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Locally Competitive Equilibrium and Properties

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■ Abstract ■

I study a solution concept which preserves the nice Nash equilibrium properties of two-person zero-sum games, and define *a locally competitive equilibrium* which is characterized by a saddle point with respect to the coordinates of strategies. I show that a locally competitive equilibrium shares the properties of uniqueness of equilibrium payoffs, interchangeablity of equilibrium strategies and convexity of the equilibrium set.

Keywords : Competitive Game, Saddle Point of Nash equilibria and Twisted equilibria

1. Introduction

If players have a strictly opposing preference pattern over the outcomes in a two-person game, we shall call it a *strictly competitive game*, and it is *strategically equivalent* to a two-person zero-sum (2PZS) game [6]. Therefore, strictly competitive games inherit all the Nash equilibrium properties of 2PZS games, such as the uniqueness of Nash equilibrium payoffs and the interchangeability of equilibrium strategies. It has been a traditional topic among game theorists to search for enlargement of the class of 2PZS games which share these nice properties (e.g., "almost strictly competitive games"[1], "order equivalent to a zero-sum game"[5], "best response equivalent to zero-sum games"[4])

Previous works, however, selected some equilibrium properties of the zero-sum games and

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enlarged the class of games which preserve these properties. None of the previous enlargements have provided some kind of equivalence between the payoff structure and some properties regarding equilibria. .

In this note, I propose a new solution concept called a *locally competitive equilibrium.* It is characterized by a saddle point which is an intersection point of the set of Nash equilibria and the set of twisted equilibria (a twisted equilibrium is such that neither player can decrease the other player's payoff by a unilateral change of strategy). Though the players are not strict adversaries of each other, locally competitive equilibrium pairs are equivalent in the sense of having the same utility payoff, and locally competitive equilibrium strategies are interchangeable in the sense that any pair, one from each player, forms a locally competitive equilibrium. In addition to these properties, I show the convexity of the locally competitive equilibrium set.

2. Locally Competitive Equilibrium

Let us consider a two-person game $\Gamma=(\{1, 2\}, S_1, S_2, \pi_1, \pi_2)$where $\{1, 2\}$ is the set of two players, S_i is the set of strategies for player i, and $\pi_i : S \rightarrow \Re$ is the payoff function for player i $(i=1, 2)$. Here S denotes the set of all possible combinations of strategies that may be chosen by the two players, that is, $S = S_1 \times S_2$. The strategy profile s^* in S is a Nash equilibrium of Γ if $\pi_i(s_i^*, s_j^*) \geq \pi_i(s_i, s_j^*)$ for every $s_i \in S_i$ and $i, j = 1, 2 (i \neq j)$. The strategy profile $\hat{s}$ in S is a twisted equilibrium of Γ if $\pi_i(\hat{s}_i, \hat{s}_j) \leq \pi_i(\hat{s}_i, s_j)$ for every $s_j \in S_j$ and $i, j=1, 2(i \neq j)$.

Definition : Strategy profile $s^\circ \in S$ is a *locally competitive equilibrium* of Γ if for all $s_i \in S_i$, $s_j \in S_j$ and $i, j=1, 2(i \neq j)$,
$\pi_i(s_i, s_j^\circ) \leq \pi_i(s_i^\circ, s_j^\circ) \leq \pi_i(s_i^\circ, s_j)$.

The inequality expressed in the above definition shows the following property of player i's payoff structure : player i's payoff π_i can only decrease with the variation of player i's strategy and can only increase with the variation of player j's strategy $(i \neq j)$. This implies the locally competitive equilibrium point corresponds to the saddle point when we describe the payoff function π_i in coordinates s_i and s_j. Clearly, any locally competitive equilibrium, if it exists, is a Nash equilibrium and a twisted equilibrium. It can be restated that a locally competitive equilibrium is an intersection point of the set of Nash equilibria and the set of twisted equilibria.

The difference between the locally competitive equilibrium and the equilibrium in almost strictly competitive games [1] is the coincidence of the set of Nash equilibrium payoff and the set of twisted equilibrium payoff. These two equilibrium sets are not necessarily same in locally competitive equilibrium as shown in the following example 1.

Player 1	Player 2	
	x_2	y_2
x_1	2, 2	0, 1
y_1	1, 0	1, 1

〈Figure 1〉 Example 1

There are three Nash equilibria in Example 1 : (x_1, x_2), (y_1, y_2) and $(\frac{1}{2}[x_1]+\frac{1}{2}[y_1], \frac{1}{2}[x_2]+\frac{1}{2}[y_2])$, and three twisted equilibria : (x_1, y_2), $(y_1,$

x_2) and (y_1, y_2). Therefore (y_1, y_2) is a locally competitive equilibrium in this example but the set of Nash equilibrium payoffs does not coincide with the set of twisted equilibrium payoffs which implies the example is not almost strictly competitive.

For any given strategic form game Γ, we can easily verify the existence of twisted equilibria [1] However, the set of Nash equilibria and the set of twisted equilibria do not necessarily intersect. This implies that the locally competitive equilibria do not always exist as shown in the following example 2.

Player 1	Player 2: x_2	y_2
x_1	2, 2	1, 3
y_1	3, 1	0, 0

〈Figure 2〉 Example 2

There are three Nash equilibria in Example 2 : (x_1, y_2), (y_1, x_2) and $(\frac{1}{2}[x_1]+\frac{1}{2}[y_1],\ \frac{1}{2}[x_2]+\frac{1}{2}[y_2])$. However, there is a unique twisted equilibrium (y_1, y_2). Hence, there cannot be any locally competitive equilibrium in this example.

Proposition 1 : *Let $s^\circ \in S$ and $t^\circ \in S$ be two distinct locally competitive equilibria of Γ. Then, for every $i=1, 2$, $\pi_i(s^\circ)=\pi_i(t^\circ)$.*

Proof : By definition, s° and t° are both Nash equilibria and twisted equilibria. Since t° is a twisted equilibrium and s° is a Nash equilibrium, $\pi_1(t_1^\circ, t_2^\circ) \le \pi_1(t_1^\circ, s_2^\circ)$ and $\pi_1(t_1^\circ, s_2^\circ) \le \pi_1(s_1^\circ, s_2^\circ)$; hence $\pi_1(t_1^\circ, t_2^\circ) \le \pi_1(s_1^\circ, s_2^\circ)$. Since s° is a twisted equilibrium and t° is a Nash equilibrium, $\pi_1(s_1^\circ, s_2^\circ) \le \pi_1(s_1^\circ, t_2^\circ)$ and $\pi_1(s_1^\circ, t_2^\circ) \le \pi_1(t_1^\circ, t_2^\circ)$; hence $\pi_1(s_1^\circ, s_2^\circ) \le \pi_1(t_1^\circ, t_2^\circ)$. This shows $\pi_1(s_1^\circ, s_2^\circ)=\pi_1(t_1^\circ, t_2^\circ)$. By the similar procedure, $\pi_2(s_1^\circ, s_2^\circ)=\pi_2(t_1^\circ, t_2^\circ)$. □

Proposition 2 : *Let $s^\circ=(s_1^\circ, s_2^\circ)$ and $t^\circ=(t_1^\circ, t_2^\circ)$ be two distinct locally competitive equilibria of Γ. Then, (s_1°, t_2°) and (t_1°, s_2°) are also locally competitive equilibria of Γ.*

Proof : First, we show that (s_1°, t_2°) is a twisted equilibrium of Γ. Since t° is a Nash equilibrium, $\pi_1(s_1^\circ, t_2^\circ) \le \pi_1(t_1^\circ, t_2^\circ)$. By proposition 1, $\pi_1(t_1^\circ, t_2^\circ)=\pi_1(s_1^\circ, s_2^\circ)$; hence, $\pi_1(s_1^\circ, t_2^\circ) \le \pi_1(s_1^\circ, s_2^\circ)$. Since s° is a twisted equilibrium, $\pi_1(s_1^\circ, s_2^\circ) \le \pi_1(s_1^\circ, s_2)$ for every $s_2 \in S_2$. Thus, we have $\pi_1(s_1^\circ, t_2^\circ) \le \pi_1(s_1^\circ, s_2)$ for every $s_2 \in S_2$. Next, we show that (s_1°, t_2°) is a Nash equilibrium of Γ. Since t° is a Nash equilibrium, $\pi_1(s_1, t_2^\circ) \le \pi_1(t_1^\circ, t_2^\circ)$ for every $s_1 \in S_1$; hence, $\pi_1(s_1, t_2^\circ) \le \pi_1(s_1^\circ, s_2^\circ)$ for every $s_1 \in S_1$. Since s° is a twisted equilibrium, $\pi_1(s_1^\circ, s_2^\circ) \le \pi_1(s_1^\circ, t_2^\circ)$; hence $\pi_1(s_1, t_2^\circ) \le \pi_1(s_1^\circ, t_2^\circ)$ for every $s_1 \in S_1$. Therefore, we have, for every $s_1 \in S_1$ and $s_2 \in S_2$, $\pi_1(s_1, t_2^\circ) \le \pi_1(s_1^\circ, t_2^\circ) \le \pi_1(s_1^\circ, s_2)$. Similarly, for every $s_1 \in S_1$ and $s_2 \in S_2$, $\pi_2(s_1^\circ, s_2) \le \pi_2(s_1^\circ, t_2^\circ) \le \pi_2(s_1, t_2^\circ)$. This shows that (s_1°, t_2°) is a locally competitive equilibrium, and by the similar procedure (t_1°, s_2°) is also a locally competitive equilibrium. □

I have shown that the uniqueness of locally competitive equilibrium payoffs and the interchangeability of equilibrium in pure strategies. These properties are easily extended to the case of mixed strategies when the players randomize their strategy profiles. The following proposition shows the convexity of locally competitive equilibrium set in mixed strategies.

Proposition 3 : *The set of locally competitive equilibria of Γ is convex.*

Proof : Let $\Delta(S_i)$ denote the set of probability distribution on S_i. Let $p^\circ = (p_1^\circ, p_2^\circ)$ and $q^\circ = (q_1^\circ, q_2^\circ)$ in $\Delta(S_1) \times \Delta(S_2)$ be two distinct locally competitive equilibria in mixed strategies. We shall show, for $0 \le \lambda \le 1$, $(\lambda p_1^\circ + (1-\lambda) q_1^\circ, \lambda p_2^\circ + (1-\lambda) q_2^\circ)$ is also a locally competitive equilibrium. First, we show that $(\lambda p_1^\circ + (1-\lambda) q_1^\circ, p_2^\circ)$is a locally competitive equilibrium. By proposition 2, (p_1°, p_2°) and (q_1°, p_2°) are Nash equilibria ; hence, for every $p_1 \in \Delta(S_1)$, $\pi_1(p_1, p_2^\circ) \le \pi_1(p_1^\circ, p_2^\circ)$ and $\pi_1(p_1, p_2^\circ) \le \pi_1(q_1^\circ, p_2^\circ)$ where $\pi_i(p_1, p_2) = \sum_{s_1 \in S_1, s_2 \in S_2} p_1(s_1) p_2(s_2) \pi_i(s_1, s_2)$. Thus, for every $p_1 \in \Delta(S_1)$ and $0 \le \lambda \le 1$, we have $\pi_1(p_1, p_2^\circ) \le \pi_1(\lambda p_1^\circ + (1-\lambda) q_1^\circ, p_2^\circ)$ Moreover, (p_1°, p_2°) and (q_1°, p_2°) are twisted equilibria ; hence, for every $p_2 \in \Delta(S_2)$, $\pi_1(p_1^\circ, p_2^\circ) \le \pi_1(p_1^\circ, p_2)$ and $\pi_1(q_1^\circ, p_2^\circ) \le \pi_1(q_1^\circ, p_2)$. Thus, $\pi_1(\lambda p_1^\circ + (1-\lambda) q_1^\circ, p_2^\circ) \le \pi_1(\lambda p_1^\circ + (1-\lambda) q_1^\circ, p_2)$ for every $p_2 \in \Delta(S_2)$ and $0 \le \lambda \le 1$. By adding up together, for every $p_1 \in \Delta(S_1)$, $p_2 \in \Delta(S_2)$ and $0 \le \lambda \le 1$, $\pi_1(p_1, p_2^\circ) \le \pi_1(\lambda p_1^\circ + (1-\lambda) q_1^\circ, p_2^\circ) \le \pi_1(\lambda p_1^\circ + (1-\lambda) q_1^\circ, p_2)$.

Through the similar procedure, we have $\pi_2(\lambda p_1^\circ + (1-\lambda) q_1^\circ, p_2) \le \pi_2(\lambda p_1^\circ + (1-\lambda) q_1^\circ, p_2^\circ) \le \pi_2(p_1, p_2^\circ)$. This implies $(\lambda p_1^\circ + (1-\lambda) q_1^\circ, p_2^\circ)$ is a locally competitive equilibrium. Similarly, $(p_1^\circ, \lambda p_2^\circ + (1-\lambda) q_2^\circ)$ is a locally competitive equilibrium. Thus, by proposition 2, $(\lambda p_1^\circ + (1-\lambda) q_1^\circ, \lambda p_2^\circ + (1-\lambda) q_2^\circ)$ is also a locally competitive equilibrium. □

3. Conclusion

In this note, I introduced a locally competitive equilibrium which corresponds to the saddle point in each player's strategy. Though the game situation is not strategically equivalent to 2PZS games, any pair of locally competitive equilibrium strategies, one for each player, is also a locally competitive equilibrium pair, and all locally competitive equilibrium pairs give rise to an outcome with the same utility payment. Moreover, I show the convexity of the locally competitive equilibrium set when the players randomize their strategies.

References

[1] Aumann, R.J., "Almost strictly competitive games," *Journal of the Society of Industrial and Applied Mathematics*, Vol.9(1961), pp. 544-550.

[2] Moeschlin, O. and D. Pallaschke, *Game Theory and Mathematical Economics*, Amsterdam, New York, Oxford : North-Holland, 1981.

[3] Moulin, H. and J. Vial, "Strategically zero-sum games : the class of games whose completely mixed equilibria cannot be impro-

ved," *International Journal of Game Theory*, Vol.7(1978), pp.201-221.

[4] Rosenthal, R.W., "Correlated equilibria in some classes of two-person games," *International Journal of Game Theory*, Vol.3(1974), pp. 119-128.

[5] Shapley, L., "*Some topics in two person games*," in Annals of Mathematical Studies (Dresher, Shapley, and Tucker, eds.), Princeton : Princeton University Press, 1964.

[6] Vorob'ev, N.N., *Game Theory : Lectures for Economists and Systems Scientists*, New York, Heidelberg, Berlin : Springer-Verlag, 1977.