First Principle Approach to Modeling of Primitive Quad Rotor

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Abstract

By the development of recent technology, a new variant of rotorcrafts having four rotors start drawing attention from aerial-robotics engineers more than before. Its potential spans from just being control device test bed to performing difficult task such as carrying surveillance device to unreachable places. In this regards, modeling a quad-rotor is significant in analyzing its dynamic behavior and in synthesizing control system for such a vehicle. This paper summarizes the modeling of a mini quad-rotor aerial vehicle. A first principle approach is considered for deriving the model based on Euler-Newton equations of motion. The result of the modeling is a simulation platform that is expected to acceptably predict the dynamic behavior of the quad-rotor in various flight conditions. Linear models associated with different flight condition can be extracted for the purpose of control synthesis.

Key words : quadrotor, modeling, first-principle approach

Introduction

Quad-rotor rotorcrafts, or quadrotor, have some advantages over conventional helicopters. Having the front and rear rotor rotates in the opposite direction than the other two, gyroscopic effects, aerodynamic torques and off-axis moments from asymmetric lift distribution on each rotor disk tend to cancel in trimmed flight. Quadrotors gain its flight control by regulating total thrust output and total torque output of its four rotors sufficiently.

The first principle modeling approach is performed the quadrotor vehicle based on previous experience in developing dynamics model of a small scale helicopter[1], [2]. Other modeling approaches to rotorcraft-based unmanned aerial vehicles (RUAVs) exist in the literature including the use of system identification[3], neural-networks [4] and linear parameter varying (LPV) identification[5]. The survey for the advances of modeling of RUAVs was reported in [6].

To construct the dynamic model of a quadrotor, we begin with the formulations needed to express all forces and moments that the quadrotor' s components may generate during flight. Then, taking the quadrotor as a rigid body, we derive its equations of motion. Linearized model is constructed by first calculating solutions of some important stationary points followed by formulating its linear response to small perturbations.

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Rotor Theory

The equations governing rotor thrust and torque are derived from the momentum theory and the blade element theory. Momentum theory provides insight into condition of a thrust generating rotor as it accelerates (or decelerates) some amount of air mass in the process. On the other hand, the blade element theory explains the mechanism of which thrust and torque are generated from each rotor blade.

2. 1 Momentum Method

For steady flow along the flow field, the mass flow is constant at any point of observation.

$$\frac{dm}{dt} = \rho \cdot A_0 \cdot w_0 = \rho \cdot A_1 \cdot w_1 = \rho \cdot A_2 \cdot w_2 \tag{1}$$

$$\begin{aligned} A_1 &= \pi \cdot \left(R^2 - r_{\text{Root}}^2\right) \\ &= \pi \cdot R^2 \cdot \left(1 - \bar{r}_0^2\right) \end{aligned} \tag{2}$$

$$\bar{r} \equiv \frac{r}{R} \tag{3}$$

The total thrust exerted by each accelerating air particle in the control volume is

$$\begin{aligned} T &= \int_{w_0}^{w_2} \frac{dm}{dt} \cdot dw \\ &= \int_{w_0}^{w_2} \rho \cdot A_1 \cdot w_1 \cdot dw \\ &= \rho \cdot A_1 \cdot w_1 \cdot (w_2 - w_0) \end{aligned} \tag{4}$$

At station−1, the velocity of the airflow has increased from w_0 to w_1 by an amount of velocity induced by the rotor, w_{ind}.

$$w_1 = w_0 + w_{\text{ind}} \tag{5}$$

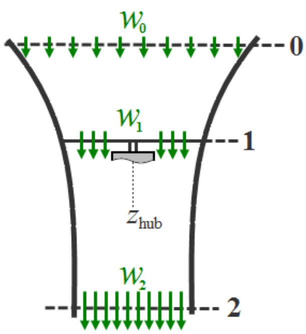

Fig. 1. Induced velocity in the vicinity of a thrust−generating rotor

The relation between flow velocities at station−1 and station−2 can be evaluated from the ideal energy rate conservation principle, and it gives

$$w_2 = 2w_{\text{ind}} - w_0 \tag{6}$$

Substituting (5) and into (4),

$$T = 2\rho \cdot \pi \cdot R^2 \cdot \left(1 - \bar{r}_0^2\right) \cdot \left(w_{\text{ind}}^2 - w_0^2\right) \tag{7}$$

2.2 Blade Element Method

A blade element is one small portion of the blade at a distance, r, from the center of rotor rotation, with a span−wise dimension dr (Figure 2).

Lift element and drag element on blade element is calculated by

$$dL = \frac{1}{2}\rho \cdot V_{\text{in}}^2 \cdot C_L \cdot c_{\text{b}} \cdot R \cdot d\bar{r} \tag{8}$$

$$dD = \frac{1}{2}\rho \cdot V_{\text{in}}^2 \cdot C_D \cdot c_{\text{b}} \cdot R \cdot d\bar{r} \tag{9}$$

V_{in} is in−plane component of blade element ' s local airspeed, which are composed of that due to rotor rotation speed and in−plane component of rotor airspeed as the whole body of the rotorcraft moves translationally.

$$\begin{aligned} V_{\text{in}} &\equiv V_{\text{in}(r,\Psi)} = \Omega \cdot r + V_{\infty\,\text{in}} \cdot \sin \Psi_{\text{a}} \\ V_{\infty\,\text{in}} &= \sqrt{u_{\text{a}}^2 + v_{\text{a}}^2} \end{aligned} \tag{10}$$

$$\begin{aligned} \Psi_{\text{a}} &= \Psi + \beta \\ \beta &\equiv \arctan\left(\frac{v_{\text{a}}}{u_{\text{a}}}\right) \end{aligned} \tag{11}$$

The lift coefficient is a linear function of the blade element ' s local angle of attack with a slope determined by blade' s cross−section airfoil. Taking the blade' s airfoil to be homogenous, the lift slope is a constant function along its span.

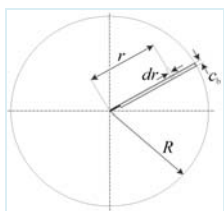

Fig. 2. Geometry of a blade element

$$C_L = a \cdot \alpha, \qquad a = \frac{\partial C_L}{\partial \alpha} \qquad \frac{\partial a}{\partial r} = 0 \tag{12}$$

The drag coefficient is not a linear function w.r.t. blade element's local angle of attack. A numerical approach would be more adequate to obtain more accurate result. For now, we take it being constant w.r.t. angle of attack.

$$\frac{\partial C_D}{\partial \alpha} \equiv 0 \tag{13}$$

The blade element's local angle of attack is determined either by the blade element's pitch angle and the direction of the blade element's local relative airspeed (Figure 3).

$$\alpha = \theta - \phi \tag{14}$$

For twisted blades, the blade element's pitch angle is a function of blade element's location along blade's span. (15) and (16) respectively are expression of blade element's pitch with constant and quadratic twist.

$$\theta \equiv \theta_{(r)} = \frac{R}{r} \cdot \theta_{\text{Tip}} = \frac{\theta_{\text{Tip}}}{\bar{r}} \tag{15}$$

$$\theta \equiv \theta_{(r)} = \frac{R^2}{r^2} \cdot \theta_{\text{Tip}} = \frac{\theta_{\text{Tip}}}{\bar{r}^2} \tag{16}$$

The inflow angle is determined by the ratio between normal and tangential component of blade element's airspeed (Figure 3).

$$\phi = \arctan\left(\frac{-V_{\text{n}}}{V_{\text{in}}}\right) \approx \frac{-V_{\text{n}}}{V_{\text{in}}}, \; V_{\text{n}} \ll V_{\text{in}} \tag{17}$$

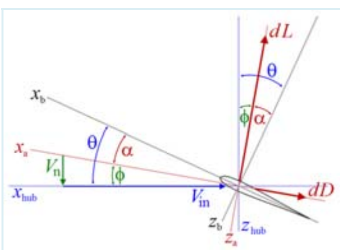

Fig. 3. Blade element environment

$$\begin{aligned} V_{\text{n}} &= w_1 \\ &= w_{\text{ind}} + w_0 \\ &= w_{\text{ind}} - w_{\text{a}}, \qquad w_{\text{a}} = -w_0 \end{aligned} \tag{18}$$

Rotor thrust is calculated by first integrating the elemental lift along rotor blade span followed by calculating mean value of blade's lift around rotor disc azimuth, then multiplying it by number of blades.

$$T = \frac{b}{2\pi} \cdot \int_0^{2\pi} \int_{\bar{r}_0}^{1} dL \cdot d\Psi_{\text{a}} \tag{19}$$

The same goes for rotor torque as well.

$$Q = \frac{b}{2\pi} \cdot \int_0^{2\pi} \int_{\bar{r}_0}^{1} dD \cdot d\Psi_{\text{a}} \tag{20}$$

The Nonlinear Model

From Newton's 2nd Law for translational motion, considering constant vehicle mass,

$$\sum \mathbf{F}_{\text{B}} + \mathbf{C}_{\text{B}/\text{I}} \cdot m \cdot \mathbf{g}_{\text{I}} = m \cdot \left(\frac{\partial \mathbf{v}_{\text{B}}}{\partial t} + \boldsymbol{\omega} \times \mathbf{v}_{\text{B}} \right) \tag{21}$$

$$\mathbf{C}_{\text{B}/\text{I}} = \begin{bmatrix} \text{c}\theta \cdot \text{c}\psi & \text{c}\theta \cdot \text{s}\psi & -\text{s}\theta \\ \text{s}\phi \cdot \text{s}\theta \cdot \text{c}\psi - \text{c}\phi \cdot \text{s}\psi & \text{s}\phi \cdot \text{s}\theta \cdot \text{s}\psi + \text{c}\phi \cdot \text{c}\psi & \text{s}\phi \cdot \text{c}\theta \\ \text{c}\phi \cdot \text{s}\theta \cdot \text{c}\psi + \text{s}\phi \cdot \text{s}\psi & \text{c}\phi \cdot \text{s}\theta \cdot \text{s}\psi - \text{s}\phi \cdot \text{c}\psi & \text{c}\phi \cdot \text{c}\theta \end{bmatrix}$$

Or, in triad formulation,

$$\begin{Bmatrix} \sum F_{x\text{B}} \\ \sum F_{y\text{B}} \\ \sum F_{z\text{B}} \end{Bmatrix} + \begin{Bmatrix} -m \cdot g \cdot \text{s}\theta \\ m \cdot g \cdot \text{s}\phi \cdot \text{c}\theta \\ m \cdot g \cdot \text{c}\phi \cdot \text{c}\theta \end{Bmatrix} = m \cdot \begin{Bmatrix} \dot{u}_{\text{B}} + q \cdot w_{\text{B}} - r \cdot v_{\text{B}} \\ \dot{v}_{\text{B}} + r \cdot u_{\text{B}} - p \cdot w_{\text{B}} \\ \dot{w}_{\text{B}} + p \cdot v_{\text{B}} - q \cdot u_{\text{B}} \end{Bmatrix} \tag{22}$$

And for angular motion,

$$\begin{aligned} \sum \mathbf{M} &= \mathbf{J} \cdot \left(\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{\omega} \right) \\ &= \mathbf{J} \cdot \frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{\omega} \times \mathbf{J} \cdot \boldsymbol{\omega} \end{aligned} \tag{23}$$

$$\begin{Bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{Bmatrix} = \begin{Bmatrix} J_{xx} \cdot \dot{p} + \left(J_{zz} - J_{yy}\right) \cdot q \cdot r \\ J_{yy} \cdot \dot{q} + \left(J_{xx} - J_{zz}\right) \cdot p \cdot r \\ J_{zz} \cdot \dot{r} + \left(J_{yy} - J_{xx}\right) \cdot p \cdot q \end{Bmatrix} \tag{24}$$

$$J_{xy} = J_{yx} = 0$$
$$J_{yz} = J_{zy} = 0$$
$$J_{xz} = J_{zx} = 0$$

Combining (22) and (24), and the triad kinematic relation, the nonlinear model is obtained.

$$\dot{u}_{\mathrm{B}} = \frac{1}{m} \cdot \sum F_{x\mathrm{B}} - g \cdot \sin\theta + r \cdot v_{\mathrm{B}} - q \cdot w_{\mathrm{B}} \quad (25)$$

$$\dot{v}_{\mathrm{B}} = \frac{1}{m} \cdot \sum F_{y\mathrm{B}} + g \cdot \sin\phi \cdot \cos\theta - r \cdot u_{\mathrm{B}} + p \cdot w_{\mathrm{B}} \quad (26)$$

$$\dot{w}_{\mathrm{B}} = \frac{1}{m} \cdot \sum F_{z\mathrm{B}} + g \cdot \cos\phi \cdot \cos\theta + q \cdot u_{\mathrm{B}} - p \cdot v_{\mathrm{B}} \quad (27)$$

$$\dot{p} = \frac{1}{J_{xx}} \cdot \sum M_x - \frac{\left(J_{zz} - J_{yy}\right)}{J_{xx}} \cdot q \cdot r \quad (28)$$

$$\dot{q} = \frac{1}{J_{yy}} \cdot \sum M_y - \frac{\left(J_{xx} - J_{zz}\right)}{J_{yy}} \cdot p \cdot r \quad (29)$$

$$\dot{r} = \frac{1}{J_{zz}} \cdot \sum M_z - \frac{\left(J_{yy} - J_{xx}\right)}{J_{zz}} \cdot p \cdot q \quad (30)$$

$$\dot{\phi} = p + \left(q \cdot \sin\phi + r \cdot \cos\phi\right) \cdot \tan\theta \quad (31)$$

$$\dot{\theta} = q \cdot \cos\phi - r \cdot \sin\phi \quad (32)$$

$$\dot{\psi} = \left(q \cdot \sin\phi + r \cdot \cos\phi\right) \cdot \sec\theta \quad (33)$$

Each of these nine expression shows the rate of time of quadrotor's motion variable states of interest, which are the triad body velocity (u_{B}, v_{B}, w_{B}), the triad body angular rate (p, q, r), and the triad body attitude (ϕ, θ, ψ). These states' rates of time are functions of its current states and driving inputs, which are the total forces and total moments. The total forces and moments come from quadrotor's propulsion system and aerodynamic forces acting on quadrotor's fuselage. In the next following subsection, the discussion will be about components of forces and moments.

3.1 Components' Forces and Moments

3.2 The Four Rotors

The four rotors are the primary components of a quadrotor. Not only it inherits the name from them, but they are the

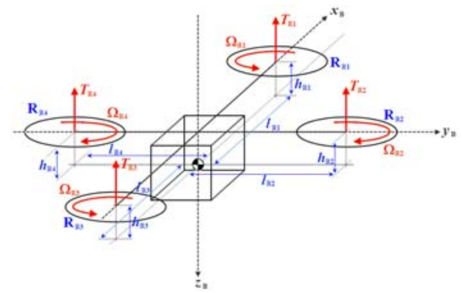

Fig. 4. Quadrotor's geometry

Table 1. Qualitative control for maneuvering quadrotor

Response\Input		Rotor Speed Increment w.r.t. Trim Setting			
		Rotor #1	Rotor #2	Rotor #3	Rotor #4
Longitudinal –Vertical	Vertical Climb/ Descend	+ / –	+ / –	+ / –	+ / –
	Pitch Up/Down	+ / –	0	– / +	0
	Move Forward/ Rearward	Coordinated Maneuver between Pitch Down/Up and Vertical Climb			
Lateral– Directional	Yaw Right/Left	+ / –	– / +	+ / –	– / +
	Roll Right/Left	0	– / +	0	+ / –
	Move Sideward Right/Left	Coordinated Maneuver between Roll Right/Left and Vertical Climb			

very instrument for it to fly and to maneuver. The generated thrusts provide mean to lift off the ground, and maneuverability is achieved by regulating thrust and torque output of each rotor to produce thrust difference and torque difference among them (Table 1). In typical quadrotor, the four rotors do not have hinges to allow their blades to flap. Therefore, in the following discussion, the direction of thrust vector is considered to be always aligned with the corresponding rotor axis. Moreover, the four rotors are considered as identical in specs.

3.2.1 Rotor Forces

Rotor force is calculated by solving rotor force equations expressed by momentum method (7) and blade element method (19).

$$T_{\mathrm{R\#}} = 2\rho \cdot \pi \cdot (\Omega \cdot R)^2 \cdot R^2 \cdot \left(1 - \bar{r}_0^2\right) \cdot \left(\lambda_0^2 - \mu_z^2\right) \quad (34)$$

$$\left(\lambda_0^2 - \mu_z^2\right) = \frac{a \cdot b \cdot c_{\mathrm{b}}}{8 \cdot \pi \cdot R} \cdot \left(\theta_{\mathrm{Tip}} \cdot \left(1 - \frac{\ln \bar{r}_0}{\left(1 - \bar{r}_0^2\right)} \cdot \mu^2\right) + \left(\lambda_0 - \mu_z\right)\right) \quad (35)$$

$$\theta = \frac{R}{r} \cdot \theta_{\text{Tip}} = \frac{\theta_{\text{Tip}}}{\overline{r}}$$

$$\left(\lambda_0^2 - \mu_z^2\right) = \frac{a \cdot b \cdot c_b}{8 \cdot \pi \cdot R} \cdot \left(\frac{2\theta_{\text{Tip}}}{(1+\overline{r}_0)} \cdot \left(1 + \frac{1}{2\overline{r}_0} \cdot \mu^2\right) + (\lambda_0 - \mu_z) \right) \quad (36)$$

$$\theta = \frac{R^2}{r^2} \cdot \theta_{\text{Tip}} = \frac{\theta_{\text{Tip}}}{\overline{r}^2}$$

And

$$\mathbf{F}_{\text{R\#}} = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix}_{\text{R\#}} = \begin{Bmatrix} 0 \\ 0 \\ -T_{\text{R\#}} \end{Bmatrix} \quad (37)$$

3.2.2 Rotor Moments

Each rotor generates moments about vehicle's center of gravity: due to its torque, due to product of its thrust vector with distance from vehicle's center of gravity, and due to asymmetric distribution of lift on each rotor disc, which is considered insignificant for the quadrotor flight regime.

$$\mathbf{M}_{\text{R1}} = \begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix}_{\text{R1}} = \begin{Bmatrix} 0 \\ l_{\text{R1}} \cdot T_{\text{R1}} \\ -Q_{\text{R1}} \end{Bmatrix}$$

$$\mathbf{M}_{\text{R2}} = \begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix}_{\text{R2}} = \begin{Bmatrix} -l_{\text{R2}} \cdot T_{\text{R2}} \\ 0 \\ -Q_{\text{R2}} \end{Bmatrix} \quad (38)$$

$$\mathbf{M}_{\text{R3}} = \begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix}_{\text{R3}} = \begin{Bmatrix} 0 \\ -l_{\text{R3}} \cdot T_{\text{R3}} \\ -Q_{\text{R3}} \end{Bmatrix}$$

$$\mathbf{M}_{\text{R4}} = \begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix}_{\text{R4}} = \begin{Bmatrix} l_{\text{R4}} \cdot T_{\text{R4}} \\ 0 \\ -Q_{\text{R4}} \end{Bmatrix}$$

$$Q_{\text{R1}} < 0 \quad Q_{\text{R2}} > 0 \qquad Q_{\text{R3}} < 0 \quad Q_{\text{R4}} > 0 \quad (39)$$

$$Q_{\text{R\#}} = \frac{1}{8}\rho \cdot b \cdot c_b \cdot (\Omega \cdot R)^2 \cdot R^2 \cdot \left(1+\mu^2\right) \cdot C_D \cdot \text{sign}(\Omega) \quad (40)$$

3.3 The Fuselage

The fuselage is part of quadrotor's body which makes the most volume of the body and contains most of the payload. The fuselage is subject to drag forces due to rotors' induced wind, air resistance as it moves, and environment's wind disturbance.

3.3.1 Fuselage Force

Fuselage force is calculated as a product of local dynamic pressure with corresponding fuselage effective wet area.

$$\mathbf{F}_{\text{Fus}} = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix}_{\text{Fus}} \quad (41)$$

$$= -\frac{1}{2}\rho \cdot \begin{Bmatrix} u_a^2 \cdot S_{\text{eff}\,x} \cdot \text{sign}(u_a) \\ v_a^2 \cdot S_{\text{eff}\,y} \cdot \text{sign}(v_a) \\ (w_a - w_{\text{ind}})^2 \cdot S_{\text{eff}\,z} \cdot \text{sign}(w_a - w_{\text{ind}}) \end{Bmatrix}$$

$$u_a = u_B - u_w, \quad v_a = v_B - v_w, \quad w_a = w_B - w_w \quad (42)$$

3.3.2 Fuselage Moment

Moment due to fuselage force may arise when fuselage aerodynamic center and vehicle body's center of gravity don't coincide. In this work however, it is considered both of them coincide at the same point. Therefore, the fuselage generates no moment about vehicle's center of gravity.

$$\mathbf{M}_{\text{Fus}} = \begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix}_{\text{Fus}} = \mathbf{0} \quad (43)$$

Solution of Force and Moment Equilibria-Stationary Condition

Stationary flight is flight condition in which net force and net moment acting on vehicle's body are zero. Therefore, stationary flight implies that motion accelerations, translational and angular, are zero.

$$\dot{u}_{\text{Bs}} = 0, \quad \dot{v}_{\text{Bs}} = 0, \quad \dot{w}_{\text{Bs}} = 0 \qquad \dot{p}_s = 0, \quad \dot{q}_s = 0, \quad \dot{r}_s = 0 \quad (44)$$

On stationary condition, the solution for every variable is called trim value since the vehicle is considered as if it is trimmed in that condition, regardless of how it may happen.

The Linearized Model

5.1 The Linearized Equation of Motion

Model linearization can be done at any point in vehicle's flight envelope;

one linearized model for each point as many as needed. At that point of interest, the dynamics of the vehicle is considered stationary. The motion of the vehicle is then formulated as perturbation about a settled point, the stationary point. The total values of all motion variables are then expanded into Taylor series. Linearization takes place when the perturbation is so small that the 2[nd] order and higher term of the expansion becomes relatively insignificant with respect to the stationary value, leaving only the constant (stationary value) and the linear term in the expansion.

From (25) to (30),

$$\begin{aligned}
&\frac{1}{m}\cdot\sum F_{x\mathrm{Bs}} - g\cdot\sin\theta_\mathrm{s} + r_\mathrm{s}\cdot v_\mathrm{Bs} - q_\mathrm{s}\cdot w_\mathrm{Bs} = 0\\
&\frac{1}{m}\cdot\sum F_{y\mathrm{Bs}} + g\cdot\sin\phi_\mathrm{s}\cdot\cos\theta_\mathrm{s} - r_\mathrm{s}\cdot u_\mathrm{Bs} + p_\mathrm{s}\cdot w_\mathrm{Bs} = 0\\
&\frac{1}{m}\cdot\sum F_{z\mathrm{Bs}} + g\cdot\cos\phi_\mathrm{s}\cdot\cos\theta_\mathrm{s} + q_\mathrm{s}\cdot u_\mathrm{Bs} - p_\mathrm{s}\cdot v_\mathrm{Bs} = 0
\end{aligned} \tag{45}$$

$$\begin{aligned}
&\frac{1}{J_{xx}}\cdot\sum M_{x\mathrm{s}} - \frac{(J_{zz}-J_{yy})}{J_{xx}}\cdot q_\mathrm{s}\cdot r_\mathrm{s} = 0\\
&\frac{1}{J_{yy}}\cdot\sum M_{y\mathrm{s}} - \frac{(J_{xx}-J_{zz})}{J_{yy}}\cdot p_\mathrm{s}\cdot r_\mathrm{s} = 0\\
&\frac{1}{J_{zz}}\cdot\sum M_{z\mathrm{s}} - \frac{(J_{yy}-J_{xx})}{J_{zz}}\cdot p_\mathrm{s}\cdot q_\mathrm{s} = 0
\end{aligned} \tag{46}$$

Applying the linear expansion to (33), and cropping the non−stationary term of them,

$$\begin{aligned}
D\dot{u}_\mathrm{Bs} = &\frac{1}{m}\cdot\sum\frac{\partial F_{x\mathrm{B}}}{\partial u}\cdot du + \left(r_\mathrm{s} + \frac{1}{m}\cdot\sum\frac{\partial F_{x\mathrm{B}}}{\partial v}\right)\cdot dv\\
&+\left(q_\mathrm{s} + \frac{1}{m}\cdot\sum\frac{\partial F_{x\mathrm{B}}}{\partial w}\right)\cdot dw\\
&+\frac{1}{m}\cdot\sum\frac{\partial F_{x\mathrm{B}}}{\partial p}\cdot dp\\
&+\left(w_\mathrm{Bs} + \frac{1}{m}\cdot\sum\frac{\partial F_{x\mathrm{B}}}{\partial q}\right)\cdot dq\\
&+\left(v_\mathrm{Bs} + \frac{1}{m}\cdot\sum\frac{\partial F_{x\mathrm{B}}}{\partial r}\right)\cdot dr\\
&+\frac{1}{m}\cdot\sum\frac{\partial F_{x\mathrm{B}}}{\partial \phi}\cdot d\phi\\
&+\left(-g\cdot\cos\theta_\mathrm{s} + \frac{1}{m}\cdot\sum\frac{\partial F_{x\mathrm{B}}}{\partial \theta}\right)\cdot d\theta\\
&+\frac{1}{m}\cdot\sum\frac{\partial F_{x\mathrm{B}}}{\partial \psi}\cdot d\psi\\
&+\frac{1}{m}\cdot\sum\frac{\partial F_{x\mathrm{B}}}{\partial \Omega_\mathrm{R1}}\cdot d\Omega_\mathrm{R1} + \frac{1}{m}\cdot\sum\frac{\partial F_{x\mathrm{B}}}{\partial \Omega_\mathrm{R2}}\cdot d\Omega_\mathrm{R2}\\
&+\frac{1}{m}\cdot\sum\frac{\partial F_{x\mathrm{B}}}{\partial \Omega_\mathrm{R3}}\cdot d\Omega_\mathrm{R3}\\
&+\frac{1}{m}\cdot\sum\frac{\partial F_{x\mathrm{B}}}{\partial \Omega_\mathrm{R4}}\cdot d\Omega_\mathrm{R4}
\end{aligned} \tag{47}$$

$$\begin{aligned}
D\dot{v}_\mathrm{B} = &\left(-r_\mathrm{s} + \frac{1}{m}\cdot\sum\frac{\partial F_{y\mathrm{B}}}{\partial u}\right)\cdot du + \frac{1}{m}\cdot\sum\frac{\partial F_{y\mathrm{B}}}{\partial v}\cdot dv\\
&+\left(p_\mathrm{s} + \frac{1}{m}\cdot\sum\frac{\partial F_{y\mathrm{B}}}{\partial w}\right)\cdot dw\\
&+\left(w_\mathrm{Bs} + \frac{1}{m}\cdot\sum\frac{\partial F_{y\mathrm{B}}}{\partial p}\right)\cdot dp + \frac{1}{m}\cdot\sum\frac{\partial F_{y\mathrm{B}}}{\partial q}\cdot dq\\
&+\left(-u_\mathrm{Bs} + \frac{1}{m}\cdot\sum\frac{\partial F_{y\mathrm{B}}}{\partial r}\right)\cdot dr\\
&+\left(g\cdot\cos\theta_\mathrm{s}\cdot\cos\phi_\mathrm{s} + \frac{1}{m}\cdot\sum\frac{\partial F_{y\mathrm{B}}}{\partial \phi}\right)\cdot d\phi\\
&+\left(-g\cdot\sin\phi_\mathrm{s}\cdot\sin\theta_\mathrm{s} + \frac{1}{m}\cdot\sum\frac{\partial F_{y\mathrm{B}}}{\partial \theta}\right)\cdot d\theta\\
&+\frac{1}{m}\cdot\sum\frac{\partial F_{y\mathrm{B}}}{\partial \psi}\cdot d\psi\\
&+\frac{1}{m}\cdot\sum\frac{\partial F_{y\mathrm{B}}}{\partial \Omega_\mathrm{R1}}\cdot d\Omega_\mathrm{R1} + \frac{1}{m}\cdot\sum\frac{\partial F_{y\mathrm{B}}}{\partial \Omega_\mathrm{R2}}\cdot d\Omega_\mathrm{R2}\\
&+\frac{1}{m}\cdot\sum\frac{\partial F_{y\mathrm{B}}}{\partial \Omega_\mathrm{R3}}\cdot d\Omega_\mathrm{R3} + \frac{1}{m}\cdot\sum\frac{\partial F_{y\mathrm{B}}}{\partial \Omega_\mathrm{R4}}\cdot d\Omega_\mathrm{R4}
\end{aligned} \tag{48}$$

$$\begin{aligned}
D\dot{w}_\mathrm{B} = &\left(q_\mathrm{s} + \frac{1}{m}\cdot\frac{\sum\partial F_{z\mathrm{B}}}{\partial u}\right)\cdot du\\
&+\left(-p_\mathrm{s} + \frac{1}{m}\cdot\frac{\sum\partial F_{z\mathrm{B}}}{\partial v}\right)\cdot dv + \frac{1}{m}\cdot\frac{\sum\partial F_{z\mathrm{B}}}{\partial w}\cdot dw\\
&+\left(-v_\mathrm{Bs} + \frac{1}{m}\cdot\frac{\sum\partial F_{z\mathrm{B}}}{\partial p}\right)\cdot dp + \left(u_\mathrm{Bs} + \frac{1}{m}\cdot\frac{\sum\partial F_{z\mathrm{B}}}{\partial q}\right)\cdot dq\\
&+\frac{1}{m}\cdot\frac{\sum\partial F_{z\mathrm{B}}}{\partial r}\cdot dr\\
&+\left(-g\cdot\cos\theta_\mathrm{s}\cdot\sin\phi_\mathrm{s} + \frac{1}{m}\cdot\frac{\sum\partial F_{z\mathrm{B}}}{\partial \phi}\right)\cdot d\phi\\
&+\left(-g\cdot\cos\phi_\mathrm{s}\cdot\sin\theta_\mathrm{s} + \frac{1}{m}\cdot\frac{\sum\partial F_{z\mathrm{B}}}{\partial \theta}\right)\cdot d\theta\\
&+\frac{1}{m}\cdot\frac{\sum\partial F_{z\mathrm{B}}}{\partial \psi}\cdot d\psi\\
&+\frac{1}{m}\cdot\frac{\sum\partial F_{z\mathrm{B}}}{\partial \Omega_\mathrm{R1}}\cdot d\Omega_\mathrm{R1} + \frac{1}{m}\cdot\frac{\sum\partial F_{z\mathrm{B}}}{\partial \Omega_\mathrm{R2}}\cdot d\Omega_\mathrm{R2}\\
&+\frac{1}{m}\cdot\frac{\sum\partial F_{z\mathrm{B}}}{\partial \Omega_\mathrm{R3}}\cdot d\Omega_\mathrm{R3} + \frac{1}{m}\cdot\frac{\sum\partial F_{z\mathrm{B}}}{\partial \Omega_\mathrm{R4}}\cdot d\Omega_\mathrm{R4}
\end{aligned} \tag{49}$$

$$\begin{aligned}
D\dot{p} = &\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial u}\cdot du + \frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial v}\cdot dv\\
&+\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial w}\cdot w + \frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial p}\cdot dp\\
&+\left(\frac{(J_{yy}-J_{zz})}{J_{xx}}\cdot r_\mathrm{s} + \frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial q}\right)\cdot dq\\
&+\left(\frac{(J_{yy}-J_{zz})}{J_{xx}}\cdot q_\mathrm{s} + \frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial r}\right)\cdot dr\\
&+\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial \phi}\cdot d\phi + \frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial \theta}\cdot d\theta\\
&+\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial \psi}\cdot d\psi\\
&+\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial \Omega_\mathrm{R1}}\cdot d\Omega_\mathrm{R1} + \frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial \Omega_\mathrm{R2}}\cdot d\Omega_\mathrm{R2}\\
&+\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial \Omega_\mathrm{R3}}\cdot d\Omega_\mathrm{R3} + \frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial \Omega_\mathrm{R4}}\cdot d\Omega_\mathrm{R4}
\end{aligned} \tag{50}$$

$$\begin{aligned} D\dot{q}_s &= \frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial u}\cdot du+\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial v}\cdot dv \\ &+\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial w}\cdot dw \\ &+\left(-\frac{(J_{xx}-J_{zz})}{J_{yy}}\cdot r_s+\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial p}\right)\cdot dp \\ &+\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial q}\cdot dq+ \\ &\left(-\frac{(J_{xx}-J_{zz})}{J_{yy}}\cdot p_s+\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial r}\right)\cdot dr \\ &+\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial \phi}\cdot d\phi+\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial \theta}\cdot d\theta \\ &+\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial \psi}\cdot d\psi \\ &+\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial \Omega_{R1}}\cdot d\Omega_{R1}+\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial \Omega_{R2}}\cdot d\Omega_{R2} \\ &+\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial \Omega_{R3}}\cdot d\Omega_{R3}+\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial \Omega_{R4}}\cdot d\Omega_{R4} \end{aligned} \quad (51)$$

$$\begin{aligned} D\dot{r} &= \frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial u}\cdot du+\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial v}\cdot dv \\ &+\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial w}\cdot dw \\ &+\left(-\frac{(J_{yy}-J_{xx})}{J_{zz}}\cdot q_s+\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial p}\right)\cdot dp \\ &+\left(-\frac{(J_{yy}-J_{xx})}{J_{zz}}\cdot p_s+\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial q}\right)\cdot dq \\ &+\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial r}\cdot dr \\ &+\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial \phi}\cdot d\phi+\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial \theta}\cdot d\theta \\ &+\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial \psi}\cdot d\psi \\ &+\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial \Omega_{R1}}\cdot d\Omega_{R1}+\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial \Omega_{R2}}\cdot d\Omega_{R2} \\ &+\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial \Omega_{R3}}\cdot d\Omega_{R3} \\ &+\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial \Omega_{R4}}\cdot d\Omega_{R4} \end{aligned} \quad (52)$$

$$\begin{aligned} D\dot{\phi} &= dp+\sin\phi_s\cdot\tan\theta_s\cdot dq+\cos\phi_s\cdot\tan\theta_s\cdot dr \\ &+\big((q_s\cdot\cos\phi_s-r_s\cdot\sin\phi_s)\cdot\tan\theta_s \\ &+(q_s\cdot\sin\phi_s+r_s\cdot\cos\phi_s)\cdot\sec^2\phi_s\big)\cdot d\phi \end{aligned} \quad (53)$$

$$\begin{aligned} D\dot{\theta} &= \cos\phi_s\cdot dq-\sin\phi_s\cdot dr \\ &-(q_s\cdot\sin\phi_s+r_s\cdot\cos\phi_s)\cdot d\phi \end{aligned} \quad (54)$$

$$\begin{aligned} D\dot{\psi} &= \sec\theta_s\cdot\sin\phi_s\cdot dq+\cos\phi_s\cdot\sec\theta_s\cdot dr \\ &+(q_s\cdot\sec\theta_s\cdot\cos\phi_s-r_s\cdot\sec\theta_s\cdot\sin\phi_s)\cdot d\phi \\ &+(q_s\cdot\sin\phi_s+r_s\cdot\cos\phi_s)\cdot\sec\theta_s\cdot\tan\theta_s\cdot d\theta \end{aligned} \quad (55)$$

(47), (48), (49), (50),(51), (52), (53), (54), and (55) can be expressed as

$$\dot{\mathbf{x}} = \mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{u} \quad (56)$$

$$\mathbf{A} = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n_{\bar{x}}} \\ \vdots & \ddots & \vdots \\ A_{n_{\bar{x}},1} & \cdots & A_{n_{\bar{x}},n_{\bar{x}}} \end{bmatrix} \quad (57)$$

$n_{\bar{x}}$ = state vector's size

$$\mathbf{B} = \begin{bmatrix} B_{1,1} & \cdots & B_{1,n_{\bar{u}}} \\ \vdots & \ddots & \vdots \\ B_{n_{\bar{x}},1} & \cdots & B_{n_{\bar{x}},n_{\bar{u}}} \end{bmatrix} \quad (58)$$

$n_{\bar{u}}$ = input vector's size

$$\mathbf{x} = \{du \quad dv \quad dw \quad dp \quad dq \quad dr \quad d\phi \quad d\theta \quad d\psi\}^T \quad (59)$$
$$\mathbf{u} = \{d\Omega_{R1} \quad d\Omega_{R2} \quad d\Omega_{R3} \quad d\Omega_{R4}\}^T$$

A is called characteristic matrix since it contains relations between states (x), and B is called input matrix since it contains relations between state (x) and driving input (u).

5.2 Force and Moment Derivatives

As it is shown in Table 2 and Table 3, quantities in characteristic and input matrices consist of forces and moments derivatives. The following sections will express those derivatives.

5.2.1 Derivative of Fx

5.2.1.1 The Four Rotors

For all rotors,

$$\frac{\partial F_x}{\partial(\bullet)} = 0 \quad (60)$$
$$(\bullet) = u, v, w, p, \ldots, \Omega_{R3}, \Omega_{R4}$$

5.2.1.2 The Fuselage

$$\frac{\partial F_{x\text{Fus}}}{\partial u} = -\rho\cdot S_{\text{eff}\,x}\cdot(u_B+u_w) \quad (61)$$

$$\frac{\partial F_{x\text{Fus}}}{\partial(\bullet)} = 0$$
$$(\bullet) = v, w, p, \ldots, \Omega_{R3}, \Omega_{R4} \quad (62)$$
$$(\bullet) \neq u$$

Table 2. Elements of Characteristic Matrix

$A_{i,j}$	$I = 1$	$I = 2$	$I = 3$	$I = 4$	$I = 5$	$I = 6$	$I = 7$	$I = 8$	$I = 9$
$j = 1$	$\frac{1}{m}\cdot\sum\frac{\partial F_{xB}}{\partial u}$	$-r_s + \frac{1}{m}\cdot\sum\frac{\partial F_{yB}}{\partial u}$	$q_s + \frac{1}{m}\cdot\frac{\sum\partial F_{zB}}{\partial u}$	$\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial u}$	$\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial u}$	$\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial u}$	0	0	0
$j = 2$	$-r_s + \frac{1}{m}\cdot\sum\frac{\partial F_{yB}}{\partial u}$	$\frac{1}{m}\cdot\sum\frac{\partial F_{yB}}{\partial v}$	$p_s + \frac{1}{m}\cdot\sum\frac{\partial F_{yB}}{\partial w}$	$\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial v}$	$\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial v}$	$\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial v}$	0	0	0
$j = 3$	$q_s + \frac{1}{m}\cdot\sum\frac{\partial F_{xB}}{\partial w}$	$p_s + \frac{1}{m}\cdot\sum\frac{\partial F_{yB}}{\partial w}$	$\frac{1}{m}\cdot\frac{\sum\partial F_{zB}}{\partial w}$	$\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial w}$	$\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial v}$	$\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial w}$	0	0	0
$j = 4$	$\frac{1}{m}\cdot\sum\frac{\partial F_{xB}}{\partial p}$	$w_{Bs} + \frac{1}{m}\cdot\sum\frac{\partial F_{yB}}{\partial p}$	$-v_{Bs} + \frac{1}{m}\cdot\frac{\sum\partial F_{zB}}{\partial p}$	$\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial p}$	$\frac{(J_{zz}-J_{xx})}{J_{yy}}\cdot r_s + \frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial p}$	$\frac{(J_{xx}-J_{yy})}{J_{zz}}\cdot q_s + \frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial p}$	1	0	0
$j = 5$	$w_{Bs} + \frac{1}{m}\cdot\sum\frac{\partial F_{xB}}{\partial q}$	$\frac{1}{m}\cdot\sum\frac{\partial F_{yB}}{\partial q}$	$u_{Bs} + \frac{1}{m}\cdot\frac{\sum\partial F_{zB}}{\partial q}$	$\frac{(J_{yy}-J_{zz})}{J_{xx}}\cdot r_s + \frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial q}$	$\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial q}$	$\frac{(J_{xx}-J_{yy})}{J_{zz}}\cdot p_s + \frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial q}$	$\sin\phi_s\cdot\tan\theta_s$	$\cos\phi_s$	$\sec\theta_s\cdot\sin\phi_s$
$j = 6$	$v_{Bs} + \frac{1}{m}\cdot\sum\frac{\partial F_{xB}}{\partial r}$	$-u_{Bs} + \frac{1}{m}\cdot\sum\frac{\partial F_{yB}}{\partial r}$	$\frac{1}{m}\cdot\frac{\sum\partial F_{zB}}{\partial r}$	$\frac{(J_{yy}-J_{zz})}{J_{xx}}\cdot q_s + \frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial r}$	$\frac{(J_{zz}-J_{xx})}{J_{yy}}\cdot p_s + \frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial r}$	$\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial r}$	$\cos\phi_s\cdot\tan\theta_s$	$-\sin\phi_s$	$\cos\phi_s\cdot\sec\theta_s$
$j = 7$	$\frac{1}{m}\cdot\sum\frac{\partial F_{xB}}{\partial \phi}$	$g\cdot\cos\theta_s\cdot\cos\phi_s + \frac{1}{m}\cdot\sum\frac{\partial F_{yB}}{\partial \phi}$	$-g\cdot\cos\theta_s\cdot\sin\phi_s + \frac{1}{m}\cdot\frac{\sum\partial F_{zB}}{\partial \phi}$	$\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial \phi}$	$\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial \phi}$	$\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial \phi}$	$\begin{pmatrix} q_s\cdot\cos\phi_s \\ -r_s\cdot\sin\phi_s \end{pmatrix}\cdot\tan\theta_s + \begin{pmatrix} q_s\cdot\sin\phi_s + \\ r_s\cdot\cos\phi_s \end{pmatrix}\cdot\sec^2\phi_s$	$-\begin{pmatrix} q_s\cdot\sin\phi_s \\ +r_s\cdot\cos\phi_s \end{pmatrix}$	$q_s\cdot\sec\theta_s\cdot\cos\phi_s - r_s\cdot\sec\theta_s\cdot\sin\phi_s$
$j = 8$	$-g\cdot\cos\theta_s + \frac{1}{m}\cdot\sum\frac{\partial F_{xB}}{\partial \theta}$	$-g\cdot\sin\phi_s\cdot\sin\theta_s + \frac{1}{m}\cdot\sum\frac{\partial F_{yB}}{\partial \theta}$	$-g\cdot\cos\phi_s\cdot\sin\theta_s + \frac{1}{m}\cdot\frac{\sum\partial F_{zB}}{\partial \theta}$	$\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial \theta}$	$\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial \theta}$	$\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial \theta}$	0	0	$(q_s\cdot\sin\phi_s + r_s\cdot\cos\phi_s)\cdot\sec\theta_s\cdot\tan\theta_s$
$j = 9$	$\frac{1}{m}\cdot\sum\frac{\partial F_{xB}}{\partial \psi}$	$\frac{1}{m}\cdot\sum\frac{\partial F_{yB}}{\partial \psi}$	$\frac{1}{m}\cdot\frac{\sum\partial F_{zB}}{\partial \psi}$	$\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial \psi}$	$\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial \psi}$	$\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial \psi}$	0	0	0

Table 3. Elements of Input Matrix

$B_{i,j}$	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$I = 1$	$\frac{1}{m}\cdot\sum\frac{\partial F_{xB}}{\partial \Omega_{R1}}$	$\frac{1}{m}\cdot\sum\frac{\partial F_{xB}}{\partial \Omega_{R2}}$	$\frac{1}{m}\cdot\sum\frac{\partial F_{xB}}{\partial \Omega_{R3}}$	$\frac{1}{m}\cdot\sum\frac{\partial F_{xB}}{\partial \Omega_{R4}}$
$I = 2$	$\frac{1}{m}\cdot\sum\frac{\partial F_{yB}}{\partial \Omega_{R1}}$	$\frac{1}{m}\cdot\sum\frac{\partial F_{yB}}{\partial \Omega_{R2}}$	$\frac{1}{m}\cdot\sum\frac{\partial F_{yB}}{\partial \Omega_{R3}}$	$\frac{1}{m}\cdot\sum\frac{\partial F_{yB}}{\partial \Omega_{R4}}$
$I = 3$	$\frac{1}{m}\cdot\frac{\sum\partial F_{zB}}{\partial \Omega_{R1}}$	$\frac{1}{m}\cdot\frac{\sum\partial F_{zB}}{\partial \Omega_{R2}}$	$\frac{1}{m}\cdot\frac{\sum\partial F_{zB}}{\partial \Omega_{R3}}$	$\frac{1}{m}\cdot\frac{\sum\partial F_{zB}}{\partial \Omega_{R4}}$
$I = 4$	$\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial \Omega_{R1}}$	$\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial \Omega_{R2}}$	$\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial \Omega_{R3}}$	$\frac{1}{J_{xx}}\cdot\sum\frac{\partial M_x}{\partial \Omega_{R4}}$
$I = 5$	$\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial \Omega_{R1}}$	$\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial \Omega_{R2}}$	$\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial \Omega_{R3}}$	$\frac{1}{J_{yy}}\cdot\sum\frac{\partial M_y}{\partial \Omega_{R4}}$
$I = 6$	$\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial \Omega_{R1}}$	$\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial \Omega_{R2}}$	$\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial \Omega_{R3}}$	$\frac{1}{J_{zz}}\cdot\sum\frac{\partial M_z}{\partial \Omega_{R4}}$
$I = 7$	0	0	0	0
$I = 8$	0	0	0	0
$I = 9$	0	0	0	0

A is called characteristic matrix since it contains relations between states (x), and B is called input matrix since it contains relations between state (x) and driving input (u).

5.2 Force and Moment Derivatives

As it is shown in Table 2 and Table 3, quantities in characteristic and input matrices consist of forces and moments derivatives. The following sections will express those derivatives.

5.2.1 Derivative of Fx

5.2.1.1 The Four Rotors

For all rotors,

$$\frac{\partial F_x}{\partial(\bullet)}=0 \qquad (60)$$
$$(\bullet)=u,v,w,p,\ldots,\Omega_{R3},\Omega_{R4}$$

5.2.1.2 The Fuselage

$$\frac{\partial F_{x\mathrm{Fus}}}{\partial u}=-\rho\cdot S_{\mathrm{eff}\,x}\cdot\left(u_B+u_w\right) \qquad (61)$$

$$\frac{\partial F_{x\mathrm{Fus}}}{\partial(\bullet)}=0 \qquad (62)$$
$$(\bullet)=v,w,p,\ldots,\Omega_{R3},\Omega_{R4}$$
$$(\bullet)\neq u$$

5.2.2 Derivative of Fy

5.2.2.1 The Four Rotors

$$\frac{\partial F_y}{\partial(\bullet)}=0 \qquad (63)$$
$$(\bullet)=u,v,w,p,\ldots,\Omega_{R3},\Omega_{R4}$$

5.2.2.2 The Fuselage

$$\frac{\partial F_{y\mathrm{Fus}}}{\partial v}=-\rho\cdot S_{\mathrm{eff}\,y}\cdot\left(v_B+v_w\right) \qquad (64)$$

$$\frac{\partial F_{y\mathrm{Fus}}}{\partial(\bullet)}=0 \qquad (65)$$
$$(\bullet)=u,w,p,\ldots,\Omega_{R3},\Omega_{R4}$$
$$(\bullet)\neq v$$

5.2.3 Derivative of Fz

5.2.3.1 The Four Rotors

$$\frac{\partial T_{R\#}}{\partial w}=4\rho\cdot(\Omega\cdot R)^2\cdot\pi\cdot R^2\cdot\left(1-\bar{r}_0^2\right)\cdot\left(\lambda_0\cdot\frac{\partial\lambda_0}{\partial w}-\frac{\mu_z}{\Omega\cdot R}\right) \qquad (66)$$

$$\frac{\partial T_{R\#}}{\partial(\bullet)}=4\rho\cdot(\Omega\cdot R)^2\cdot\pi\cdot R^2\cdot\left(1-\bar{r}_0^2\right)\cdot\lambda_0\cdot\frac{\partial\lambda_0}{\partial(\bullet)} \qquad (67)$$
$$(\bullet)=u,v,p,\ldots,\Omega_{R3},\Omega_{R4}$$
$$(\bullet)\neq w$$

$$\frac{\partial T_{Ri}}{\partial\Omega_{Rj}}=4\rho\cdot\pi\cdot R^4\cdot\left(1-\bar{r}_0^2\right)\cdot\left(\Omega_{Rj}\cdot\left(\lambda_0^2-\mu_z^2\right)\cdot\frac{\partial\Omega_{Ri}}{\partial\Omega_{Rj}}+\Omega_{Rj}^2\cdot\left(\lambda_0\cdot\frac{\partial\lambda_{0Ri}}{\partial\Omega_{Rj}}-\mu_z\cdot\frac{\partial\mu_z}{\partial\Omega_{Rj}}\right)\right)$$
$$=\begin{cases}4\rho\cdot\pi\cdot R^4\cdot\left(1-\bar{r}_0^2\right)\cdot\Omega_{Rj}\cdot\left(\left(\lambda_0^2-\mu_z^2\right)+\Omega_{Rj}\cdot\lambda_0\cdot\dfrac{\partial\lambda_{0Ri}}{\partial\Omega_{Rj}}\right), & i=j\\ 0, & i\neq j\end{cases} \qquad (68)$$

5.2.3.2 The Fuselage

$$\frac{\partial F_{z\mathrm{Fus}}}{\partial(\bullet)}=\rho\cdot S_{\mathrm{eff}\,z}\cdot\left(w_a-w_{\mathrm{ind}}\right)\cdot\frac{\partial w_{\mathrm{ind}}}{\partial(\bullet)} \qquad (69)$$
$$(\bullet)=u,v,w,p,\ldots,\psi$$
$$(\bullet)\neq\Omega_{R1},\Omega_{R2},\Omega_{R3},\Omega_{R4}$$

$$\frac{\partial F_{z\mathrm{Fus}}}{\partial\Omega_{Rj}}=\frac{1}{4}\rho\cdot S_{\mathrm{eff}\,z}\cdot\left(w_a-w_{\mathrm{ind}}\right)\cdot\frac{\partial w_{\mathrm{ind}Ri}}{\partial\Omega_{Rj}},\quad i=j \qquad (70)$$

5.2.4 Derivative of Mx

5.2.4.1 The Four Rotors

$$\frac{\partial M_{xRi}}{\partial(\bullet)}=0,\qquad i=1,3 \qquad (71)$$
$$(\bullet)=u,v,w,p,\ldots,\Omega_{R3},\Omega_{R4}$$

$$\frac{\partial M_{xRi}}{\partial(\bullet)}=-l_{Ri}\cdot\frac{\partial T_{Ri}}{\partial(\bullet)},\qquad i=2,4 \qquad (72)$$
$$(\bullet)=u,v,w,p,\ldots,\Omega_{R3},\Omega_{R4}$$

5.2.4.2 The Fuselage

$$\frac{\partial M_{x\mathrm{Fus}}}{\partial(\bullet)}=0 \qquad (73)$$
$$(\bullet)=u,v,w,p,\ldots,\Omega_{R3},\Omega_{R4}$$

5.2.5 Derivative of My

5.2.5.1 The Four Rotors

$$\frac{\partial M_{y\mathrm{R}i}}{\partial(\bullet)} = -l_{\mathrm{R}i} \cdot \frac{\partial T_{\mathrm{R}i}}{\partial(\bullet)}, \qquad i = 1,3 \tag{74}$$

$$(\bullet) = u, v, w, p, \ldots, \Omega_{\mathrm{R3}}, \Omega_{\mathrm{R4}}$$

$$\frac{\partial M_{x\mathrm{R}i}}{\partial(\bullet)} = 0, \qquad i = 2,4 \tag{75}$$

$$(\bullet) = u, v, w, p, \ldots, \Omega_{\mathrm{R3}}, \Omega_{\mathrm{R4}}$$

5.2.5.2 The Fuselage

$$\frac{\partial M_{y\mathrm{Fus}}}{\partial(\bullet)} = 0 \tag{76}$$

$$(\bullet) = u, v, w, p, \ldots, \Omega_{\mathrm{R3}}, \Omega_{\mathrm{R4}}$$

5.2.6 Derivative of Mz

5.2.6.1 The Four Rotors

$$\frac{\partial M_{z\mathrm{R}\#}}{\partial u} = \tag{77}$$

$$-\frac{1}{4}\rho \cdot b \cdot c_{\mathrm{b}} \cdot R^4 \cdot C_D \cdot \Omega_{\mathrm{R}\#}^2 \cdot \mu \cdot \frac{u}{\sqrt{u^2+v^2}} \cdot \mathrm{sign}(\Omega_{\mathrm{R}\#})$$

$$\frac{\partial M_{z\mathrm{R}\#}}{\partial v} = \tag{78}$$

$$-\frac{1}{4}\rho \cdot b \cdot c_{\mathrm{b}} \cdot R^4 \cdot C_D \cdot \Omega_{\mathrm{R}\#}^2 \cdot \mu \cdot \frac{v}{\sqrt{u^2+v^2}} \cdot \mathrm{sign}(\Omega_{\mathrm{R}\#})$$

$$\frac{\partial M_{z\mathrm{R}\#}}{\partial(\bullet)} = 0 \tag{79}$$

$$(\bullet) = w, p, \ldots, \Omega_{\mathrm{R3}}, \Omega_{\mathrm{R4}}$$

$$(\bullet) \neq u, v$$

$$\frac{\partial M_{z\mathrm{R}i}}{\partial \Omega_{\mathrm{R}j}} = -\frac{1}{4}\rho \cdot b \cdot c_{\mathrm{b}} \cdot R^4 \cdot C_D \cdot \Omega_{\mathrm{R}i} \cdot \left((1+\mu^2) \cdot \frac{\partial \Omega_{\mathrm{R}i}}{\partial \Omega_{\mathrm{R}j}} + \Omega_{\mathrm{R}i} \cdot \mu \cdot \frac{\partial \mu}{\partial \Omega_{\mathrm{R}j}} \right) \cdot \mathrm{sign}(\Omega_{\mathrm{R}i}) \tag{80}$$

$$= \begin{cases} -\frac{1}{4}\rho \cdot b \cdot c_{\mathrm{b}} \cdot R^4 \cdot C_D \cdot \Omega_{\mathrm{R}i} \cdot (1 + \mu^2) \cdot \mathrm{sign}(\Omega_{\mathrm{R}i}), & i = j \\ 0, & i \neq j \end{cases}$$

5.2.6.2 The Fuselage

$$\frac{\partial M_{z\mathrm{Fus}}}{\partial(\bullet)} = 0 \tag{81}$$

$$(\bullet) = u, v, w, p, \ldots, \Omega_{\mathrm{R3}}, \Omega_{\mathrm{R4}}$$

Results and Conclusions

Trim solutions of several flight conditions are tabulated in Table 5. Variations of characteristic roots are plotted with respect to corresponding forward speed variation in Fig. 5 to Fig. 13

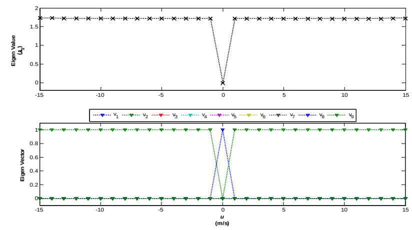

Fig. 5. 1st eigen value and the corresponding eigen vector

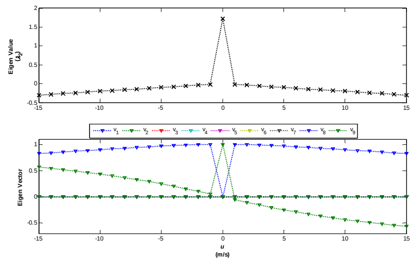

Fig. 6. 2nd eigen value and the corresponding eigen vector

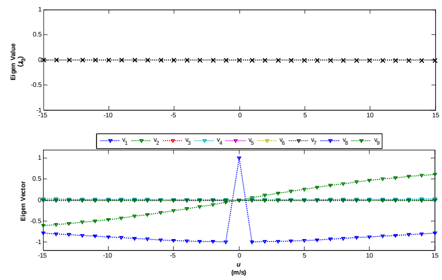

Fig. 7. 3rd eigen value and the corresponding eigen vector

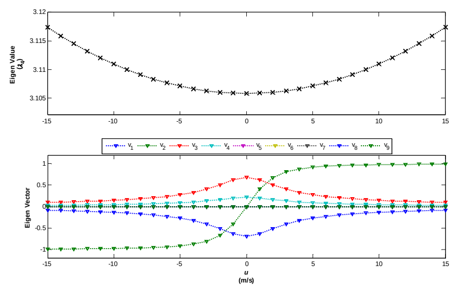

Fig. 8. 4th eigen value and the corresponding eigen vector

Table 4. Quadrotor's basic parameters

Parameter		Value	Description
Symbol	Unit		
Overall body			
I_{xx}	kg m^2/s	0.0125	Quadrotor' s principal moment of inertia about x_B−axis
I_{yy}	kg m^2/s	0.0125	Quadrotor' s principal moment of inertia about y_B−axis
I_{zz}	kg m^2/s	0.0287	Quadrotor' s principal moment of inertia about z_B−axis
m	kg	1.02	Quadrotor' s mass
Individual Rotor			
a_R	1/rad	5.0	Rotor blades' lift coefficient gradient
b_R	1	3	Number of rotor blades
c_R	m	0.029	Rotor blades' chord length
l_R	m	0.200	Horizontal distance of rotor axis from vehicle center of gravity
R_R	m	0.1300	Rotor blades' span, rotor radius
$\theta_{Tip\ R}$	rad	0.0873	Rotor blades' pitch angle at tip
Fuselage			
$S_{eff\ x}$	m^2	0.0168	Fuselage' s effective wet area normal to x_B−axis
$S_{eff\ x}$	m^2	0.0168	Fuselage' s effective wet area normal to y_B−axis
$S_{eff\ x}$	m^2	0.0235	Fuselage' s effective wet area normal to z_B−axis
Environment parameters			
ρ	kg/m^3	1.225	Ambient air density
g	m/s^2	9.80665	Gravitational constant

Table 5. Trim solutions of several flight conditions

	v = 0	v = 5	v = 10	v = 15
u = 0	$\phi_s = 0$ $\theta_s = 0$ $\lvert\Omega_{R\#s}\rvert = 154.8187$ rad/s	$\phi_s = 0.0257$ rad $= 1.4737°$ $\theta_s = 0$ $\lvert\Omega_{R\#s}\rvert = 154.8636$ rad/s	$\phi_s = 0.1031$ rad $= 5.9045°$ $\theta_s = 0$ $\lvert\Omega_{R\#s}\rvert = 154.9985$ rad/s	$\phi_s = 0.2336$ rad $= 13.3831°$ $\theta_s = 0$ $\lvert\Omega_{R\#s}\rvert = 155.2241$ rad/s
u = 5	$\phi_s = 0$ $\theta_s = -0.0257$ rad $= -1.4737°$ $\lvert\Omega_{R\#s}\rvert = 154.8636$ rad/s	$\phi_s = 0.0257$ rad $= 1.4742°$ $\theta_s = -0.0257$ rad $= -1.4737°$ $\lvert\Omega_{R\#s}\rvert = 154.9085$ rad/s	$\phi_s = 0.1031$ rad $= 5.9065°$ $\theta_s = -0.0257$ rad $= -1.4737°$ $\lvert\Omega_{R\#s}\rvert = 155.0435$ rad/s	$\phi_s = 0.2337$ rad $= 13.3876°$ $\theta_s = -0.0257$ rad $= -1.4737°$ $\lvert\Omega_{R\#s}\rvert = 155.2694$ rad/s
u = 10	$\phi_s = 0$ $\theta_s = -0.1031$ rad $= -5.9045°$ $\lvert\Omega_{R\#s}\rvert = 154.9985$ rad/s	$\phi_s = 0.0259$ rad $= 1.4815°$ $\theta_s = -0.1031$ rad $= -5.9045°$ $\lvert\Omega_{R\#s}\rvert = 155.0435$ rad/s	$\phi_s = 0.1036$ rad $= 5.9361°$ $\theta_s = -0.1031$ rad $= -5.9045°$ $\lvert\Omega_{R\#s}\rvert = 155.1789$ rad/s	$\phi_s = 0.2348$ rad $= 13.4558°$ $\theta_s = -0.1031$ rad $= -5.9045°$ $\lvert\Omega_{R\#s}\rvert = 155.4055$ rad/s
u = 15	$\phi_s = 0$ $\theta_s = -0.2336$ rad $= -13.3831°$ $\lvert\Omega_{R\#s}\rvert = 154.2241$ rad/s	$\phi_s = 0.0264$ rad $= 1.5148°$ $\theta_s = -0.2336$ rad $= -13.3831°$ $\lvert\Omega_{R\#s}\rvert = 155.2694$ rad/s	$\phi_s = 0.1059$ rad $= 6.0700°$ $\theta_s = -0.2336$ rad $= -13.3831°$ $\lvert\Omega_{R\#s}\rvert = 155.4054$ rad/s	$\phi_s = 0.2402$ rad $= 13.7639°$ $\theta_s = -0.2336$ rad $= -13.3831°$ $\lvert\Omega_{R\#s}\rvert = 155.6331$ rad/s

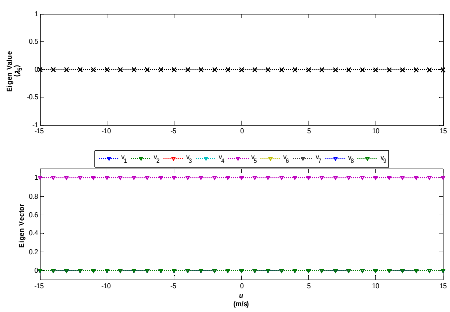

Fig. 9. 5th eigen value and the corresponding ponding eigen vector

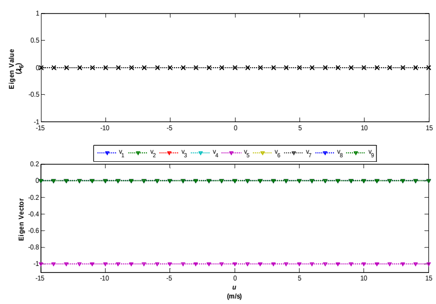

Fig. 10. 6th eigen value and the corresponding eigen vector

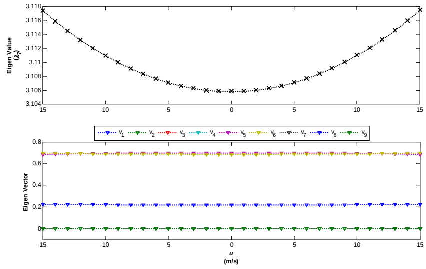

Fig. 11. 7th eigen value and the corresponding eigen vector

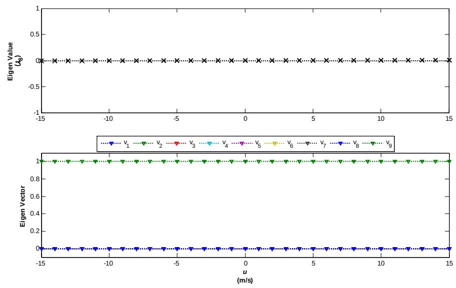

Fig. 12. 8th eigen value and the corresponding eigen vector

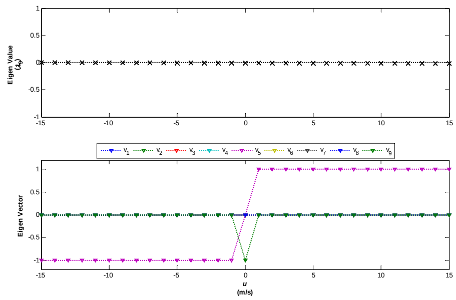

Fig. 13. 9th eigen value and the corresponding eigen vector

It has shown that a quadrotor has symmetricity on x_B−axis and y_B−axis, which is one axis more than a conventional helicopter does. This gives the advantage of a quadrotor to change direction in level flight without having to change its heading. The downside shown in Table 5 is that to achieve higher cruise velocity, the quadrotor has to increase its corresponding attitude with respect to the level line, which in practical sense, less favorable.

From stability analysis, the quadrotor generally shows unstable characteristics. A control strategy will have to deal with almost all of the plant’ s characteristic roots and get them to the stable zone. The first root becomes fully unstable as quadrotor gain cruise speed. The second root shows quite the opposite, which is naturally unstable when the quadrotor hovers. Two unstable roots appear as a double and they get more unstable as the quadrotor’ s speed gets higher. The other five roots have indifferent value and invariant with respect to the quadrotor’ s cruise speed. The discontinuities shown by the variations of 1st, 2nd, 4th, 5th, 6th, and 7th roots suggest the quadrotor has distinctive characteristics between hover and cruise flight that a single control strategy may not be sufficient to cope with. Considering only cruise flight, only one of the 4 non−indifferent roots shows stability increase as the quadrotor’ s cruise speed increases, while the other three roots show increment in instability.

Overall, the intent of the study is analytical model development of quadrotor vehicle. The developed model should be ultimately verified against the flight data throughout the applicable range of velocities in the flight envelope. The tests to gather such data is currently underway.

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