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FATOU THEOREM AND EMBEDDING THEOREMS FOR THE MEAN LIPSCHITZ FUNCTIONS ON THE UNIT BALL

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Abstract. We investigate the boundary values of the holomorphic mean Lipschitz function. In fact, we prove that the admissible limit exists at every boundary point of the unit ball for the holomorphic mean Lipschitz functions under some assumptions on the Lipschitz order. Moreover, we get embedding theorems of holomorphic mean Lipschitz spaces into Hardy spaces or into the Bloch space on the unit ball in $\mathbb{C}^n$.

1. Introduction and results

The purpose of this paper is to study the boundary values and embedding theorems for the holomorphic mean Lipschitz functions on the unit ball. In fact, we prove that the admissible limit exists at every boundary point of the unit ball for the holomorphic mean Lipschitz functions under some assumptions on the Lipschitz order. Moreover, we get embedding theorems of holomorphic mean Lipschitz spaces into Hardy spaces or into the Bloch space on the unit ball in $\mathbb{C}^n$.

Let $\mathbf{B}$ be the unit ball in $\mathbb{C}^n$. Let $\mathbf{S}$ be the boundary of $\mathbf{B}$. Let σ denote the surface area measure on $\mathbf{S}$ normalized to be $\sigma(\mathbf{S}) = 1$. If $0 < r < 1$ and f is a holomorphic function in $\mathbf{B}$, we define

$$M_p(r, f) = \Big(\int_{\mathbf{S}} |f(r\zeta)|^p d\sigma(\zeta) \Big)^{1/p}, \quad 1 \le p < \infty,$$
$$M_\infty(r, f) = \sup\{|f(r\zeta)| : \zeta \in \mathbf{S}\}.$$

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For $1 \le p \le \infty$ the Hardy space $H^p(\mathbf{B})$ consists of those function f, holomorphic in $\mathbf{B}$, for which

$$||f||_{H^p(\mathbf{B})} := \sup_{0<r<1} M_p(r,f) < \infty.$$

If f is a function in $\mathbf{B}$ and has a boundary value at almost everywhere on $\mathbf{S}$, we define the L^p modulus of continuity of f as following

$$\omega_p(t,f) = \sup_{||U-I||\le t} \Big(\int_{\mathbf{S}} |f(U\zeta) - f(\zeta)|^p d\sigma(\zeta) \Big)^{1/p}, \quad t > 0, \quad 1 \le p < \infty,$$
$$\omega_\infty(t,f) = \sup_{||U-I||\le t} \operatorname*{ess\,sup}_{\zeta\in\mathbf{S}} |f(U\zeta) - f(\zeta)|, \quad t > 0,$$

where U is a unitary operator and I the identity operator of $\mathbb{C}^n$. In [3], authors introduce the definition of the mean Lipschitz space on the unit ball in $\mathbb{C}^n$. We think that this is the first definition of the mean Lipschitz space on the unit ball in $\mathbb{C}^n$.

Definition 1.1. For $0 < \alpha < 1$ and $1 \le p \le \infty$, we say that $f \in \Lambda^p_\alpha(\mathbf{S})$ if $f \in H^p(\mathbf{B})$ and

$$\|f\|_{\Lambda^p_\alpha(\mathbf{S})} := \|f\|_{H^p(\mathbf{B})} + \sup_{0<t<1} \frac{\omega_p(t,f)}{t^\alpha} < \infty.$$

When $p = \infty$, we write $\Lambda_\alpha(\mathbf{S})$.

The approach region $D_\theta(\zeta)$ is defined for $\theta > 1, \zeta \in \mathbf{S}$

$$D_\theta(\zeta) = \Big\{ z \in \mathbf{B} : |1 - \langle z, \zeta\rangle| < \frac{\theta}{2}(1-|z|^2) \Big\}.$$

It is defined that f has an admissible limit at $\zeta \in \mathbf{S}$ if $f(z)$ has a limit as z approaches ζ through $D_\theta(\zeta)$ for all $\theta > 1$, i.e., there exists

$$\lim_{D_\theta(\zeta)\ni z\to\zeta} |f(z)|.$$

For any function f defined in $\mathbf{B}$ we define the exceptional set $E(f)$ by the set of all $\zeta \in \mathbf{S}$ such that f fails to have an admissible limit at ζ. In [1], Ahern-Cohn studied exceptional sets for Hardy-Sobolev functions.

We want to find the condition on the Lipschitz order α such that exceptional set $E(f)$ is empty.

Theorem 1.2. *Let $1 < p \le \infty$ and $0 < \alpha < 1$. Then $E(f) = \emptyset$ for all $f \in \Lambda^p_\alpha(\mathbf{S})$ if and only if $n/p < \alpha$.*

We introduce the Hardy-Littlewood type characterization of the mean Lipschitz function that we need in the sequel. Let $\mathcal{R}f$ be the radial derivative of holomorphic functions f in $\mathbf{B}$ defined by

$$\mathcal{R}f = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j} f.$$

Let

$$HL^p_\alpha(f) = \sup_{0<r<1}(1-r)^{1-\alpha} M_p(r, \mathcal{R}f),$$

where HL means the Hardy-Littlewood quantity.

Theorem 1.3 ([3]). *Let $0 < \alpha < 1$ and $1 \le p < \infty$. For $f \in \Lambda^p_\alpha(\mathbf{S})$ we have*

$$\|f\|_{\Lambda^p_\alpha(\mathbf{S})} \simeq \|f\|_{H^p} + HL^p_\alpha(f).$$

The Bloch space $\mathfrak{B}(\mathbf{B})$ is consisting of the holomorphic functions such that

$$\sup_{z\in\mathbf{B}}(1-|z|^2)|\mathcal{R}f(z)| < \infty.$$

We prove the following results.

Theorem 1.4. *Let $1 < p < q < \infty$ and $0 < \alpha < 1$. Then we have*

(i) $\Lambda^p_\alpha(\mathbf{S}) \subset H^q(\mathbf{B})$ *if and only if* $n(1/p - 1/q) < \alpha$.
(ii) $\Lambda^p_\alpha(\mathbf{S}) \subset \Lambda_{\alpha-n/p}(\mathbf{S})$ *if and only if* $n/p < \alpha$.
(iii) $\Lambda^p_\alpha(\mathbf{S}) \subset \mathfrak{B}(\mathbf{B})$ *if and only if* $n/p \le \alpha$.

In the case of $n = 1$, the embedding theorems were proved in ([2], [4]).

We can compare the above results with those of the Besov spaces. We define $A^{p,\alpha}(\mathbf{B})$ the space of all holomorphic functions f on $\mathbf{B}$ satisfying

$$\|f\|_{A^{p,\alpha}} = \|f\|_{A^p} + \left(\int_{\mathbf{B}} |\mathcal{R}f(z)|^p (1-|z|^2)^{(1-\alpha)p} dV(z)\right)^{1/p}.$$

Let $1 < p < q < \infty$ and $0 < \alpha < 1$. Then we have

(i) $A^{p,\alpha}(\mathbf{B}) \subset A^q(\mathbf{B})$ if and only if $(n+1)(1/p - 1/q) < \alpha$.
(ii) $A^{p,\alpha}(\mathbf{B}) \subset \Lambda_{\alpha-(n+1)/p}(\mathbf{B})$ if and only if $(n+1)/p < \alpha$.
(iii) $A^{p,\alpha}(\mathbf{B}) \subset \mathfrak{B}(\mathbf{B})$ if and only if $(n+1)/p \le \alpha$.

2. Proof of Theorem 1.2

We first assume that $f \in \Lambda^p_\alpha(\mathbf{S})$, where $n/p < \alpha$. For $z \in \mathbf{B}$ we have

$$f(z) - f(0) = \int_0^1 \mathcal{R}f(tz)\frac{dt}{t}. \tag{2.1}$$

Since $|\mathcal{R}f(tz)| \lesssim t|\nabla f(tz)|$, by Cauchy integral formula, we have

$$|\mathcal{R}f(tz)| \lesssim t \sup_{|z|\le 2/3} |f(z)| \quad \text{for} \quad 0 < t < 1/2$$

so that

$$\begin{aligned} \left|\int_0^{1/2} \mathcal{R}f(tz)\frac{dt}{t}\right| &\lesssim \int_0^{1/2} |\mathcal{R}f(tz)|\frac{dt}{t} \\ &\lesssim \int_0^{1/2} t \sup_{|w|\le 2/3}|f(w)|\frac{dt}{t} \qquad (2.2) \\ &\lesssim \sup_{|w|\le 2/3}|f(w)|. \end{aligned}$$

From (2.1) and (2.2) we have that

$$|f(z)| \lesssim \sup_{|w|\le 2/3} |f(w)| + \int_{1/2}^{1} |\mathcal{R}f(tz)|dt. \tag{2.3}$$

From (2.3) for $\zeta \in \mathbf{S}$ we have

$$\lim_{D_\theta(\zeta)\ni z\to\zeta} |f(z)| \lesssim \sup_{|w|\le 2/3} |f(w)| + \lim_{D_\theta(\zeta)\ni z\to\zeta} \int_{1/2}^{1} |\mathcal{R}f(tz)|dt.$$

Since $\mathcal{R}f$ is holomorphic, we have the following Cauchy integral formula (see [6])

$$\begin{aligned} &\mathcal{R}f(z) \\ &= \frac{1}{(2\pi i)^n} \int_{|\zeta|=\rho} \frac{\mathcal{R}f(\zeta)\left(\sum_{j=1}^n \bar{\zeta}_j d\zeta_j\right) \wedge \left(\sum_{j=1}^n d\bar{\zeta}_j \wedge d\zeta_j\right)^{n-1}}{(\rho^2 - \langle z,\zeta\rangle)^n}, \ |z| < \rho. \end{aligned} \tag{2.4}$$

By (2.4) and Hölder's inequality, it follows that

$$|\mathcal{R}f(tz)| \lesssim M_p(\rho, \mathcal{R}f) \frac{1}{(\rho - t|z|)^{n/p}}, \quad t|z| < \rho.$$

Take $\rho = (1 + t|z|)/2$. Then

$$|\mathcal{R}f(tz)| \lesssim \frac{(1-\rho)^{-1+\alpha}}{(\rho - t|z|)^{n/p}} \lesssim (1 - t|z|)^{-1+\alpha-n/p}.$$

Thus we have

$$\begin{aligned} \int_{1/2}^{1} |\mathcal{R}f(tz)|dt &\lesssim \int_{1/2}^{1} \frac{dt}{(1-t|z|)^{1-\alpha+n/p}} \\ &\lesssim \int_{1/2}^{1} \frac{dt}{(1-t)^{1-\alpha+n/p}} \\ &\lesssim 1, \end{aligned}$$

since $n/p < \alpha$. Thus f has a admissible limit at $\zeta \in \mathbf{S}$ and so that $E(f) = \emptyset$.

Set

$$f(z) = \int_0^1 \frac{1}{t^{1-\alpha}(1+t-z_1)^{n/p}} \ dt, \quad z \in \mathbf{B}.$$

Then we have

$$\mathcal{R}f(z) = \frac{n}{p} z_1 \int_0^1 \frac{1}{t^{1-\alpha}(1+t-z_1)^{1+n/p}} \ dt, \quad z \in \mathbf{B}.$$

It is clear that f is holomorphic in $\mathbf{B}$. By Minkowski's inequality, we have

$$\begin{aligned} M_p(r, \mathcal{R}f) &= \left(\int_{\mathbf{S}} |\mathcal{R}f(r\zeta)|^p \, d\sigma(\zeta) \right)^{1/p} \\ &\lesssim \left(\int_{\mathbf{S}} \left(\int_0^1 \frac{1}{|1+t-r\zeta_1|^{1+n/p}} \frac{dt}{t^{1-\alpha}} \right)^p \, d\sigma(\zeta) \right)^{1/p} \\ &\lesssim \int_0^1 \left(\int_{\mathbf{S}} \frac{1}{|1+t-r\zeta_1|^{p+n}} \, d\sigma(\zeta) \right)^{1/p} \frac{1}{t^{1-\alpha}} \, dt. \end{aligned}$$

We calculate

$$\begin{aligned} \int_{\mathbf{S}} \frac{d\sigma(\zeta)}{|1+t-r\zeta_1|^{p+n}} &\lesssim \frac{1}{(1+t)^{p+n}} \int_{\mathbf{S}} \frac{d\sigma(\zeta)}{|1-\langle r/(1+t)\vec{e_1}, \zeta\rangle|^{p+n}} \\ &\lesssim \frac{1}{(1+t)^{p+n}} \frac{1}{(1-r/(1+t))^p} \\ &\lesssim \frac{1}{(1+t-r)^p}, \end{aligned}$$

where $\vec{e_1} = (1, 0, \ldots, 0)$. Thus we have

$$\begin{aligned} M_p(r, \mathcal{R}f) &\lesssim \int_0^1 \frac{1}{t^{1-\alpha}(1+t-r)} \, dt \\ &= \int_0^{1-r} \frac{1}{t^{1-\alpha}(1+t-r)} \, dt + \int_{1-r}^1 \frac{1}{t^{1-\alpha}(1+t-r)} \, dt. \end{aligned}$$

For the first term we have

$$\int_0^{1-r} \frac{1}{t^{1-\alpha}(1+t-r)} \, dt \le \frac{1}{1-r} \int_0^{1-r} \frac{1}{t^{1-\alpha}} \, dt \lesssim \frac{1}{(1-r)^{1-\alpha}}. \tag{2.5}$$

On the other hand, for the second term we have

$$\int_{1-r}^1 \frac{1}{t^{1-\alpha}(1+t-r)} \, dt \lesssim \int_{1-r}^1 \frac{1}{t^{2-\alpha}} \, dt \lesssim \frac{1}{(1-r)^{1-\alpha}}. \tag{2.6}$$

Then (2.5) and (2.6) imply

$$M_p(r, \mathcal{R}f) \lesssim \frac{1}{(1-r)^{1-\alpha}},$$

and it implies that $f \in \Lambda^p_\alpha(\mathbf{S})$.

For $1/2 < r < 1$ we have

$$\begin{aligned}\mathcal{R}f(r,0,\dots,0) &= \frac{n}{p} r \int_0^1 \frac{dt}{t^{1-\alpha}(1+t-r)^{1+n/p}} \\ &\gtrsim \int_{1-r}^{2(1-r)} \frac{1}{t^{1-\alpha}(1+t-r)^{1+n/p}}\, dt \\ &\gtrsim \frac{1}{(1-r)^{1+n/p}} \int_{1-r}^{2(1-r)} \frac{dt}{t^{1-\alpha}} \\ &\gtrsim (1-r)^{\alpha-(1+n/p)}.\end{aligned}$$

By Fatou's lemma, we have

$$\begin{aligned}\liminf_{r\to 1} f(r,0,\dots,0)) &= f(0) + \liminf_{r\to 1} \int_0^1 \mathcal{R}f(sr,0,\dots,0)\frac{ds}{s} \\ &\geq f(0) + \int_0^1 \liminf_{r\to 1} \mathcal{R}f(sr,0,\dots,0)\frac{ds}{s} \\ &\gtrsim f(0) + \int_0^1 \frac{ds}{(1-s)^{1+(n/p-\alpha)}}\frac{ds}{s}.\end{aligned}$$

This implies that

$$\lim_{D_\theta(\zeta)\ni z\to(1,0,\dots,0)} |f(z)| = \infty \quad \text{for all} \quad \theta > 1.$$

Thus $(1,0,\dots,0) \in E(f)$ and it is contradiction.

3. Proof of Theorem 1.4

For the proof of Theorem 1.4 we need a generalization of the Fejér-Riesz inequality as follows.

Lemma 3.1 ([5])**.** *Let $L_{j,k} = \mathbb{R}^j \times \mathbb{C}^k \times \{0\} \times \cdots \times \{0\} \subset \mathbb{C}^n$, where $1 \leq j \leq n$ and $1 \leq j+k \leq n$. For $0 < p < \infty$ we have*

$$\int_{\mathbf{B}\cap L_{j,k}} |f(z)|^p (1-|z|^2)^{n-1/2(j+2k+1)} dz \lesssim \|f\|_p^p,$$

where dz is the Lebesgue measure on $L_{j,k}$.

Proof of Theorem 1.4. (i) We first assume that $\Lambda_\alpha^p(\mathbf{S}) \subset H^q(\mathbf{B})$. Let

$$f(z) = \int_0^1 \frac{1}{t^{1-\alpha}(1+t-z_1)^{n/p}}\, dt, \quad z = (z_1, z_2, \dots, z_n) \in \mathbf{B}. \tag{3.1}$$

Since $f \in \Lambda_\alpha^p(\mathbf{S})$, $f \in H^q(\mathbf{B})$ by assumption. Applying Lemma 3.1, we have that

$$\int_0^1 |f(x)|^q (1-x^2)^{n-1}\, dx \lesssim ||f||_{H^q}^q < \infty.$$

On the other hand, we obtain

$$\int_0^1 |f(x)|^q (1-x^2)^{n-1}\,dx = \int_0^1 \Big(\int_0^1 \frac{1}{t^{1-\alpha}(1+t-x)^{n/p}}\,dt\Big)^q (1-x^2)^{n-1}\,dx$$
$$\gtrsim \int_0^1 \Big(\int_0^1 \frac{1}{t^{1-\alpha}(s+t)^{n/p}}\,dt\Big)^q s^{n-1}\,ds$$

by putting $1-x=s$. We know that

$$\int_0^1 \frac{dt}{t^{1-\alpha}(s+t)^{n/p}} \ge \int_0^1 \frac{dt}{(s+t)^{1-\alpha+n/p}} \gtrsim s^{\alpha-n/p}.$$

Thus we have

$$\int_0^1 |f(x)|^q (1-x^2)^{n-1}\,dx \gtrsim \int_0^1 s^{(\alpha-n/p)q+n-1}\,ds$$

and so that $s^{(\alpha-n/p)q+n-1}$ is integrable on $(0,1)$. Thus $n(1/p-1/q)<\alpha$.

Conversely, let us suppose that $n(1/p-1/q)<\alpha$. Let $f\in\Lambda^p_\alpha(\mathbf{S})$. By (2.4) and Hölder's inequality, it follows that

$$|\mathcal{R}f(z)| \lesssim M_p(\rho,\mathcal{R}f)\,\frac{1}{(\rho^2-|z|^2)^{n/p}},\quad |z|<\rho. \tag{3.2}$$

For $0<r<1$ we take $\rho=(1+r)/2$. From (3.2) we have

$$\begin{aligned} M_\infty(r,\mathcal{R}f) &\lesssim M_p(\rho,\mathcal{R}f)(1-r)^{-n/p}\\ &\lesssim \|f\|_{\Lambda^p_\alpha(\mathbf{S})}(1-\rho)^{-1+\alpha}(1-r)^{-n/p}\\ &\lesssim \|f\|_{\Lambda^p_\alpha(\mathbf{S})}(1-r)^{-1+(\alpha-n/p)}. \end{aligned}$$

Thus we obtain

$$\begin{aligned} M_q(r,\mathcal{R}f) &\lesssim \left(\int_{|\zeta|=r} |\mathcal{R}f(\zeta)|^p |\mathcal{R}f(\zeta)|^{q-p} d\sigma\right)^{1/q}\\ &\lesssim (M_\infty(r,\mathcal{R}f))^{1-p/q}(M_p(r,\mathcal{R}f))^{q/p}\\ &\lesssim \|f\|_{\Lambda^p_\alpha(\mathbf{S})}(1-r)^{n(1/q-1/p)-1+\alpha}. \end{aligned}$$

We note that

$$f(\zeta)-f(0) = \int_0^1 \mathcal{R}f(r\zeta)\frac{dr}{r}. \tag{3.3}$$

Since $|\mathcal{R}f(r\zeta)| \lesssim r|\nabla f(r\zeta)|$ by Cauchy integral formula, we have

$$|\mathcal{R}f(r\zeta)| \lesssim r \sup_{|z|\le 2/3} |f(z)| \quad \text{for} \quad 0<r<1/2$$

so that

$$\left|\int_0^{1/2} \mathcal{R}f(r\zeta)\frac{dr}{r}\right| \lesssim \int_0^{1/2} |\mathcal{R}f(r\zeta)|\frac{dr}{r} \tag{3.4}$$
$$\lesssim \sup_{|z|\le 2/3} |f(z)|.$$

From (3.3) and (3.4) we have that

$$|f(\zeta)| \lesssim \sup_{|z|\le 2/3} |f(z)| + \int_{1/2}^1 |\mathcal{R}f(r\zeta)|dr.$$

By Minkowski's inequality, it follows that

$$\begin{aligned}\left(\int_{\mathbf{S}} |f(\zeta)|^q d\sigma(\zeta)\right)^{1/q} &\lesssim \sup_{|z|\le 2/3} |f(z)| + \left(\int_{\mathbf{S}} \left(\int_{1/2}^1 |\mathcal{R}f(r\zeta)|dr\right)^q d\sigma(\zeta)\right)^{1/q} \\ &\lesssim \sup_{|z|\le 2/3} |f(z)| + \int_{1/2}^1 M_q(r, \mathcal{R}f)dr \\ &\lesssim \|f\|_{H^p(\mathbf{B})} + \|f\|_{\Lambda^p_\alpha(\mathbf{S})} \int_{1/2}^1 (1-r)^{n(1/q-1/p)-1+\alpha} dr.\end{aligned}$$

Note that $n(1/p - 1/q) < \alpha$ and so that $n(1/q - 1/p) - 1 + \alpha > -1$. Thus the integral

$$\int_{1/2}^1 (1-r)^{n(1/q-1/p)-1+\alpha} dr$$

converges and so it follows that $\|f\|_{H^q(\mathbf{B})} \lesssim \|f\|_{\Lambda^p_\alpha(\mathbf{S})}$. Hence we get (i).

Now we prove (ii) and (iii). Let $0 < \rho < 1$. Let $f \in \Lambda^p_\alpha$. By (3.2), we have

$$|\mathcal{R}f(z)| \lesssim M_p(\rho, \mathcal{R}f)\ \frac{1}{(\rho^2 - |z|^2)^{n/p}}.$$

Take $\rho = (1 + |z|)/2$. Then we obtain

$$\begin{aligned}|\mathcal{R}f(z)| &\lesssim M_p(\rho, \mathcal{R}f)\ (1 - |z|^2)^{-n/p} \\ &\lesssim (1-\rho)^{-1+\alpha}\ (1 - |z|^2)^{-n/p} \\ &\lesssim (1 - |z|^2)^{-1+(\alpha-n/p)}.\end{aligned}$$

Therefore $f \in \Lambda_{\alpha-n/p}$ if $n/p < \alpha$ by Theorem 1.3.

In particular, if $\alpha = n/p$, then $f \in \mathfrak{B}(\mathbf{B})$.

Conversely, for $\alpha < n/p$, it is enough to find a function $f \in \Lambda^p_\alpha(\mathbf{S})$ which is not a Bloch function. Taking the same function as (3.1), we have shown that the function f is in $\Lambda^p_\alpha(\mathbf{S})$ and

$$\mathcal{R}f(r, 0, \ldots, 0) \gtrsim (1-r)^{\alpha-(1+n/p)}$$

for $1/2 < r < 1$. Hence

$$(1-r)|\mathcal{R}f(r, 0, \ldots, 0)| \gtrsim (1-r)^{\alpha-n/p} \to \infty \quad \text{as} \quad r \to 1,$$

and it implies that $f \notin \mathfrak{B}(\mathbf{B})$. This completes the proof of Theorem 1.4. $\square$

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