韓國經營科學會誌
第34권 第1號
2009年 3月

단축-축차관리도의 설계*

장 영 순**

Design of a Curtailed-SPRT Control Chart*

Young Soon Chang**

■ Abstract ■

This paper proposes a curtailed-sequential probability ratio test (SPRT) control chart. For using the conventional SPRT control chart, the number of items inspected in a sampling point should have no restriction since items in a sampling point are inspected one by one until an SPRT is terminated. The number of observations taken in a sampling point, however, has an upper bound since sampling and testing of an item is time-consuming or expensive. When the sample size reaches the upper bound without evidence of an in-control or out-of-control state of a process, the proposed chart makes a decision using the sample mean of all observations taken in a sampling point. The properties of the proposed chart are obtained by a Markov chain approach and the performance of the chart is compared with fixed sample size (FSS) and variable sample size (VSS) control charts. A comparative study shows that the proposed chart performs better than VSS control charts as well as conventional FSS control charts.

Keyword : SPRT Control Chart, VSS Control Chart, Sample Size Upper Bound

1. Introduction

Control charts are used to monitor a process and to make sure that it stays in-control. A conventional approach to sampling for control charts is to take samples of a fixed size (FSS; fixed sample size) with a fixed time interval between samples (FSI; fixed sampling interval), and the

논문접수일 : 2008년 06월 10일　논문게재확정일 : 2009년 01월 21일
논문수정일(1차 : 2008년 12월 15일)
* 이 논문은 2006년도 정부재원(교육과학기술부 학술연구조성사업비)으로 한국학술진흥재단의 지원을 받아 연구되었음(KRF-2006-332-B00125).
** 명지대학교 경영학과

resulting charts are called fixed sampling rate (FSR) control charts. In recent years, variable sampling rate (VSR) control charts have been proposed, which vary the sample size or the sampling interval for the next sampling point based on the value of the current charting statistic. The VSR can be achieved by using variable sample size (VSS) or/and variable sampling interval (VSI) schemes-the VSS scheme varies the sample size and the VSI scheme varies the sampling interval. By increasing the sampling rate during times in which there is evidence of an out-of-control state of the process, the VSR charts can detect an out-of-control state faster than the traditional FSR charts. See Costa [3], Tagaras [11], Reynolds et al. [7], Runger and Montgomery [8], and Zimmer et al. [13] for VSR control charts.

The size of the next sample or the sampling interval to the next sampling point of the VSR control charts is determined in advance. Stoumbos and Reynolds [9], Reynolds and Stoumbos [6], Stoumbos and Reynolds [10], and Reynolds and Kim [5] proposed sequential probability ratio test (SPRT) control charts which determine the size of a sample depending on the current data instead determining the sample size in advance. The SPRT chart applies a sequential probability ratio test at each sampling point which is a highly efficient statistical hypothesis testing method developed by Wald [12].

The SPRT chart can be considered as a kind of VSS chart since the sample size of a sampling point changes. The VSS chart uses one among a few sorts of sample sizes determined previously. The sample size of the SPRT chart, however, can vary from one to infinite extremely. Therefore, the SPRT chart can detect changes in a process much faster than the VSS charts as well as the traditional FSR charts. See Stoumbos and Reynolds [9] for a detailed discussion.

The SPRT chart can be used when there is no restriction on the number of observations at a sampling point. For situations where sampling and testing of an item is time-consuming or expensive, the SPRT chart may not be directly applied. Therefore the SPRT scheme should be modified for using the chart practically. The objective of this paper is to propose an SPRT control chart for a situation in which there exists an upper bound on the sample size and to evaluate the performance of the proposed chart. When the number of observations in a sampling point reaches the upper bound, the proposed chart makes an in-control/out-of-control decision for a process based on the sample mean of all observations obtained in a sampling point.

The remainder of this study is organized as follows. The next section proposes a modified SPRT chart under the curtailed sequential sampling plan. In Section 3, some statistical properties of the proposed chart are evaluated using a Markov chain approach, and the performance of the charts is compared with those of FSR and VSS control charts in Section 4.

2. Curtailed-SPRT Chart

The SPRT control chart applies an SPRT at each sampling point, which is to decide to accept a statistical hypothesis, to reject it, or to obtain an additional observation. The SPRT needs the smallest sample size among statistical testing methods having same type I and type II errors.

When the process mean and the standard deviation at the in-control state are μ_0 and σ, respectively, the charting statistic of the SPRT chart with j observations $X_{i1},\cdots,X_{ij}$ at the ith sampling point to detect a mean shift is

$$U_{ij}=\sum_{k=1}^{j}(Z_{ik}-\gamma), \tag{1}$$

where $Z_{ik}=(X_{ik}-\mu_0)/\sigma$, and the reference value $\gamma=\delta/2$ is used to quickly detect the mean shift to $\mu_1=\mu_0+\delta\sigma(\delta>0)$. When the jth observation is obtained at the ith sampling point, i) if $U_{ij}>h$, the chart issues an out-of-control signal and the process is examined, ii) if $U_{ij}\le g$, sampling at the ith sampling point is stopped and another SPRT is started at the next $(i+1)$th sampling point, or iii) if $g<U_{ij}\le h$, $U_{i,j+1}$ is calculated after obtaining an additional observation $X_{i,j+1}$ and the SPRT is continued. See Stoumbos and Reynolds [9] on the detailed discussion of the chart.

The SPRT chart detects a process change much faster than the traditional control charts, however, it can be used only when there is no restriction on the number of observations at any sampling point due to the characteristic of the SPRT. For situations where the time (or cost) required to take an individual observation is too long (or high), it can not be directly applied. <Table 1> presents the probability distribution and the average number of observations in a sampling point when the SPRT chart with $\gamma=0.15$, $h=16.01$, and $g=0.00$ is used. See Chang and Bai [2] on the formula for obtaining the probability distribution and the average of the sample size. The table shows that the number of observations is fairly large especially when $\delta\le 1.0$. Therefore, the existing SPRT chart may not be used when the limitation of the sample size exists.

〈Table 1〉 The probability distribution and the average of the sample size of the SPRT chart
($\gamma=0.15$, $h=16.01$, and $g=0.00$)

δ \ Num*	> 5	> 10	> 15	> 20	> 25	average
0.00	0.17	0.10	0.07	0.05	0.04	5.00
0.25	0.31	0.24	0.21	0.20	0.18	17.04
0.50	0.46	0.43	0.41	0.39	0.35	18.21
1.00	0.74	0.73	0.56	0.25	0.08	14.48
2.00	0.97	0.20	0.00	0.00	0.00	8.97

Note) * *Num* : the number of observations in a sampling point.

To overcome a week point of the SPRT control chart, this paper proposes a curtailed-SPRT control chart. If the upper bound of the sample size in a sampling point is N and any distinct evidence of the mean shift is not found by the SPRT until the $(N-1)$th observation is obtained, then it is reasonable to decide the state of a process with the sample mean of N observations after the last Nth observation is inspected. The proposed chart uses the SPRT to detect an out-of-control state until the $(N-1)$th observation is obtained. If the decision that the process is in-control or out-of-control is impossible, it is made using the mean of all N observations, i.e., the chart issues an out-of-control signal if $\overline{Z}_i>\zeta\cdot(1/\sqrt{N})$, where $\overline{Z}_i$ is the mean of N standardized observations, $Z_{i1},\cdots,Z_{iN}$, in the ith sampling point, and the mean and the standard deviation of $\overline{Z}_i$ at the in-control state are 0 and $1/\sqrt{N}$, respectively. From Equation (1), $U_{iN}=N\cdot\overline{Z}_i-N\cdot\gamma$. Therefore, if the charting statistic evaluated with the Nth observation in the ith sampling point,

$$U_{iN}>\eta=\zeta\cdot\sqrt{N}-N\cdot\gamma, \tag{2}$$

the chart issues an out-of-control alarm. When the jth observation is taken at the ith sampling point, the operating procedure of the proposed curtailed-SPRT chart is as follows.

i) In case that $j \le N-1$,

(a) if $U_{ij} > h$, the chart issues an out-of-control signal.

(b) if $U_{ij} \le g$, the chart concludes that the process is in-control and go to the next $(i+1)$th sampling point.

(c) if $g < U_{ij} \le h$ and $j < N-1$, (a)-(c) is repeated after obtaining the $(j+1)$th additional observation.

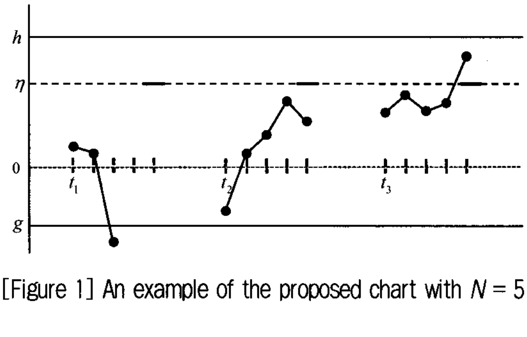
(d) if $g < U_{ij} \le h$ and $j = N-1$, the following procedure ii) is applied after calculating U_{iN} with the last Nth observation.

ii) In case that $j = N$,

(a) if $U_{iN} > \eta$, the chart issues an out-of-control signal.

(b) if $U_{iN} \le \eta$, the chart concludes that the process is in-control and go to the next $(i+1)$th sampling point.

[Figure 1] shows an example of the proposed chart with $N=5$. In the first sampling point t_1, the chart concludes that the process is in-control after the 3rd observation is obtained. In the second sampling point t_2, the chart can not make a decision on the state of the process with first 4 observations but concludes that the process is in-control after the last 5th observation is inspected. In the 3rd sampling point t_3, the chart issues an out-of-control signal since $U_{3,5} > \eta$.



[Figure 1] An example of the proposed chart with $N = 5$

3. Statistical Properties of the Curtailed-SPRT Control Chart

To assess FSS control charts, the average number of sample to signal (*ANSS*) is used in general. VSS control charts, however, should be evaluated with the average sample number (*ASN*) and the average number of observations to signal (*ANOS*) as well as *ANSS* due to the variety of the sample size in a sampling point. In this section, the performance measures of the proposed chart-*ANSS*, *ASN*, and *ANOS*-are derived by the Markov chain approach proposed by Brook and Evans [1].

To compute the performance measures using the Markov chain approach, the state of the chain should be defined. Let E_I, E_o, E_1, E_2, $\cdots$, E_w be the states of a Markov chain, where $E_I = (-\infty, g]$ and $E_o = (h, \infty)$ are absorbing states and $E_1, \cdots, E_w$ are transient states defined by dividing $(g, h]$ into w regions. Let e_u be the midpoint of E_u, $u = 1, \cdots, w$, and the size of the region be $2\Delta = (h-g)/w$. Then, $E_u = (e_u - \Delta, e_u + \Delta]$, $u = 1, \cdots, w$. The transition probabilities matrix in a sampling point is

$$\mathbf{R} = \begin{bmatrix} \mathbf{P} & \mathbf{L} & \mathbf{U} \\ \mathbf{0}^T & 1 & 0 \\ \mathbf{0}^T & 0 & 1 \end{bmatrix}, \tag{3}$$

where 'T' means the transpose of a matrix, $\mathbf{0}$ is the column vector whose all elements are zero, $\mathbf{P} = \{p_{uv}\}$, $\mathbf{L} = \{p_{uI}\}$, and $\mathbf{U} = \{p_{uO}\}$, $u, v = 1, \cdots, w$. p_{uv} is the transition probability that the charting statistic will, when in state E_u, next make a transition into state E_v. p_{uI} and p_{uO} represent the transition probabilities that the charting statistic will make a transition into absorbing states E_I and E_O, respectively. See <Appendix> for the derivation of the transition probabilities.

The performance measures of the proposed chart can be calculated using the transition probabilities matrix of Formula (3). Let $\mathbf{p}_0$ be the initial probability vector whose elements mean the probabilities that the first charting statistic of a sampling point is located in transient state E_u, $u=1,\cdots v$, and p_β be the probability that the first charting statistic is in absorbing state E_I. If a process follows a normal distribution, then *ANSS*, *ASN*, and *ANOS* can be obtained approximately as

$$ANSS=\frac{1}{1-OC},$$
$$ASN=1+\mathbf{p}_0^T(\mathbf{I}-\mathbf{P})^{-1}(\mathbf{I}-\mathbf{P}^{N-1})\mathbf{1}, \quad (4)$$
$$ANOS=ASN\cdot ANSS,$$

where $OC=p_\beta+\mathbf{p}_0^T(\mathbf{I}-\mathbf{P})^{-1}(\mathbf{I}-\mathbf{P}^{N-2})\mathbf{L}+\sum_{i=1}^{w}(\Phi(\eta-e_v+\gamma-\delta)\cdot\psi_v)$, $\Phi(\cdot)$ is the cumulative standard normal distribution function, and ψ_v is the vth element of $\mathbf{p}_0^T\mathbf{P}^{N-2}$. For the detailed derivation, see <Appendix>.

For using the proposed curtailed-SPRT chart, reference value γ and control limits g, h, and η should be calculated considering predetermined N, δ, and in-control *ANSS* and *ASN* ($ANSS_0$ and ASN_0). γ can be determined as $\delta/2$ and $\eta=\zeta\sqrt{N}-N\cdot\gamma$, where ζ is determined as the $100(1-1/ANSS_0)$th percentile of a standard normal random variable $z_{1/ANSS_0}$ such as the control limit of a standardized one-sided $\bar{X}$ control chart. Then g and h should be determined by solving Equation (4). For example, assume that we design the curtailed-SPRT chart with $N=10$, $\delta=0.5$, $ANSS_0=740.8$, and $ASN_0=3.0$. In this case, $\gamma=\delta/2=0.5/2=0.25$, $\zeta=z_{1/740.8}=z_{0.00135}\simeq 3.0$, and $\eta=3\sqrt{N}-N\cdot\gamma=3\sqrt{10}-10\cdot 0.25\simeq 6.99$. By solving Equation (4) with a numerical search method of IMSL subroutines [4], g and h are obtained as $g=-0.29$ and $h=7.59$.

4. Performance of the Curtailed-SPRT Control Chart

This section compares the performance of the proposed chart with those of the FSS and VSS $\bar{X}$ control charts, the conventional SPRT chart, and the cumulative sequential chart [2]. If in-control *ANSS*s (and *ASN*s) of control charts are same, the control chart with smaller out-of-control *ANSS* and *ASN* has a better performance. Therefore, out-of-control *ANSS*, *ANOS*, and *ASN* ($ANSS_1$, $ANOS_1$, and ASN_1, respectively) are used for a comparison after adjusting the control limits of control charts whose in-control *ANSS* and *ASN* are $ANSS_0$ and ASN_0, respectively.

<Table 2> and <Table 3> show $ANSS_1$s, $ANOS_1$s, and ASN_1s of the proposed curtailed-SPRT control chart, the conventional SPRT chart, the cumulative sequential chart, and the FSS and VSS $\bar{X}$ charts according to the mean shift from μ_0 to $\mu_0+\delta\sigma$ when the upper bound of the sample size $N=10$. Note that the conventional SPRT chart cannot be used practically in this case due to the limitation of the sample size in a sampling point. <Table 2> is obtained when $ANSS_0=740.8$ and $ASN_0=3.0$, and <Table 3> is constructed when $ANSS_0=740.8$ and $ASN_0=5.0$. All charts issue an out-of-control alarm when the standardized charting statistic is greater than h. VSS $\bar{X}$ charts use the sample size n_1 at the next sampling point if a charting statistic is less than the threshold

limit ξ, otherwise, they use n_2. γ is the reference value used in the SPRT chart, the cumulative sequential chart, and the proposed chart. [Figure 2] depicts the change of $ANSS_1$ and $ANOS_1$ of the proposed chart according to the upper bound of the sample size N.

The tables and figure imply that :

i) The performance of the proposed curtailed-SPRT chart is better than those of FSS and VSS $\bar{X}$ charts especially when δ is small. In the case of $\delta=0.5$ in <Table 2>, for example, $ANSS_1$ of the FSS $\bar{X}$ chart is 60.89 and those of the VSS $\bar{X}$ charts are 36.66 and 20.94. Whereas $ANSS_1$ of the proposed chart is 14.18. This means that the proposed chart can detect the mean shift as fast as 75% and 30~60% than the FSS and VSS $\bar{X}$ charts, respectively.

ii) ASN_1 of the proposed chart is larger than those of the FSS or VSS charts especially when the mean shift is small. $ANOS_1$ of the curtailed-SPRT chart is, however, smaller than those of the FSS or VSS charts for most cases. This means that the proposed chart increases the efficiency by obtaining more observations in a sampling point if there is a strong possibility of an out-of-control state.

iii) The proposed chart performs better than the cumulative sequential chart when $\delta \geq 1.0$. The performance of the cumulative sequential chart, however, is better than that of the proposed chart when the size of the mean shift is small. This is because the cumulative sequential chart uses a cumulative statistic such as CUSUM (cumula-

<Table 2> $ANSS_1$, $ANOS_1$, ASN_1 of the curtailed-SPRT, conventional SPRT, and $\bar{X}$ charts ($ANSS_0=740.8$ and $ASN_0=3.0$)

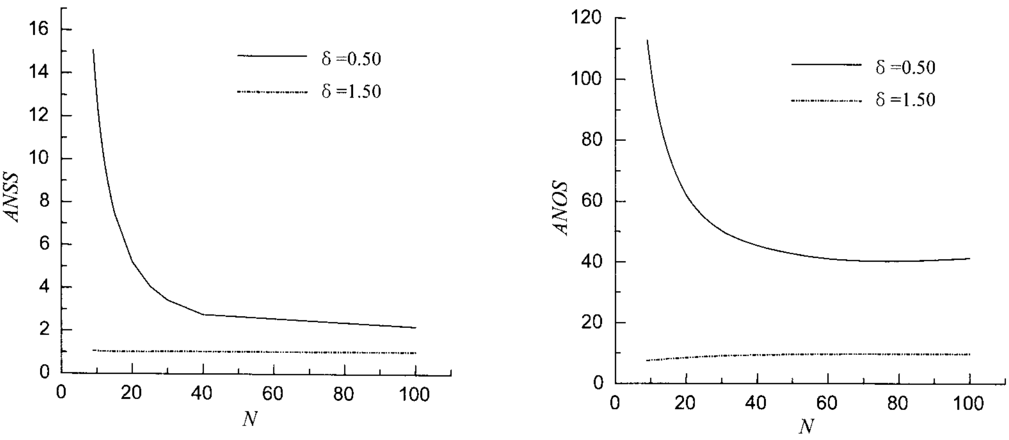
	curtailed-SPRT chart			conventional SPRT chart			cumulative sequential chart			FSS $\bar{X}$ chart			VSS $\bar{X}$ chart ($n_1=2$, $n_2=6$)			VSS $\bar{X}$ chart ($n_1=1$, $n_2=10$)		
δ	ANSS	ANOS	ASN	ANSS	ANOS	ASN	ANSS	ANOS	ASN	ANSS	ANOS	ASN	ANSS	ANOS	ASN	ANSS	ANOS	ASN
0.00	740.80	2222.40	3.00	740.80	2222.40	3.00	740.80	2222.40	3.00	740.80	2222.40	3.00	740.80	2222.40	3.00	740.80	2222.40	3.00
0.25	78.58	331.92	4.22	17.31	127.75	7.38	23.92	128.16	5.36	194.96	584.88	3.00	170.06	592.41	3.48	143.76	530.86	3.69
0.50	14.18	79.40	5.60	3.49	37.26	10.68	5.83	37.13	6.40	60.89	182.67	3.00	36.66	145.74	3.98	20.94	88.74	4.24
0.75	4.24	28.35	6.69	1.99	20.64	10.37	3.20	20.65	6.45	22.48	67.44	3.00	10.48	45.28	4.32	6.08	26.19	4.31
1.00	1.98	14.12	7.13	1.51	14.24	9.43	2.23	14.24	6.39	9.76	29.28	3.00	4.61	20.42	4.43	3.41	14.64	4.29
1.25	1.33	9.08	6.83	1.28	10.88	8.50	1.69	10.88	6.44	4.95	14.85	3.00	2.84	12.47	4.39	2.59	11.62	4.49
1.50	1.12	6.91	6.17	1.16	8.82	7.60	1.32	8.82	6.68	2.91	8.73	3.00	2.14	9.22	4.31	2.25	10.85	4.82
1.75	1.05	5.75	5.48	1.09	7.43	6.82	1.12	7.43	6.63	1.95	5.85	3.00	1.80	7.56	4.20	2.07	10.71	5.17
2.00	1.02	4.98	4.88	1.05	6.43	6.12	1.05	6.43	6.12	1.47	4.41	3.00	1.59	6.46	4.06	1.94	10.55	5.44
2.25	1.01	4.41	4.37	1.03	5.68	5.51	1.02	5.68	5.57	1.23	3.69	3.00	1.43	5.56	3.89	1.83	10.18	5.56
2.50	1.01	3.97	3.93	1.02	5.10	5.00	1.01	5.10	5.05	1.10	3.30	3.00	1.30	4.77	3.67	1.72	9.57	5.56
2.75	1.00	3.61	3.61	1.01	4.63	4.58	1.01	4.63	4.58	1.04	3.12	3.00	1.19	4.12	3.46	1.61	8.79	5.46
3.00	1.00	3.33	3.33	1.00	4.25	4.25	1.00	4.25	4.25	1.01	3.03	3.00	1.11	3.64	3.28	1.51	7.89	5.23
γ	0.25			0.25			0.25											
h	7.59			10.14			10.13			3.00			3.00			3.00		
g	-0.29			0.08			-0.05											
η	6.99																	
ξ													0.67			0.76		

〈Table 3〉 $ANSS_1$, $ANOS_1$, ASN_1 of the curtailed-SPRT, conventional SPRT, and $\bar{X}$ charts
($ANSS_0$ = 740.8 and ASN_0 = 5.0)

	curtailed-SPRT chart			conventional SPRT chart			cumulative sequential chart			FSS $\bar{X}$ chart			VSS $\bar{X}$ chart (n_1 = 2, n_2 = 6)			VSS $\bar{X}$ chart (n_1 = 1, n_2 = 10)		
δ	ANSS	ANOS	ASN	ANSS	ANOS	ASN	ANSS	ANOS	ASN	ANSS	ANOS	ASN	ANSS	ANOS	ASN	ANSS	ANOS	ASN
0.00	740.80	3704.00	5.00	740.80	3704.00	5.00	740.80	3704.00	5.00	740.80	3704.00	5.00	740.80	3704.00	5.00	740.80	3704.00	5.00
0.25	74.35	489.30	6.58	10.75	151.81	14.12	20.22	153.03	7.57	136.54	682.72	5.00	125.30	674.75	5.39	99.56	582.60	5.85
0.50	13.01	103.62	7.96	2.22	41.24	18.58	5.12	41.23	8.05	33.42	167.10	5.00	27.72	156.06	5.63	15.61	100.69	6.45
0.75	3.83	33.66	8.79	1.43	22.59	15.80	2.89	22.47	7.78	10.76	53.80	5.00	9.03	51.67	5.72	5.14	33.93	6.60
1.00	1.80	15.90	8.83	1.19	15.51	13.03	2.09	15.39	7.36	4.50	22.50	5.00	4.26	24.25	5.69	2.97	19.47	6.56
1.25	1.22	10.03	8.22	1.09	11.82	10.84	1.65	11.71	7.10	2.39	11.95	5.00	2.70	15.15	5.61	2.30	15.27	6.64
1.50	1.05	7.61	7.25	1.05	9.57	9.11	1.31	9.46	7.22	1.57	7.85	5.00	2.08	11.47	5.51	2.06	14.00	6.80
1.75	1.01	6.33	6.27	1.02	8.05	7.89	1.10	7.95	7.23	1.22	6.10	5.00	1.78	9.66	5.43	1.95	13.56	6.95
2.00	1.00	5.48	5.48	1.01	6.96	6.89	1.02	6.87	6.74	1.08	5.40	5.00	1.59	8.51	5.35	1.87	13.16	7.04
2.25	1.00	4.84	4.84	1.01	6.14	6.08	1.00	6.07	6.07	1.02	5.10	5.00	1.43	7.58	5.30	1.79	12.59	7.03
2.50	1.00	4.35	4.35	1.00	5.50	5.50	1.00	5.44	5.44	1.00	5.00	5.00	1.30	6.78	5.22	1.70	11.84	6.96
2.75	1.00	3.96	3.96	1.00	5.00	5.00	1.00	4.94	4.94	1.00	5.00	5.00	1.19	6.12	5.14	1.60	10.95	6.84
3.00	1.00	3.64	3.64	1.00	4.58	4.58	1.00	4.53	4.53	1.00	5.00	5.00	1.11	5.64	5.08	1.50	9.98	6.65
γ	0.25			0.25			0.25											
h	8.44			11.04			10.89			3.00			3.00			3.00		
g	-1.27			-0.58			-0.99											
η	6.99																	
ξ													-0.68			0.14		

tive sum) control charts.

iv) As N increases, the performance of the proposed chart is improved exponentially. If the mean shift is small, the conventional SPRT chart performs better than the proposed curtailed-SPRT chart with the small upper bound of the



[Figure 2] $ANSS_1$ and $ANOS_1$ of the curtailed-SPRT chart according to N

sample size. Otherwise, the performance of the proposed chart is similar to that of the conventional SPRT chart.

5. Conclusions

This paper proposes a curtailed-SRPT control chart, where items are inspected one by one and a sequential probability ratio test at each sampling point is applied. The proposed chart can be used practically when the restriction of the sample size exists, whereas the conventional SPRT chart cannot be used in such a case. The performance measures to evaluate the proposed chart are developed using a Markov chain approach. The comparative study shows that the proposed chart performs better than the FSS and VSS $\overline{X}$ charts. The performance of the curtailed-SPRT chart is dramatically improved as the upper bound of the sample size increases. Further, the optimal decision problem of the upper bound of the sample size and the control limits should be studied.

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⟨Appendix⟩

Let the process mean be $\mu = \mu_0 + \delta\sigma$, then the transition probabilities

$$\begin{aligned} p_{uv} &= \Pr\{U_{i,j+1} \in E_v \mid U_{i,j} \in E_u\} \quad (A.1) \\ &\simeq \Pr\{e_v - \Delta < U_{i,j+1} \le e_v + \Delta \mid U_{i,j} = e_u\} \\ &= \Phi(e_v - e_u + \Delta + \gamma - \delta) - \Phi(e_v - e_u - \Delta + \gamma - \delta), \\ & u, v = 1, 2, \cdots, w, \end{aligned}$$

$$\begin{aligned} p_{uI} &= \Pr\{U_{i,j+1} \in E_I \mid U_{i,j} \in E_u\} \quad (A.2) \\ &\simeq \Pr\{U_{i,j+1} \le g \mid U_{i,j} = e_u\} \\ &= \Phi(g - e_u + \gamma - \delta), \\ & u = 1, 2, \cdots, w, \end{aligned}$$

$$\begin{aligned} p_{uO} &= \Pr\{U_{i,j+1} \in E_O \mid U_{i,j} \in E_u\} \quad (A.3) \\ &\simeq \Pr\{U_{i,j+1} > h \mid U_{i,j} = e_u\} \\ &= 1 - \Phi(h - e_u + \gamma - \delta), \\ & u = 1, 2, \cdots, w. \end{aligned}$$

As a similar way, initial probabilities p_v, $v = 1, \cdots, w$ and p_β can be calculated as $p_v = \Pr\{U_{i1} \in E_v\} = \Phi(e_v + \Delta + \gamma - \delta) - \Phi(e_v - \Delta + \gamma - \delta)$ and $p_\beta = \Pr\{U_{i1} \in E_I\} = \Phi(g + \gamma - \delta)$, respectively.

Using Chapman-Komogorov Equation, the n step transition probabilities matrix, which consists of the probabilities that the nth charting statistic $U_{i,n}$ will make a transition into state E_v when $U_{i,1} \in E_u$, is

$$R^n = \begin{bmatrix} \mathbf{P}^n & \mathbf{L}_n & \mathbf{U}_n \\ \mathbf{0}^T & 1 & 0 \\ \mathbf{0}^T & 0 & 1 \end{bmatrix}, \quad (A.4)$$

where $\mathbf{L}_n = \mathbf{P}^{n-1}\mathbf{L} + \mathbf{L}_{n-1} = (\mathbf{I}-\mathbf{P})^{-1}(\mathbf{I}-\mathbf{P}^n)\mathbf{L}$ and $\mathbf{U}_n = (\mathbf{I}-\mathbf{P})^{-1}(\mathbf{I}-\mathbf{P}^n)\mathbf{U}$.

i) Derivation of *ANSS*

Since the proposed chart performs the SPRT independently at each sampling point, *ANSS* can be easily obtained with the probability *OC* that the chart concludes the process is in-control at a sampling point. The probability *OC* is obtained as

$$\begin{aligned} OC &= \Pr\{\text{with } U_{i,1}\} + \Pr\{\text{with } U_{i,2} \sim U_{i,N-1}\} \quad (A.5) \\ &\quad + \Pr\{\text{with } U_{i,N}\} \\ &= p_\beta + \mathbf{p}_0^T \mathbf{L}_{N-2} + \textstyle\sum_{v=1}^{w} [\Phi(\eta - e_v + \gamma - \delta)\psi_v] \\ &= p_\beta + \mathbf{p}_0^T (\mathbf{I}-\mathbf{P})^{-1}(\mathbf{I}-\mathbf{P}^{N-2})\mathbf{L} \\ &\quad + \textstyle\sum_{v=1}^{w} [\Phi(\eta - e_v + \gamma - \delta)\psi_v], \end{aligned}$$

where ψ_v is the vth element of $\mathbf{p}_0^T \mathbf{P}^{N-2}$. Therefore, $ANSS = 1/(1-OC)$.

ii) Derivation of *ASN*

Let *SN* be the number of observations obtained in a sampling point, then the cumulative distribution function of *SN* is

$$\begin{aligned} \Pr\{SN \le n\} &= \mathbf{p}_0^T(\mathbf{L}_{n-1} + \mathbf{U}_{n-1}) + p_\beta + p_\alpha \quad (A.6) \\ &= \mathbf{p}_0^T(\mathbf{I} - \mathbf{P}^{n-1})\mathbf{1} + (1 - \mathbf{p}_0^T\mathbf{1}) \\ &= 1 - \mathbf{p}_0^T \mathbf{P}^{n-1}\mathbf{1}, \quad n = 1, \cdots N-1, \\ \Pr\{SN \le N\} &= 1, \end{aligned}$$

where $p_\alpha = \Pr\{U_{i1} \in E_O\} = 1 - \Phi(h + \gamma - \delta)$. Using (A.6), *ASN* can be obtained as

$$\begin{aligned} ASN &= \textstyle\sum_{n=1}^{N} \cdot \Pr\{SN = n\} \quad (A.7) \\ &= 1 - \mathbf{p}_0^T\mathbf{1} + \textstyle\sum_{n=2}^{N-1} n \cdot \mathbf{p}_0^T(\mathbf{P}^{n-2} - \mathbf{P}^{n-1})\mathbf{1} \\ &\quad + N \cdot \mathbf{p}_0^T \mathbf{P}^{N-2}\mathbf{1} \\ &= 1 + \mathbf{p}_0^T(\mathbf{I}-\mathbf{P})^{-1}(\mathbf{I}-\mathbf{P}^{N-1})\mathbf{1}. \end{aligned}$$