+1}, t)$  for all  $t > 0$ , then we prove that  $\{d_m(t)\}$  is increasing with respect to  $m$  and  $\{d_m^*(t)\}$  is decreasing with respect to  $m$ . Setting  $m = 2n$ , then we have

$$\begin{aligned} d_{2n}(kt) &= M(y_{2n}, y_{2n+1}, kt) = M(Ax_{2n}, Bx_{2n+1}, kt) \\ &\geq \phi \left( \begin{array}{cc} M(Sx_{2n}, Tx_{2n+1}, t), & M(Ax_{2n}, Sx_{2n}, t), \\ M(Bx_{2n+1}, Tx_{2n+1}, t), & M(Ax_{2n}, Tx_{2n+1}, (1-q)t), \\ M(Bx_{2n+1}, Sx_{2n}, (1+q)t) & \end{array} \right) \\ &= \phi \left( \begin{array}{cc} M(y_{2n-1}, y_{2n}, t), & M(y_{2n}, y_{2n-1}, t), \\ M(y_{2n+1}, y_{2n}, t), & M(y_{2n}, y_{2n}, (1-q)t), \\ M(y_{2n+1}, y_{2n-1}, (1+q)t) & \end{array} \right) \\ &= \phi(d_{2n-1}(t), d_{2n-1}(t), d_{2n}(t), 1, M(y_{2n+1}, y_{2n-1}, (1+q)t)), \end{aligned}$$

that is,

$$(1.5) \quad d_{2n}(kt) \geq \phi(d_{2n-1}(qt), d_{2n-1}(qt), d_{2n}(qt), 1, d_{2n-1}(t) * d_{2n}(qt)).$$

The above inequality is true since  $\phi$  is an increasing function and

$$\begin{aligned} M(y_{2n-1}, y_{2n+1}, (1+q)t) &\geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, qt) \\ &= d_{2n-1}(t) * d_{2n}(qt). \end{aligned}$$

Also we have

$$\begin{aligned}
d_{2n}^i(kt) &= N(y_{2n}, y_{2n+1}, kt) = N(Ax_{2n}, Bx_{2n+1}, kt) \\
&\leq \psi \left( \begin{array}{cc} N(Sx_{2n}, Tx_{2n+1}, t), & N(Ax_{2n}, Sx_{2n}, t), \\ N(Bx_{2n+1}, Tx_{2n+1}, t), & N(Ax_{2n}, Tx_{2n+1}, (1-q)t), \\ N(Bx_{2n+1}, Sx_{2n}, (1+q)t) & \end{array} \right) \\
&= \psi \left( \begin{array}{cc} N(y_{2n-1}, y_{2n}, t), & N(y_{2n}, y_{2n-1}, t), \\ N(y_{2n+1}, y_{2n}, t), & N(y_{2n}, y_{2n}, (1-q)t), \\ N(y_{2n+1}, y_{2n-1}, (1+q)t) & \end{array} \right) \\
&= \phi(d_{2n-1}^i(t), d_{2n-1}^i(t), d_{2n}^i(t), 0, N(y_{2n+1}, y_{2n-1}, (1+q)t)),
\end{aligned}$$

that is,

$$(1.6) \quad d_{2n}^i(kt) \leq \psi(d_{2n-1}^i(qt), d_{2n-1}^i(qt), d_{2n}^i(qt), 0, d_{2n-1}^i(t) \diamond d_{2n}^i(qt)).$$

The above inequality is true since  $\psi$  is an increasing function and

$$\begin{aligned}
N(y_{2n-1}, y_{2n+1}, (1+q)t) &\leq N(y_{2n-1}, y_{2n}, t) \diamond N(y_{2n}, y_{2n+1}, qt) \\
&= d_{2n-1}(t) * d_{2n}(qt).
\end{aligned}$$

We claim that for all  $n \in N$ ,  $d_{2n}(t) \geq d_{2n-1}(t)$ . In fact if  $d_{2n}(t) < d_{2n-1}(t)$ , then since  $d_{2n}(qt) * d_{2n-1}(t) \geq d_{2n}(qt) * d_{2n}(qt) = d_{2n}(qt)$ .

By the inequality (1.5), we have

$$d_{2n}(kt) \geq \phi(d_{2n}(qt), d_{2n}(qt), d_{2n}(qt), d_{2n}(qt), d_{2n}(qt), ) > d_{2n}(qt),$$

that is,  $d_{2n}(qt) > d_{2n}(qt)$ , which is a contradiction. Hence  $d_{2n}(t) \geq d_{2n-1}(t)$  for all  $n \in N$  and  $t > 0$ .

Similarly, for  $m = 2n + 1$ , we have  $d_{2n+1}(t) \geq d_{2n}(t)$  and so  $\{d_n(t)\}$  is an increasing sequence in  $[0, 1]$ .

Now we claim that for all  $n \in N$ ,  $d_{2n}^i(t) \leq d_{2n-1}^i(t)$ . In fact if  $d_{2n}^i(t) > d_{2n-1}^i(t)$ , then since  $d_{2n}^i(qt) \diamond d_{2n-1}^i(t) \leq d_{2n}^i(qt) \diamond d_{2n}^i(qt) = d_{2n}^i(qt)$ .

By the inequality (1.6), we have

$$d_{2n}^i(kt) \leq \psi(d_{2n}^i(qt), d_{2n}^i(qt), d_{2n}^i(qt), d_{2n}^i(qt), d_{2n}^i(qt), ) < d_{2n}^i(qt),$$

that is,  $d_{2n}^i(qt) < d_{2n}^i(qt)$ , which is a contradiction. Hence  $d_{2n}^i(t) \geq d_{2n-1}^i(t)$  for all  $n \in N$  and  $t > 0$ .

Similarly, for  $m = 2n + 1$ , we have  $d_{2n+1}^i(t) \geq d_{2n}^i(t)$  and so  $\{d_n(t)\}$  is an decreasing sequence in  $[0, 1]$ .

By inequality (1.5), we have

$$\begin{aligned}
d_{2n}(kt) &\geq \phi(d_{2n-1}(qt), d_{2n-1}(qt), d_{2n-1}(qt), d_{2n-1}(qt), d_{2n-1}(t) * d_{2n-1}(qt)) \\
&> d_{2n-1}(qt).
\end{aligned}$$

Similarly for  $m = 2n + 1$ , we have  $d_{2n+1}(kt) \geq d_{2n}(qt)$  and so  $d_n(kt) \geq d_{n-1}(qt)$  for all  $n \in N$ , that is,

$$(1.7) \quad M(y_n, y_{n+1}, t) \geq M\left(y_{n-1}, y_n, \frac{q}{k}t\right) \geq \cdots \geq M\left(y_0, y_1, \left(\frac{q}{k}\right)^n t\right).$$

By inequality (1.6), we have

$$d_{2n}^i(kt) \leq \psi(d_{2n-1}^i(qt), d_{2n-1}^i(qt), d_{2n-1}^i(qt), d_{2n-1}^i(qt), d_{2n-1}^i(t) * d_{2n-1}^i(qt)) < d_{2n-1}^i(qt).$$

Similarly for  $m = 2n + 1$ , we have  $d_{2n+1}^i(kt) \leq d_{2n}^i(qt)$  and so  $d_n^i(kt) \leq d_{n-1}^i(qt)$  for all  $n \in N$ , that is

$$(1.8) \quad N(y_n, y_{n+1}, t) \leq N\left(y_{n-1}, y_n, \frac{q}{k}t\right) \leq \dots \leq N\left(y_0, y_1, \left(\frac{q}{k}\right)^n t\right).$$

With the help of Lemma 3, it is clear from (1.7) and (1.8) that  $\{y_n\}$  is a Cauchy sequence and by completeness of  $X$ ,  $\{y_n\}$  converges to a point in  $X$ . Let  $y_n \rightarrow z$  as  $n \rightarrow \infty$ . Hence we have

$$\begin{aligned} \lim_{n \rightarrow \infty} y_n &= \lim_{n \rightarrow \infty} Ax_{2n} = \lim_{n \rightarrow \infty} Tx_{2n+1} \\ &= \lim_{n \rightarrow \infty} y_{2n+1} = \lim_{n \rightarrow \infty} Bx_{2n+1} = \lim_{n \rightarrow \infty} Sx_{2n+1} = z. \end{aligned}$$

Now suppose that  $T$  is continuous and the pair  $(B, T)$  is compatible of type (I). Hence we have

$$\lim_{n \rightarrow \infty} TTx_{2n+1} = Tz, \quad M(Tz, z, t) \geq \lim_{n \rightarrow \infty} M(BTx_{2n+1}, z, t)$$

and

$$N(Tz, z, t) \leq \lim_{n \rightarrow \infty} N(BTx_{2n+1}, z, t).$$

Now for  $\alpha = 1$ , setting  $x = x_{2n}$  and  $y = Tx_{2n+1}$  in the inequality (1.2), we obtain

$$M(Ax_{2n}, BTx_{2n+1}, kt) \geq \phi \left( \begin{array}{cc} M(Sx_{2n}, TTx_{2n+1}, t), & M(Ax_{2n}, Sx_{2n}, t), \\ M(BTx_{2n+1}, TTx_{2n+1}, t), & M(Ax_{2n}, TTx_{2n+1}, t), \\ M(BTx_{2n+1}, Sx_{2n}, t) & \end{array} \right)$$

and

$$N(Ax_{2n}, BTx_{2n+1}, kt) \leq \psi \left( \begin{array}{cc} N(Sx_{2n}, TTx_{2n+1}, t), & N(Ax_{2n}, Sx_{2n}, t), \\ N(BTx_{2n+1}, TTx_{2n+1}, t), & N(Ax_{2n}, TTx_{2n+1}, t), \\ N(BTx_{2n+1}, Sx_{2n}, t) & \end{array} \right).$$

Letting  $n \rightarrow \infty$ , we have

$$\begin{aligned} M(z, \lim_{n \rightarrow \infty} BTx_{2n+1}, kt) &\geq \phi \left( \begin{array}{cc} M(z, Tz, t), & M(z, z, t), \\ M(\lim_{n \rightarrow \infty} BTx_{2n+1}, Tz, t), & M(z, Tz, t), \\ M(\lim_{n \rightarrow \infty} BTx_{2n+1}, z, t) & \end{array} \right) \\ &\geq \phi \left( \begin{array}{cc} M(z, Tz, \frac{t}{2}), & M(z, z, \frac{t}{2}), \\ M(\lim_{n \rightarrow \infty} BTx_{2n+1}, Tz, \frac{t}{2}), & M(z, Tz, \frac{t}{2}), \\ M(\lim_{n \rightarrow \infty} BTx_{2n+1}, z, \frac{t}{2}) & \end{array} \right) \end{aligned}$$

and

$$N(z, \lim_{n \rightarrow \infty} BTx_{2n+1}, kt) \leq \psi \left( \begin{array}{cc} N(z, Tz, t), & N(z, z, t), \\ N(\lim_{n \rightarrow \infty} BTx_{2n+1}, Tz, t), & N(z, Tz, t), \\ N(\lim_{n \rightarrow \infty} BTx_{2n+1}, z, t) & \end{array} \right) \\ \leq \psi \left( \begin{array}{cc} N(z, Tz, \frac{t}{2}), & N(z, z, \frac{t}{2}), \\ N(\lim_{n \rightarrow \infty} BTx_{2n+1}, Tz, \frac{t}{2}), & N(z, Tz, \frac{t}{2}), \\ N(\lim_{n \rightarrow \infty} BTx_{2n+1}, z, \frac{t}{2}) & \end{array} \right).$$

Thus it follows that

$$\lim_{n \rightarrow \infty} M(BTx_{2n+1}, Tz, t) \geq \lim_{n \rightarrow \infty} M(BTx_{2n+1}, z, \frac{t}{2}) * \lim_{n \rightarrow \infty} M(z, Tz, \frac{t}{2}), \\ \lim_{n \rightarrow \infty} N(BTx_{2n+1}, Tz, t) \leq \lim_{n \rightarrow \infty} N(BTx_{2n+1}, z, \frac{t}{2}) \diamond \lim_{n \rightarrow \infty} N(z, Tz, \frac{t}{2}).$$

So,

$$\lim_{n \rightarrow \infty} M(BTx_{2n+1}, Tz, t) \geq \lim_{n \rightarrow \infty} M(BTx_{2n+1}, z, \frac{t}{2}), \\ \lim_{n \rightarrow \infty} N(BTx_{2n+1}, Tz, t) \leq \lim_{n \rightarrow \infty} N(BTx_{2n+1}, z, \frac{t}{2}).$$

Hence since  $\phi(t, t, t, t, t) > t$  and  $\psi(t, t, t, t, t) < t$  by the above inequalities we have

$$M(z, \lim_{n \rightarrow \infty} BTx_{2n+1}, kt) > M(z, \lim_{n \rightarrow \infty} BTx_{2n+1}, \frac{t}{2})$$

and

$$N(z, \lim_{n \rightarrow \infty} BTx_{2n+1}, kt) < N(z, \lim_{n \rightarrow \infty} BTx_{2n+1}, \frac{t}{2}),$$

which is a contradiction. It follows that  $\lim_{n \rightarrow \infty} BTx_{2n+1} = z$ .

Now using the compatibility of type (I), we have

$$M(Tz, z, t) \geq \lim_{n \rightarrow \infty} M(z, BTx_{2n+1}, t) = 1$$

and

$$N(Tz, z, t) \leq \lim_{n \rightarrow \infty} N(z, BTx_{2n+1}, t) = 0.$$

So it follows that  $Tz = z$ .

Again replacing  $x$  by  $x_{2n}$  and  $y$  by  $z$  in (1.2), with  $\alpha = 1$ , we have

$$M(Ax_{2n}, Bz, kt) \geq \phi \left( \begin{array}{cc} M(Sx_{2n}, Tz, t), & M(Ax_{2n}, Sx_{2n}, t), \\ M(Bz, Tz, t), & M(Ax_{2n}, Tz, t), \\ M(Bz, Sx_{2n}, t) & \end{array} \right)$$

and

$$N(Ax_{2n}, Bz, kt) \leq \psi \left( \begin{array}{cc} N(Sx_{2n}, Tz, t), & N(Ax_{2n}, Sx_{2n}, t), \\ N(Bz, Tz, t), & N(Ax_{2n}, Tz, t), \\ N(Bz, Sx_{2n}, t) & \end{array} \right)$$

and so letting  $n \rightarrow \infty$ , we have

$$M(Bz, z, kt) > M(Bz, z, t), \quad N(Bz, z, kt) < N(Bz, z, t),$$

which implies that  $Bz = z$ . Since  $B(X) \sqsubseteq S(X)$ , there exists  $u \in X$  such that  $Su = z = Bz$ . So by (1.2) with  $\alpha = 1$ , we have

$$M(Au, Bz, kt) \geq \phi \left( \begin{array}{cc} M(Su, Tz, t), & M(Au, Su, t), \\ M(Bz, Tz, t), & M(Au, Tz, t), \\ M(Bz, Su, t) \end{array} \right)$$

and

$$N(Au, Bz, kt) \leq \psi \left( \begin{array}{cc} N(Su, Tz, t), & N(Au, Su, t), \\ N(Bz, Tz, t), & N(Au, Tz, t), \\ N(Bz, Su, t) \end{array} \right).$$

Therefore

$$M(Au, z, kt) > M(z, Au, t) \quad \text{and} \quad N(Au, z, kt) < N(z, Au, t),$$

which implies that  $Au = z$ . Since the pair  $(A, S)$  is compatible of type (I) and  $Au = Su = z$ , by Proposition 1, we have

$$M(Au, SSu, t) \geq M(Au, ASu, t) \quad \text{and} \quad N(Au, SSu, t) \leq N(Au, ASu, t).$$

Thus

$$M(z, Sz, t) \geq M(z, Az, t) \quad \text{and} \quad N(z, Sz, t) \leq N(z, Az, t).$$

Again by (1.2) with  $\alpha = 1$ , we have

$$M(Az, Bz, kt) \geq \phi \left( \begin{array}{cc} M(Sz, Tz, t), & M(Az, Sz, t), \\ M(Bz, Tz, t), & M(Az, Tz, t), \\ M(Bz, Sz, t) \end{array} \right)$$

and

$$N(Az, Bz, kt) \leq \psi \left( \begin{array}{cc} N(Sz, Tz, t), & N(Az, Sz, t), \\ N(Bz, Tz, t), & N(Az, Tz, t), \\ N(Bz, Sz, t) \end{array} \right).$$

Thus it follows that

$$\begin{aligned} M(Az, Sz, t) &\geq M(Az, z, \frac{t}{2}) * M(z, Sz, \frac{t}{2}) \\ &\geq M(Az, z, \frac{t}{2}) * M(z, Az, \frac{t}{2}) \\ &= M(Az, z, \frac{t}{2}) \end{aligned}$$

and

$$\begin{aligned} N(Az, Sz, t) &\leq N(Az, z, \frac{t}{2}) \diamond N(z, Sz, \frac{t}{2}) \\ &\leq N(Az, z, \frac{t}{2}) \diamond N(z, Az, \frac{t}{2}) \\ &= N(Az, z, \frac{t}{2}). \end{aligned}$$

Hence we have

$$M(Az, z, kt) \geq \phi \left( \begin{array}{cc} M(Az, z, \frac{t}{2}), & M(Az, z, \frac{t}{2}), \\ M(Az, z, \frac{t}{2}), & M(z, Az, \frac{t}{2}), \\ M(z, Az, \frac{t}{2}) & \end{array} \right) > M(z, Az, \frac{t}{2})$$

and

$$N(Az, z, kt) \leq \psi \left( \begin{array}{cc} N(Az, z, \frac{t}{2}), & N(Az, z, \frac{t}{2}), \\ N(Az, z, \frac{t}{2}), & N(z, Az, \frac{t}{2}), \\ N(z, Az, \frac{t}{2}) & \end{array} \right) < N(z, Az, \frac{t}{2}),$$

and so  $Az = z$ . Therefore  $Az = Bz = Tz = z$  and  $z$  is a common fixed point of the self mappings  $A, B, S$  and  $T$ . The uniqueness of a common fixed point of the mappings  $A, B, S$  and  $T$  can be easily verified by using (1.2). In fact if  $w$  is another common fixed point for  $A, B, S$  and  $T$ , then for  $\alpha = 1$ , we have

$$\begin{aligned} M(z, w, t) &= M(Az, Bw, kt) \\ &\geq \phi \left( \begin{array}{cc} M(Sz, Tw, t), & M(Az, Sz, t), \\ M(Bw, Tw, t), & M(Az, Tw, t), \\ M(Bw, Sz, t) & \end{array} \right) > M(z, w, t) \end{aligned}$$

and

$$\begin{aligned} N(z, w, t) &= N(Az, Bw, kt) \\ &\leq \psi \left( \begin{array}{cc} N(Sz, Tw, t), & N(Az, Sz, t), \\ N(Bw, Tw, t), & N(Az, Tw, t), \\ N(Bw, Sz, t) & \end{array} \right) < N(z, w, t). \end{aligned}$$

Thus  $z = w$ . □

**Example 3.** Let  $X = [0, 1]$  with the metric  $d$  defined by  $d(x, y) = |x - y|$  and for each  $t \in [0, 1]$  define

$$\begin{aligned} M(x, y, t) &= \frac{t}{t + |x - y|}, & N(x, y, t) &= \frac{|x - y|}{t + |x - y|}, \\ M(x, y, 0) &= 0, & N(x, y, 0) &= 1 \end{aligned}$$

for all  $x, y \in X$ .

Clearly  $(X, M, N, *, \diamond)$  is a complete intuitionistic fuzzy metric space where  $*$  is defined by  $a * b = \min\{a, b\}$  and  $\diamond$  is defined by  $a \diamond b = \max\{a, b\}$ .

Define the self mappings  $A, B, S$  and  $T$  on  $X$  by

$$\begin{aligned} Ax &= Bx = 0 \quad \text{for all } x \in X, \\ Sx &= \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1, \end{cases} \\ Tx &= x \quad \text{for all } x \in X. \end{aligned}$$

If we define a sequence  $\{x_n\}$  in  $X$  by  $x_n = \{\frac{1}{n}\}$ , then we have

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = 0,$$

$$\begin{aligned} \lim_{n \rightarrow \infty} M(SAx_n, 0, t) &\leq M(A0, 0, t) = 1, \\ \lim_{n \rightarrow \infty} N(SAx_n, 0, t) &\leq N(A0, 0, t) = 0, \text{ and} \\ \lim_{n \rightarrow \infty} Bx_n &= \lim_{n \rightarrow \infty} Tx_n = 0, \\ \lim_{n \rightarrow \infty} M(TBx_n, 0, t) &\leq M(B0, 0, t) = 1, \\ \lim_{n \rightarrow \infty} N(TBx_n, 0, t) &\leq N(B0, 0, t) = 0. \end{aligned}$$

That is the pairs  $(A, S)$ ,  $(B, T)$  are compatible of type (II) and  $A, B$  are continuous. Consider the functions  $\phi : [0, 1]^5 \rightarrow [0, 1]$  defined by

$$\phi(x_1, x_2, x_3, x_4, x_5) = (\min\{x_i\})^h \quad \text{for some } 0 < h < 1$$

and  $\psi : [0, 1]^5 \rightarrow [0, 1]$  defined by

$$\psi(x_1, x_2, x_3, x_4, x_5) = (\max\{x_i\})^h \quad \text{for some } h > 1.$$

Then we have

$$M(Ax, By, t) \geq \phi(x_1, x_2, x_3, x_4, x_5), \quad N(Ax, By, t) \leq \psi(x_1, x_2, x_3, x_4, x_5).$$

Therefore all the conditions of Theorem 1 are satisfied and so  $A, B, S$  and  $T$  have a unique common fixed point 0 in  $X$ .

**Corollary 2.** *Let  $(X, M, N, *, \diamond)$  be a complete intuitionistic fuzzy metric space. Let  $A, B, S$  and  $T$  be mappings from  $X$  into itself such that*

$$(2.1) \quad A(X) \subseteq T(X), B(X) \subseteq S(X),$$

(2.2) *there exists a constant  $k \in (0, \frac{1}{2})$  such that*

$$\begin{aligned} M(Ax, By, kt) &\geq [a_1(t)M(Sx, Ty, t) + a_2(t)M(Ax, Sx, t) + a_3(t)M(By, Ty, t) \\ &\quad + a_4(t)M(Ax, Ty, \alpha t) + a_5(t)M(By, Sx, (2 - \alpha)t)]^{\frac{1}{2}}, \\ N(Ax, By, kt) &\leq [a_1(t)N(Sx, Ty, t) + a_2(t)N(Ax, Sx, t) + a_3(t)N(By, Ty, t) \\ &\quad + a_4(t)N(Ax, Ty, \alpha t) + a_5(t)N(By, Sx, (2 - \alpha)t)]^{\frac{1}{2}} \end{aligned}$$

for all  $x, y \in X$ ,  $\alpha \in (0, 2)$ ,  $t > 0$  and  $a_i : R^+ \rightarrow (0, 1]$  such that  $\sum_{i=1}^5 a_i(t) = 1$ .

If the mappings  $A, B, S$  and  $T$  satisfy any one of the following conditions:

(2.3) *the pairs  $(A, S)$  and  $(B, T)$  are compatible of type (II) and  $A$  or  $B$  is continuous,*

(2.4) *the pairs  $(A, S)$  and  $(B, T)$  are compatible of type (I) and  $S$  or  $T$  is continuous. Then  $A, B, S$  and  $T$  have a unique common fixed point in  $X$ .*

*Proof.* By Theorem 1, if we define

$$\phi(x_1, x_2, x_3, x_4, x_5) \geq [a_1(t)x_1 + a_2(t)x_2 + a_3(t)x_3 + a_4(t)x_4 + a_5(t)x_5]^{\frac{1}{2}}$$

and

$$\psi(x_1, x_2, x_3, x_4, x_5) \leq [a_1(t)x_1 + a_2(t)x_2 + a_3(t)x_3 + a_4(t)x_4 + a_5(t)x_5]^{\frac{1}{2}},$$

then we have the conclusion. □

**Corollary 3.** *Let  $(X, M, N, *, \diamond)$  be a complete intuitionistic fuzzy metric space. Let  $A, B, R, S, H$  and  $T$  be mappings from  $X$  into itself such that*

$$(3.1) \quad A(X) \sqsubseteq TH(X), B(X) \sqsubseteq SR(X),$$

(3.2) *there exists a constant  $k \in (0, \frac{1}{2})$  such that*

$$M(Ax, By, kt) \geq \phi \left( \begin{array}{cc} M(SRx, THy, t), & M(Ax, SRx, t), \\ M(By, THy, t), & M(Ax, THy, \alpha t), \\ M(By, SRx, (2-\alpha)t) & \end{array} \right)$$

and

$$N(Ax, By, kt) \leq \psi \left( \begin{array}{cc} N(SRx, THy, t), & N(Ax, SRx, t), \\ N(By, THy, t), & N(Ax, THy, \alpha t), \\ N(By, SRx, (2-\alpha)t) & \end{array} \right)$$

for all  $x, y \in X, \alpha \in (0, 2), t > 0$  and  $\phi \in \Phi, \psi \in \Psi$ ,

$$(3.4) \quad TH = HT, AR = RA, BH = HB \text{ and } SR = RS.$$

*If the mappings  $A, B, SR$  and  $TH$  satisfy any one of the following conditions:*

(3.5) *the pairs  $(A, SR)$  and  $(B, TH)$  are compatible of type (II) and  $A$  or  $B$  is continuous,*

(3.6) *the pairs  $(A, SR)$  and  $(B, TH)$  are compatible of type (I) and  $SR$  or  $TH$  is continuous. Then  $A, B, R, S, H$  and  $T$  have a unique common fixed point in  $X$ .*

*Proof.* By Theorem 1,  $A, B, TH$  and  $SR$  have a unique common fixed point in  $X$ . That is there exists  $z \in X$  such that  $Az = Bz = THz = SRz = z$ . Now, we prove that  $R(z) = z$ . In fact by the condition (3.2), we have

$$M(ARz, Bz, kt) \geq \phi \left( \begin{array}{cc} M(SRRz, THz, t), & M(ARz, SRRz, t), \\ M(Bz, THz, t), & M(ARz, THz, \alpha t), \\ M(Bz, SRRz, (2-\alpha)t) & \end{array} \right)$$

and

$$N(ARz, Bz, kt) \leq \psi \left( \begin{array}{cc} N(SRRz, THz, t), & N(ARz, SRRz, t), \\ N(Bz, THz, t), & N(ARz, THz, \alpha t), \\ N(Bz, SRRz, (2-\alpha)t) & \end{array} \right).$$

For  $\alpha = 1$ , we have

$$M(Rz, z, kt) \geq \phi \left( \begin{array}{cc} M(Rz, z, t), & M(Rz, Rz, t), \\ M(z, z, t), & M(Rz, z, t), \\ M(z, Rz, t) & \end{array} \right) > M(Rz, z, t)$$

and

$$N(Rz, z, kt) \leq \psi \left( \begin{array}{cc} N(Rz, z, t), & N(Rz, Rz, t), \\ N(z, z, t), & N(Rz, z, t), \\ N(z, Rz, t) & \end{array} \right) < N(Rz, z, t)$$

which is a contradiction. Therefore, it follows that  $Rz = z$ . Hence  $Sz = SRz = z$ . Similarly, we get  $Tz = Hz = z$ .  $\square$

**Corollary 4.** Let  $(X, M, N, *, \diamond)$  be a complete intuitionistic fuzzy metric space. Let  $S$  and  $T$  be mappings from  $X$  into itself such that

(4.1) there exists a constant  $k \in (0, \frac{1}{2})$  such that

$$M(x, y, kt) \geq \phi \begin{pmatrix} M(Sx, Ty, t), & M(x, Sx, t), \\ M(y, Ty, t), & M(x, Ty, \alpha t), \\ M(y, Sx, (2-\alpha)t) \end{pmatrix}$$

and

$$N(x, y, kt) \leq \psi \begin{pmatrix} N(Sx, Ty, t), & N(x, Sx, t), \\ N(y, Ty, t), & N(x, Ty, \alpha t), \\ N(y, Sx, (2-\alpha)t) \end{pmatrix}$$

for all  $x, y \in X, \alpha \in (0, 2), t > 0$  and  $\phi \in \Phi, \psi \in \Psi$ . Then  $S$  and  $T$  have a unique common fixed point in  $X$ .

*Proof.* If we set  $A = B = I$  (the identity mapping) in Theorem 1, then it is easy to check that the pairs  $(I, S)$  and  $(I, T)$  are compatible of type (II) and the identity mapping  $I$  is continuous. Hence by Theorem 1,  $T$  and  $S$  have a unique common fixed point in  $X$ .  $\square$

**Corollary 5.** Let  $(X, M, N, *, \diamond)$  be a complete intuitionistic fuzzy metric space. Let  $A$  and  $S$  be mappings from  $X$  into itself such that

$$(5.1) \quad A^n(X) \sqsubseteq S^m(X),$$

(5.2) there exists a constant  $k \in (0, 1)$  such that

$$M(A^n x, A^n y, kt) \geq \phi \begin{pmatrix} M(S^m x, S^m y, t), & M(A^n x, S^m x, t), \\ M(A^n y, S^m y, t), & M(A^n x, S^m y, \alpha t), \\ M(A^n y, S^m x, (2-\alpha)t) \end{pmatrix}$$

and

$$N(A^n x, A^n y, kt) \leq \psi \begin{pmatrix} N(S^m x, S^m y, t), & N(A^n x, S^m x, t), \\ N(A^n y, S^m y, t), & N(A^n x, S^m y, \alpha t), \\ N(A^n y, S^m x, (2-\alpha)t) \end{pmatrix}$$

for all  $x, y \in X, \alpha \in (0, 2), t > 0, \phi \in \Phi, \psi \in \Psi$  and for some  $m, n \in \mathbb{N}$ .

$$(5.3) \quad A^n S = S A^n \quad \text{and} \quad A S^m = S^m A.$$

If the mappings  $A^n$  and  $S^m$  satisfy any one of the following conditions:

(5.4) the pairs  $(A^n, S^m)$  is compatible of type (II) and  $A^n$  is continuous,

(5.5) the pairs  $(A^n, S^m)$  is compatible of type (I) and  $S^m$  is continuous. Then  $A$  and  $S$  have a unique common fixed point in  $X$ .

*Proof.* If we set  $A = B = A^n$  and  $S = T = S^m$  in Theorem 1, then  $A^n$  and  $S^m$  have a unique common fixed point in  $X$ . That is there exists  $z \in X$  such that  $A^n(z) = A S^m(z) = z$ . Since  $A^n(Az) = A(A^n z) = Az$ , and  $S^m(Az) = A(S^m z) = Az$ . It follows that  $Az$  is a fixed point of  $A^n$  and  $S^m$  and hence  $Az = z$ . Similarly we have  $Sz = z$ .  $\square$

**Corollary 6.** Let  $(X, M, N, *, \diamond)$  be a complete intuitionistic fuzzy metric space. Let  $S, T$  and two sequences  $\{A_i\}, \{B_j\}$  for all  $i, j \in N$  be mappings from  $X$  into itself such that

(6.1) there exists  $i_0, j_0 \in N$  such that  $A_{i_0}(X) \subseteq T(X), B_{j_0}(X) \subseteq S(X)$ ,

(6.2) there exists a constant  $k \in (0, \frac{1}{2})$  such that

$$M(A_i x, B_j y, kt) \geq \phi \left( \begin{array}{cc} M(Sx, Ty, t), & M(A_i x, Sx, t), \\ M(B_j y, Ty, t), & M(A_i x, Ty, \alpha t), \\ M(B_j y, Sx, (2-\alpha)t) & \end{array} \right)$$

and

$$N(A_i x, B_j y, kt) \leq \psi \left( \begin{array}{cc} N(Sx, Ty, t), & N(A_i x, Sx, t), \\ N(B_j y, Ty, t), & N(A_i x, Ty, \alpha t), \\ N(B_j y, Sx, (2-\alpha)t) & \end{array} \right)$$

for all  $x, y \in X$ ,  $\alpha \in (0, 2)$ ,  $t > 0$  and  $\phi \in \Phi, \psi \in \Psi$ .

If the mappings  $A_{i_0}, B_{j_0}, S$  and  $T$  satisfy any one of the following conditions:

(6.3) the pairs  $(A_{i_0}, S)$  and  $(B_{j_0}, T)$  are compatible of type (II) and  $A_{i_0}$  or  $B_{j_0}$  is continuous,

(6.4) the pairs  $(A_{i_0}, S)$  and  $(B_{j_0}, T)$  are compatible of type (I) and  $S$  or  $T$  is continuous. Then  $A_i, B_j, S$  and  $T$  have a unique common fixed point in  $X$  for  $i = j = 1, 2, \dots$

*Proof.* By Theorem 1, the mappings  $S, T$  and  $A_{i_0}$  and  $B_{j_0}$  for some  $i_0, j_0 \in N$  have a unique common fixed point in  $X$ . That is there exists a unique point  $z \in X$  such that  $S(z) = T(z) = A_{i_0}(z) = B_{j_0}(z) = z$ .

Suppose that there exists  $i \in N$  such that  $i \neq j_0$ . Then by (6.2), with  $\alpha = 1$ , we have

$$\begin{aligned} M(A_i z, z, kt) &= M(A_i z, B_{j_0} z, kt) \\ &\geq \phi \left( \begin{array}{cc} M(Sz, Tz, t), & M(A_i z, Sz, t), \\ M(B_{j_0} z, Tz, t), & M(A_i z, Tz, t), \\ M(B_{j_0} z, Sz, t) & \end{array} \right) > M(A_i z, z, kt) \end{aligned}$$

and

$$\begin{aligned} N(A_i z, z, kt) &= N(A_i z, B_{j_0} z, kt) \\ &\leq \psi \left( \begin{array}{cc} N(Sz, Tz, t), & N(A_i z, Sz, t), \\ N(B_{j_0} z, Tz, t), & N(A_i z, Tz, t), \\ N(B_{j_0} z, Sz, t) & \end{array} \right) < N(A_i z, z, kt), \end{aligned}$$

which is a contradiction. Hence for all  $i \in N$ , it follows that  $A_i(z) = z$ . Similarly for all  $j \in N$ , we have  $B_j(z) = z$ . Therefore for all  $i, j \in N$ , we have  $A_i(z) = B_j(z) = S(z) = T(z) = z$ .  $\square$

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