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A GENERALIZED IDEAL BASED-ZERO DIVISOR GRAPHS OF NEAR-RINGS

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Abstract. In this paper, we introduce the generalized ideal-based zero-divisor graph structure of near-ring N, denoted by $\widehat{\Gamma_I(N)}$. It is shown that if I is a completely reflexive ideal of N, then every two vertices in $\widehat{\Gamma_I(N)}$ are connected by a path of length at most 3, and if $\widehat{\Gamma_I(N)}$ contains a cycle, then the core K of $\widehat{\Gamma_I(N)}$ is a union of triangles and rectangles. We have shown that if $\widehat{\Gamma_I(N)}$ is a bipartite graph for a completely semiprime ideal I of N, then N has two prime ideals whose intersection is I.

1. Preliminaries

Throughout this paper, N denotes a zero-symmetric near-ring not necessarily with identity unless otherwise stated. For any vertices x, y in a graph G, if x and y are adjacent, we denote it as $x \approx y$. In [3], Beck introduced the concept of a zero-divisor graph of a commutative ring with identity, but this work was mostly concerned with coloring of rings. In [2], Anderson and Livingston associate to a commutative ring with identity a (simple) graph $\Gamma(R)$, whose vertex set is $Z(R)^* = Z(R)\backslash\{0\}$, the set of nonzero-divisor of R, in which two distinct $x, y \in Z(R)^*$ are joined by an edge if and only if $xy = 0$. They investigated the interplay between the ring-theoretic properties of R and the graph-theoretics properties of $\Gamma(R)$. The zero-divisor graph has also been introduced and studied for semigroups by DeMeyer et al. in [7].

In [11], Redmond has generalized the notion of the zero-divisor graph. For a given ideal I of a commutative ring R, he defined an undirected graph $\Gamma_I(R)$ with vertices $\{x \in R\backslash I : xy \in I \text{ for some } y \in R\backslash I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. In [8], Dheena and Elavarasan extended this graph structure to near-rings. Following [8], let I be a completely reflexive ideal (i.e., $ab \in I$ implies $ba \in I$ for $a, b \in N$) of N. Then the ideal-based zero-divisor graph, denoted by $\Gamma_I(N)$, is the graph whose vertices are the set

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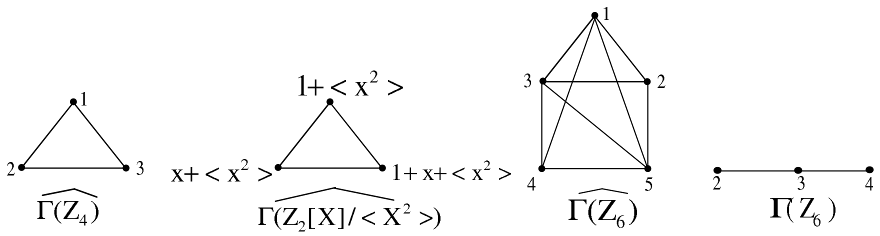
$\{x \in N \backslash I : xy \in I$ for some $y \in N \backslash I\}$ with distinct vertices x and y are adjacent if and only if $xy \in I$.

In this paper, we define a generalized ideal-based zero-divisor graph structure of the near-ring N. Let N be a near-ring and I be a completely reflexive ideal of N. We define an undirected graph $\widehat{\Gamma_I(N)}$ with vertices $\{x \in N \backslash I$: there exists $y \in N \backslash I$ such that $x_1 y_1 \in I$ for some $x_1 \in \langle x \rangle \backslash I$ and $y_1 \in \langle y \rangle \backslash I\}$, where distinct vertices x and y are adjacent if and only if $x_1 y_1 \in I$ for some $x_1 \in \langle x \rangle \backslash I$ and $y_1 \in \langle y \rangle \backslash I$, where $\langle x \rangle$ denotes the ideal of N generated by x.

Clearly $\Gamma_I(N)$ is a induced subgraph of $\widehat{\Gamma_I(N)}$, and if $I = \{0\}$, then $\widehat{\Gamma_I(N)}$ will be denoted simply by $\widehat{\Gamma(N)}$. Also, $\widehat{\Gamma_I(N)} = \phi$ if and only if I is a prime ideal of N. That is, $V(\widehat{\Gamma_I(N)}) = \phi$ if and only if $V(\widehat{\Gamma(N/I)}) = \phi$. Observe that $|V(\Gamma_I(N))| = 0$ if and only if $|V(\widehat{\Gamma_I(N)})| = 0$. Also $|V(\Gamma_I(N))| \leq |V(\widehat{\Gamma_I(N)})|$.

Example 1.1. Below are the generalized zero-divisor graphs for several near-rings. Note that these examples show that the graph structures $\Gamma_I(N)$ and $\widehat{\Gamma_I(N)}$ are not isomorphic and non-isomorphic near-rings may have the isomorphic generalized zero-divisor graph.



Given a graph G, for distinct vertices x and y of G, let $d(x, y)$ be the length of the shortest path from x to y. The diameter of a connected graph is the supremum of the distances between vertices. The core K of G is the union of all cycles of G. From [8], for any subset S and ideal I of N, we define $I_S = \{n \in N : nS \subseteq I\}$. If $S = \{a\}$, then we denote $I_{\{a\}}$ by I_a. In this paper the notations of graph theory are from [5], the notations of near-ring are from [10].

2. Main results

Theorem 2.1. *Let I be a completely reflexive ideal of N. Then $\Gamma_I(N)$ is a connected graph and $diam(\Gamma_I(N)) \leq 3$.*

Proof. The sketch of this proof follows in the similar manner to the proof of Theorem 2.4 of [11]. □

Lemma 2.2. *Let I be a completely reflexive ideal of N. For any $x, y \in \widehat{\Gamma_I(N)}$, if $x \approx y$ is an edge in $\widehat{\Gamma_I(N)}$, then for each $n \in N \backslash I$, either $n \approx y$ or $x \approx y'$ is an edge in $\widehat{\Gamma_I(N)}$ for some $y' \in \langle y \rangle \backslash I$.*

Proof. Let $x, y \in N \backslash I$ with $x \approx y$ be an edge in $\widehat{\Gamma_I(N)}$ and suppose that $n \approx y$ is not an edge in $\widehat{\Gamma_I(N)}$ for some $n \in N \backslash I$. Then $x_1 y_1 \in I$ for some $x_1 \in \langle x \rangle \backslash I; y_1 \in \langle y \rangle \backslash I$ and $n y_1 \notin I$. But $(n y_1) x_1 \in I$. So $x \approx y^{'}$ is an edge in $\widehat{\Gamma_I(N)}$ for some $y^{'} \in \langle y \rangle \backslash I$. □

Theorem 2.3. *Let I be a completely reflexive ideal of N. Then $\widehat{\Gamma_I(N)}$ is connected graph with $diam(\widehat{\Gamma_I(N)}) \leq 3$.*

Proof. Let $x, y \in \widehat{\Gamma_I(N)}$. If $x_1 y_1 \in I$ for some $x_1 \in \langle x \rangle \backslash I$ and $y_1 \in \langle y \rangle \backslash I$, then $d(x, y) = 1$. Let us assume that $x_1 y_1 \notin I$ for all $x_1 \in \langle x \rangle \backslash I$ and for all $y_1 \in \langle y \rangle \backslash I$. Then $x_1^2 \notin I$ and $y_1^2 \notin I$ for all $x_1 \in \langle x \rangle \backslash I$ and for all $y_1 \in \langle y \rangle \backslash I$. Since $x, y \in \widehat{\Gamma_I(N)}$, there exist $x_2 \in \langle x \rangle \backslash I; y_2 \in \langle y \rangle \backslash I$ and $a_1, b_1 \in N \backslash (I \cup \{x_2, y_2\})$ such that $a_1 x_2 \in I$ and $b_1 y_2 \in I$.

If $a_1 = b_1$, then $x \approx a_1 \approx y$ is a path of length 2. So assume that $a_1 \neq b_1$.

If $a_1 b_1 \in I$, then $x \approx a_1 \approx b_1 \approx y$ is a path of length 3. Otherwise $a_1 b_1 \notin I$. Then $\langle a_1 \rangle \cap \langle b_1 \rangle \nsubseteq I$. Now for every $d \in \langle a_1 \rangle \cap \langle b_1 \rangle \backslash (I \cup \{x_2, y_2\})$, we have $d x_2 \in \langle d \rangle \langle x_2 \rangle \subseteq \langle a_1 \rangle \langle x_2 \rangle \subseteq I$ and $d y_2 \in \langle b_1 \rangle \langle y_2 \rangle \subseteq I$. Thus $x \approx d \approx y$ is a path of length 2 and hence $\Gamma_I(N)$ is connected and $diam(\widehat{\Gamma_I(N)} \leq 3$. □

Theorem 2.4. *Let I be a completely reflexive ideal of N and if $a \approx x \approx b$ is a path in $\widehat{\Gamma_I(N)}$, then either $I \cup \{x_1\}$ is an ideal of N for some $x_1 \in \langle x \rangle \backslash I$ or $a \approx x \approx b$ is contained in a cycle of length ≤ 4.*

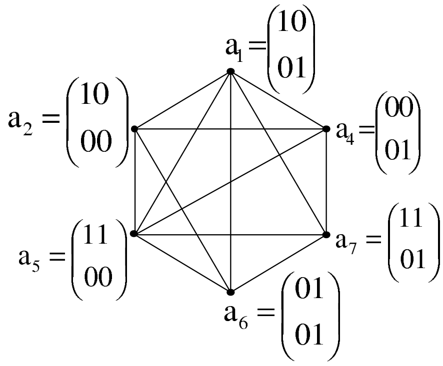
Proof. Let $a \approx x \approx b$ be a path in $\widehat{\Gamma_I(N)}$. Then there exist $x_1, x_2 \in \langle x \rangle \backslash I$;$a_1 \in \langle a \rangle \backslash I$ and $b_1 \in \langle b \rangle \backslash I$ such that $a_1 x_1 \in I$ and $b_1 x_2 \in I$. If $a^{'} b^{'} \in I$ for some $a^{'} \in \langle a \rangle \backslash I$; for some $\langle b \rangle \backslash I$, then $a \approx x \approx b \approx a$ is contained in a cycle of length ≤ 4. So let us assume that $a_1 b_1 \notin I$ for all $a_1 \in \langle a \rangle \backslash I$ and $b_1 \in \langle b \rangle \backslash I$.

Case (i) Let $x_1 = x_2$. Then either $I_{a_1} \cap I_{b_1} = I \cup \{x_1\}$ or there exists $c \in I_{a_1} \cap I_{b_1}$ such that $c \notin I \cup \{x_1\}$. Then $c a_1, c b_1 \in I$. In the first case, $I \cup \{x_1\}$ is an ideal. In the second case $a \approx x \approx b \approx c \approx a$ is contained in a cycle of length ≤ 4.

Case (ii) Let $x_1 \neq x_2$. Then clearly $\langle a_1 \rangle \cap \langle b_1 \rangle \nsubseteq I$. Then for each $z \in \langle a_1 \rangle \cap \langle b_1 \rangle \backslash I$, we have $z x_1 \in \langle a_1 \rangle \langle x_1 \rangle \subseteq I$ and $z x_2 \in I$. Clearly either $x_1 \neq x$ or $x_2 \neq x$. Say $x_1 \neq x$. Then we have a path $a \approx x_1 \approx b$ and hence $a \approx x \approx b \approx x_1 \approx a$ is contained in a cycle of length ≤ 4. □

In Theorem 2.4, the bound for the length of the cycle is sharp as the following example shows.

Example 2.5. Let $N = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$, where $F = \{0, 1\}$ is the field under addition and multiplication modulo 2. Then it's prime radical $P = \{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\}$ is a completely reflexive ideal of the near-ring N and its generalized zero-divisor graph $\widehat{\Gamma_P(N)}$ is:



It is easy to verify that $P \cup \{a\}$ is not an ideal of N for any $a \in \langle a_2 \rangle \backslash P$ and $a_4 \approx a_2 \approx a_6$ is not contained in cycle of length 3.

Corollary 2.6. *Let I be a completely reflexive ideal of N and $|V(\Gamma_I(N))| > 2$. If $I \cup \{x\}$ is not an ideal of N for any $x \in N \backslash I$, then every edge in $\widehat{\Gamma_I(N)}$ is contained in a cycle of length ≤ 4, and therefore $\widehat{\Gamma_I(N)}$ is a union of triangles and squares.*

Lemma 2.7. *Let I be a completely reflexive ideal of N. Then, $\widehat{\Gamma_I(N)}$ can be neither a pentagon nor a hexagon.*

Proof. Suppose that $\widehat{\Gamma_I(N)}$ is $a \approx b \approx c \approx d \approx e \approx a$, a pentagon. Then by Theorem 2.4, for one of the vertices (say b), $I \cup \{b_1\}$ is an ideal of N for some $b_1 \in \langle b \rangle \backslash I$. Then in the pentagon, there exist $d_1 \in \langle d \rangle \backslash I$ and $e_1 \in \langle e \rangle \backslash I$ such that $d_1 e_1 \in I$. Since $I \cup \{b_1\}$ is ideal, $b_1 d_1 = b_1 = b_1 e_1$. But $b_1(d_1 e_1) \in I$, then $b_1 \in I$, a contradiction. The proof for the hexagon is the same. □

Theorem 2.8. *Let I be a completely reflexive ideal of N. Then the following hold*:

(i) *If N has identity, then $\widehat{\Gamma_I(N)}$ has no cut-vertices.*

(ii) *If N has no identity and if I is non-zero ideal of N, then $\widehat{\Gamma_I(N)}$ has no cut-vertices.*

Proof. Suppose that the vertex x of $\widehat{\Gamma_I(N)}$ is a cut vertex. Let $u \approx x \approx w$ be a path in $\widehat{\Gamma_I(N)}$. Since x is a cut-vertex, x lies in every path from u to w.

(i) Assume that N is a near-ring with identity.

For any $u, v \in \widehat{\Gamma_I(N)}$, there exist a path $u \approx 1 \approx w$ which shows $x(\neq 1)$ in $\widehat{\Gamma_I(N)}$ is not a cut vertex. Suppose $x = 1$. Then there exist $u_1 \in \langle u \rangle \backslash I; w_1 \in \langle w \rangle \backslash I$ and $t_1, t_2 \in N \backslash I$ such that $u_1 t_1, w_1 t_2 \in I$ which implies $u_1, w_1 \in \Gamma_I(N)$. Since $\Gamma_I(N)$ is connected, there exist $n, n_1 \in N \backslash (I \cup \{x\})$ such that $u_1 \approx n \approx w_1$ or $u_1 \approx n \approx n_1 \approx w_1$ is a path in $\Gamma_I(N)$ which implies $u \approx n \approx w \approx 1 \approx u$ or $u \approx n \approx n_1 \approx w \approx 1 \approx u$ is a cycle in $\widehat{\Gamma_I(N)}$, contradicting $x = 1$ is a cut-vertex.

(ii) Let N be a near-ring without identity and I be a non-zero ideal of N.

Since $u \approx x \approx w$ is a path from u to w, then there exist $u_1 \in \langle u\rangle\backslash I; w_1 \in \langle w\rangle\backslash I$ and $x_1, x_2 \in \langle x\rangle\backslash I$ such that $u_1x_1 \in I$ and $w_1x_2 \in I$.

Case (i) $x_1 = x_2$

If $u_1+I = x_1+I$, then $u_1w_1 \in I$ which implies u is adjacent to w. Similarly, if $x_2 + I = w_1 + I$, u is adjacent to w. So assume that $u_1 + I \neq x_1 + I$ and $x_2 + I \neq w_1 + I$. Let $0 \neq i \in I$. Then $u_1x_1 \in I$ and $w_1x_2 \in I$ which imply that $u_1(x_1+i), w_1(x_1+i) \in I$. If $x = x_1+i$, then $x \neq x_1$ which implies $u \approx x_1 \approx w$ is a path in $\widehat{\Gamma_I(N)}$. Otherwise $u \approx (x_1+i) \approx w$ is a path in $\widehat{\Gamma_I(R)}$. Thus there exist a path from u to w not passing through x, a contradiction.

Case (ii) Either x_1 or x_2 equal to x.

Without loss of generality, let us assume that $x_1 = x$ and $x_2 \neq x$. Then $u_1x \in I$ and $x_2w_1 \in I$ which implies $u_1x_2 \in I$ and $x_2w_1 \in I$, and so we have a path $u \approx x_2 \approx w$, a contradiction.

Case (iii) Neither x_1 nor x_2 equal to x.

If $x_1x_2 \in I$, then we have a path $u \approx x_1 \approx x_2 \approx w$, a contradiction. Otherwise $x_1x_2 \notin I$.

If $x_1x_2 = x$, then $u_1x \in I$ and $w_1x \in I$. By sub case (i), we have a contradiction.

So assume that $x_1x_2 \neq x$, then we have a path $u \approx x_1x_2 \approx w$, a contradiction.

Thus x can not be a cut-vertex. □

From Theorem 2.8, we have the following question. If N is a near-ring without identity and $\{0\}$ is a completely reflexive ideal of N, then whether $\widehat{\Gamma(N)}$ has a cut-vertex.

Theorem 2.9. *Let I be a completely reflexive ideal of N. If $\widehat{\Gamma_I(N)}$ contains a cycle, then the core K of $\widehat{\Gamma_I(N)}$ is a union of triangles and rectangles. Moreover, any vertex in $\widehat{\Gamma_I(N)}$ is either a vertex of the core K of $\widehat{\Gamma_I(N)}$ or else is an end vertex of $\widehat{\Gamma_I(N)}$.*

Proof. Let $a \in K$ and assume that a is not in any square or rectangle in $\widehat{\Gamma_I(N)}$. Then a is part of a cycle $a \approx b \approx c \approx d \approx \cdots \approx a$ which implies $c_1d_1 \in I$ for some $c_1 \in \langle c\rangle\backslash I$ and $d_1 \in \langle d\rangle\backslash I$. Also, by Lemma 2.4, $I \cup \{a_1\}$ is an ideal of N for some $a_1 \in \langle a\rangle\backslash I$. Then $d_1a_1 = a_1 = c_1a_1$ and $a_1(d_1c_1) \in I$ which implies $a_1 \in I$, a contradiction.

For the "moreover"statement, we can assume $\left|\widehat{\Gamma_I(N)}\right| \geq 3$. If x is a vertex in $\widehat{\Gamma_I(N)}$, then one of the following is true:

1. x is in the core;
2. x is an end vertex of $\widehat{\Gamma_I(N)}$;
3. $a \approx x \approx b$ is a path in $\widehat{\Gamma_I(N)}$ where a is an end vertex and $b \in K$;

4. $a \approx x \approx y \approx b$ or $a \approx y \approx x \approx b$ is a path in $\widehat{\Gamma_I(N)}$, where a is an end vertex and $b \in K$.

In the first two cases, we are done. Let us assume that $a \approx x \approx b$ is a path with $b \in K$. Then by Lemma 2.4, $I \cup \{x_1\}$ is an ideal of N for some $x_1 \in \langle x \rangle \backslash I$ and $x \approx b \approx c \approx d \approx b$ or $x \approx b \approx c \approx d \approx e \approx b$ is a path in $\widehat{\Gamma_I(N)}$ which implies $c_1 d_1 \in I$ for some $c_1 \in \langle c \rangle \backslash I$ and $d_1 \in \langle d \rangle \backslash I$. Since $x \notin K$, we have $x_1 c_1 = x_1$ and so x is a vertex in the cycle $x \approx b \approx c \approx d \approx x$, a contradiction.

Although the proof of case 4 is just a slight modification of that for Theorem 2.4 given in [6], we include a sketch of the proof to illustrate the style.

Without loss of generality, assume $a \approx x \approx y \approx b$ is a path in $\widehat{\Gamma_I(N)}$. Since $b \in K$, there is some $c \in K$ such that $c \neq b$ and $b \approx c$ is part of a cycle. Then $a \approx x \approx y \approx b \approx c$ is a path in $\widehat{\Gamma_I(N)}$. But the distance from a to c is four, a contradiction unless $y \approx c$ or $x \approx c$ is an edge. However, if $y \approx c$ is an edge, then $y \in K$. By case 3, x is also in the core. If instead, $x \approx c$ is an edge, then $x \approx y \approx b \approx c \approx x$ is a cycle. Thus $x, y \in K$.

Hence it must be the case that any vertex x of $\widehat{\Gamma_I(N)}$ is either an end or in the core. □

Corollary 2.10. *Let I be a completely reflexive ideal of N. If N has identity with $|V(\Gamma_I(N))| \geq 2$ or if I is non-zero ideal of N with $|V(\Gamma_I(N))| > 2$, then $\widehat{\Gamma_I(N)} = K$, where K is the core of $\widehat{\Gamma_I(N)}$.*

Corollary 2.11. *Let I be a completely reflexive ideal of N with $|V(\Gamma_I(N))| > 2$. If $I \cup \{x\}$ is not an ideal of N, then every pair of vertices in $\widehat{\Gamma_I(N)}$ is contained in a cycle of length ≤ 6.*

Proof. Let a, b be vertices of $\widehat{\Gamma_I(N)}$. If $a \approx b$ is an edge in $\widehat{\Gamma_I(N)}$, then $a \approx b$ is the edge of a triangle or rectangle by Corollary 2.6. If $a \approx x \approx y$ is a path in $\widehat{\Gamma_I(N)}$, then $a \approx x \approx b$ is contained in a cycle of length ≤ 4 by Theorem 2.4. If $a \approx x \approx y \approx b$ is a path in $\widehat{\Gamma_I(N)}$, then by Lemma 2.4 we can find cycles $a \approx x \approx y \approx c \approx a$ and $b \approx y \approx x \approx d \approx b$, where $c \neq x$ and $d \neq y$. This gives a cycle $a \approx x \approx d \approx b \approx y \approx c \approx a$ of length ≤ 6. □

Recall that a bipartite graph is one whose vertex set can be partitioned into two subsets so that no edge has both ends in any one subset. We now obtain the properties of a near-ring implied by its generalized ideal-based zero-divisor graph.

Theorem 2.12. *Let I be a completely reflexive ideal of N and I is completely semiprime. If $\widehat{\Gamma_I(N)}$ is a bipartite graph, then there exist prime ideals P_1 and P_2 of N such that $I = P_1 \cap P_2$.*

Proof. Let A, B be the partition of the graph $\widehat{\Gamma_I(N)}$. Let $V_1 = \{x \in A$ is a vertex in $\widehat{\Gamma_I(N)}$ such that $xy \in I$ for some $y \in B\}$ and $V_2 = \{y \in B$ is a vertex in $\widehat{\Gamma_I(N)}$ such that $xy \in I$ for some $y \in A\}$. Observe that V_1 and V_2 are non-empty and also $V_1 \cap I = V_2 \cap I = \phi$. Let $P_1 = V_1 \cup I$ and $P_2 = V_2 \cup I$. Then $I = P_1 \cap P_2$. Let us show that P_1 is an ideal of N. Let $x_1, x_2 \in P_1$.

Case (i): If x_1 and x_2 are in I, then $x_1 - x_2 \in I \subseteq P_1$.

Case (ii): Let $x_1, x_2 \in V_1$. If $x_1 - x_2 \in I$, then $x_1 - x_2 \in P_1$. Hence let us assume that $x_1 - x_2 \notin I$. Now there exist $y_1, y_2 \in V_2$ such that $x_1y_1 \in I$ and $x_1y_2 \in I$. Hence $x_1y_1y_2 \in I$ and $x_2y_1y_2 \in I$. If $y_1y_2 \in I$, then $y_1 \approx y_2$ contradicts to the fact that no two vertices in B are adjacent. Hence $y_1y_2 \notin I$. Clearly $x_1 - x_2 \notin V_2$ and $x_1 - x_2 \neq y_1y_2$. Indeed, if $x_1 - x_2 \in V_2$ or $x_1 - x_2 = y_1y_2$, then $(x_1 - x_2)y_1y_2 \in I$ or $(y_1y_2)^2 \in I$, a contradiction since $y_1y_2 \in V_2$. Now $(x_1 - x_2)y_1y_2 \in I$ with $x_1 - x_2 \notin I, y_1y_2 \notin I$. Thus $x_1 - x_2 \in A$ and $y_1y_2 \in B$ and hence $x_1 - x_2 \in V_1 \subseteq P_1$.

Case (iii): Suppose $x_1 \in V_1$ and $x_2 \in I$. Clearly $x_1 - x_2 \notin I$. Since $x_1 \in V_1$, there exists $y_1 \in V_2$ such that $x_1y_1 \in I$. Clearly $y_1 \notin I$ as $V_2 \cap I = \phi$. Now $(x_1 - x_2)y_1 \in I$ with $x_1 - x_2 \notin I$ and $y_1 \notin I$. Thus $x_1 - x_2 \in A$ and $y_1 \in B$. Hence $x_1 - x_2 \in V_1 \subseteq P_1$. Thus P_1 is an additive subgroup of N.

Now let us show that P_1 is a normal subgroup of N. Let $x \in P_1$ and $n \in N$. If $x \in I$, then $n + x - n \in I \subseteq P_1$. Let us assume $x \in V_1$. Then there exists $y \in V_2$ such that $xy \in I$. If $n + x - n \in I$, we are done. Let us assume $n + x - n \notin I$. Now $(n + x - n)y = ny - xy - ny \in I$. Thus $n + x - n \in A$ and hence $n + x - n \in V_1 \subseteq P_1$. So P_1 is a normal subgroup of N.

Now we claim that P_1 is a right ideal of N. Let $x \in P_1$ and $n \in N$. If $x \in I$, then $xn \in I \subseteq P_1$. If $nx \in I$, we are done. So $nx \notin I$ and $x \in V_1$. Then there exists $y \in V_2$ such that $xy \in I$. Now $(nx)y \in I$ with $nx \notin I$ and $y \notin I$. Thus $nx \in A$ and hence $nx \in V_1 \subseteq P_1$. So P_1 is a right ideal of N.

Now let us show that P_1 is a left ideal of N. Let $x \in P_1$ and $n, n' \in N$. If $x \in I$, then $n(n' + x) - nn' \in I \subseteq P_1$. Let us assume that $x \in V_1$. If $n(n' + x) - nn' \in I$, then we are done. Hence let us assume $n(n' + x) - nn' \notin I$. Since $x \in V_1$, there exists $y \in V_2$ such that $xy \in I$. Clearly $y \notin I$. Now $(n(n' + x) - nn')y = n(n'y + xy) - n(n'y) \in I$ as $xy \in I$. Thus $n(n' + x) - nn' \in V_1 \subseteq P_1$ and hence P_1 is an ideal of N. So P_1 is an ideal of N. Similarly P_2 is an ideal of N.

We now show that P_1 is a prime ideal of N. Let J and K be ideals of N such that $JK \subseteq P_1$ and suppose that $J \nsubseteq P_1$. Let $j \in J$ but $j \notin P_1$. Let $k \in K$. If $k \in I$, then $k \in P_1$. Let us assume that $k \notin I$. Clearly $jk \in P_1$. If $jk \in I$, then $j \in B$ and $k \in A$ since $j \notin V_1$, and hence $k \in V_1 \subseteq P_1$. If $jk \notin I$, then $jk \in V_1$ and there exists $y \in V_2$ such that $jky \in I$. Since P_2 is an ideal, we have $jy \in V_2$, and so $k \in V_1 \subseteq P_1$. Thus P_1 is a prime ideal of N. Similarly P_2 is a prime ideal of N. □

Note that in Theorem 2.12, the converse is not true in general, as the following example shows.

Example 2.13. In $N = \mathbb{Z}_6$, $\{0\}$ is a completely reflexive ideal and completely semiprime ideal and $\mathbb{Z}_6$ has only two prime ideals, but its generalized ideal-based zero-divisor graph $\widehat{\Gamma(N)}$ is not a bipartite.

We now also show by an example that the Theorem 2.12 will fail if I is not completely semiprime.

Example 2.14. Let $(N, +)$ (where $N = \{0, a, b, c\}$) be the Klein's four group. Define multiplication in N as follows:

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	0	a

Then $(N, +, \cdot)$ is a near-ring (see Pilz [10], P-408, Scheme-14). If $I = \{0, a\}$, then I is completely reflexive, but not completely semiprime. Here $\widehat{\Gamma_I(N)}$ is a complete bipartite graph but I cannot be written as the intersection of two prime ideals.

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