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A Study of Two Preservice Teachers' Development of Covariational Reasoning

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This article describes the interview data with two preservice teachers where they dealt with five water-filling problems for the investigation of their covariational thinking. The study's results revealed that two students developed their covariation levels from Direction level to Instantaneous Rate with an aid of the pre-constructed GSP simulations for the problem situations. However, this study also points out that there is a missing important feature for a function notion, 'causality' in the covariation framework and suggests that future research should combine students' conception of causality with their covariational thinking for the investigation of their development of a function concept.

Key Words : Function, Covariational reasoning, GSP simulation

Introduction

The concept of a function is without doubt one of the most important concepts in modern mathematics and it has various aspects and associated sub-concepts, which may account for some of the difficulties observed in school mathematics. Thompson (1994) commented that it is impractical to speak strictly about the development of a single concept like a function. For the analysis of students' concepts of a function, we should be aware that "the imagery and understanding evoked in students by our probing is going to be textured by their pre-understandings of such things as expressions, variables, arithmetic operations, and quantity" (Thompson, 1994, p. 1). Due to the complexity of the "understanding function concepts," many researchers have tried to interpret function concepts and students' difficulties with the concepts from various points of view. Among those perspectives, this study focuses on students' covariational reasoning and investigates a pair of preservice secondary mathematics teachers' notion of covariational reasoning and its development when their interpreting and representing

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dynamic functional events. Students' understanding of covariation is, as Saldanha and Thompson (1998) describe, "holding in mind a sustained image of two quantities' values (magnitudes) simultaneously" (p. 298). Actually, the focus on covariational reasoning as a foundational study for function concepts is initiated by the observations that (a) the literature on early function instruction supports the promotion of conceptual thinking about functions that includes investigations of patterns of change, (b) recent research has demonstrated that even academically talented undergraduate students had difficulty in modeling functional relationships of situations involving the rate of change of one variable as it continuously varied in a dependent relationship on another variable, and (c) little research on preservice secondary mathematics teachers′ understanding of functions has examined fine-grained features in terms of its developmental aspect although there has been some cross-sectional reports about students′ covariational reasoning. Therefore, the results will provide a new insight into preservice teachers′ development of a function conception in terms of covariational reasoning and further contribute to a well-defined structure for designing a course and creating curriculum for preservice secondary mathematics teachers.

Literature Review

Development of function concepts

There are several theoretical perspectives in the literature where a stable consensus seems to have developed regarding the importance for students' understanding of a function. Although these are not totally separate one another, close examination on each perspective could benefit the understanding of the issues of learning and teaching the notion of a function. In higher level of mathematics like functions as well as in arithmetic, many studies have reported that there exists individuals' own different understanding of the same 'formal' symbol or name in mathematics textbooks. That is, it is improper to say that symbols and names themselves contain the meaning of concepts. Tall and Vinner (1981) used the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is constructed over the years through experiences of all kinds and changed whenever the individual meets new stimuli and materials. Thus, it need not be coherent at all times through the process of development. The portion of the concept image activated at a particular time is called the evoked concept image. At different times, seemingly conflicting images may be evoked. There is no actual sense of conflict or confusion until conflicting aspects are evoked simultaneously. On the other hand, the concept definition is a form of words used to specify that concept, that is, a form of words that the student uses for one's own explanation of evoked concept image. A personal concept definition can differ from a formal concept definition, which is accepted by the mathematical community at large (Tall & Vinner, 1981). Thompson (1994) also argued that a predominant image evoked in students by the word "function" is of two written expressions separated by an equal sign, citing an instance of Thompson's undergraduate mathematics class where not a single student thought that there was anything wrong with the formulation f(x)=n(n+1)(2n+1)/6 because it fitted within students' concept image of function.

Skemp (1971) stated that functions can be defined not only as a set of ordered pairs, but also as a certain computational process or as a method for getting from one system to another. Sfard (1991) also indicated that the latter type of description speaks about processes, algorithms and actions rather than about objects. Therefore, it reflects an operational conception of a function. In general, interpreting a notion as a process implies regarding it as a potential rather than an actual entity, which comes into existence upon request in a sequence of actions. Thus, the operational is dynamic, sequential, and detailed. On the other hand, seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing - a static structure, existing somewhere in space and time (Sfard, 1991). The ability of seeing a function both as a process and as an object is indispensable for a deep understanding of mathematics. She also argued that the structural approach should be regarded as the more advanced stage of concept development historically as well as psychologically. According to Sfard, the model of learning can be refined into three stages: interiorization, condensation and reification. At the stage of interiorizationa a learner gets acquainted with the processes which will eventually give rise to a new concept. These processes are operations performed on lower-level mathematical objects. Gradually, the learner becomes skilled at performing these processes. In the case of function, it is when the idea of variable is learned and the ability of using formula to find values of the "dependent" variable is acquired. The phase of condensation is a period of "squeezing: lengthy sequences of operations into more manageable units." At this stage a person becomes more and more capable of thinking about a given process as a whole, without feeling a necessity to go into details. When a function being considered, the more capable the person becomes of playing with a mapping as a whole, without actually looking into its specific values, the more advanced in the process of condensation he or she should be regarded. Eventually, the learner can investigate functions, draw their graphs, combine couples of functions (e.g. by composition), even to find the inverse of a given function. Only when a person becomes capable of conceiving the notion as a fully-fledged object, it is said that the concept has been reified. Reification, therefore, is defined as an ontological shift - a sudden ability to see something familiar in a totally new light. In the case of function, reification may be evidenced by proficiency in solving equations in which "unknowns" are functions (differential and functional equations with parameters), by ability to talk about general properties of different processes performed on functions (such as composition or inversion), and by ultimate recognition that computability is not a necessary characteristic of the sets of ordered pairs (Sfard, 1991).

Dubinsky and Harel (1992) adopted the terms prefunction, action, process and object conceptions for describing a function conception. At the stage of prefunction the subject really does not display very much of a function concept. Whatever the term means to such a subject, this meaning is not very useful in solving the tasks that are requested in mathematical activities related to functions. An action is a repeatable mental or physical manipulation of objects. Such a conception of function would involve, for example, the ability to plug numbers into an algebraic expression and calculate. It is a static conception in that the subject will tend to think about it one step at a time. A student whose the function conception is limited to actions might be able to form the composition of two given functions by replacing each occurrence of the variable in one expression by the other expression and then simplifying, but he or she would probably be unable to compose two functions in more general situations, e.g., when they are given by different expressions on different parts of their domains, or if they are not given by expressions at all, but by algorithms. A process conception of function involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done. Notions such as 'one to one' or 'onto' become more accessible as the subject's process conception strengthens (Dubinsky & Harel, 1992). A function is comprehended as an object if it is possible to perform actions on it. At the point where students have solidified a process conception of function so that a representation of the process is sufficient to support their reasoning about it, they can begin to reason formally about functions - they can reason about functions as if they were objects. To reason formally about functions seems to involve a scheme of conceptual operations which grow from a great deal of reflection on functional processes. An image of functional process as defining a correspondence between two sets is crucial among the conceptual operations. One hallmark of a student's object conception of functions is her ability to reason about operations on sets of functions(Breidenbach, Dubinsky, Hawks, & Nichols, 1992). However, when individuals would be in transition from action to process and, as with all cognitive transitions, the progress is never in a single direction. This makes it quite difficult to determine with certainty that a function concept of an individual is limited to action or that he or she has a process conception.

Covariation framework

Realizing the lack of fine-grained analysis for the development of function notions, Carlson, Jacob, Coe, Larsen, and Hsu (2002) conducted their study based on five mental actions of covariational reasoning (see Table 1). The mental actions of the covariation framework provide a means of classifying behaviors as students engage in covariation tasks.

Table 1: Mental actions of the covariation framework (from Carlson et al., 2002, p. 357)

Mental Actions of the Covariation Framework

Mental Action	Description of mental action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	Constructing an increasing straight line Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	Plotting points/constructing secant lines Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	Constructing contiguous secant lines for the domain Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	Constructing a smooth curve with clear indications of concavity changes Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

A student is given a level classification according to the overall image that appears to support the various mental actions that he or she exhibits in the context of a problem or a task. The covariation framework contains five distinct developmental levels (see Table 2). Carlson et al. (2002) argue that one's covariational reasoning ability should go under a given level of development when it supports the mental actions associated with that level and the actions associated with all lower levels.

Tabel 2: Levels of the covariation framework (from Carlson et al., 2002, p. 358)

Levels of the Covariation Framework

Covariational Reasoning Levels

The covariation framework describes five levels of development of images of covariation. These images of covariation are presented in terms of the mental actions supported by each image.

Level 1 (L1). Coordination

At the coordination level, the images of covariation can support the mental action of coordinating the change of one variable with changes in the other variable (MA1).

Level 2 (L2). Direction

At the direction level, the images of covariation can support the mental actions of coordinating the direction of change of one variable with changes in the other variable. The mental actions identified as MA1 and MA2 are both supported by L2 images.

Level 3 (L3). Quantitative Coordination

At the quantitative coordination level, the images of covariation can support the mental actions of coordination the amount of change in one variable with changes in the other variable. The mental actions identified as MA1, MA2 and MA3 are supported by L3 images.

Level 4 (L4). Average Rate

At the average rate level, the images of covariation can support the mental actions of coordinating the average rate of change of the function with uniform changes in the input variable. The average rate of change can be unpacked th coordinate the amount of change of the output variable with changes in the input variable. The mental actions identified as MA1 through MA4 are supported by L4 images.

Level 5 (L5). Instantaneous Rate

At the instantaneous rate level, the images of covariation can support the mental actions of coordinating the instantaneous rate of change of the function with continuous changes in the input variable. This level includes an awareness that the instantaneous rate of change resulted from smaller refinements of the average rate of change. It also includes awareness that the inflection point is where the rate of change changes from increasing to decreasing, or decreasing to increasing. The mental actions identified as MA1 through MA5 are supported by L5 images.

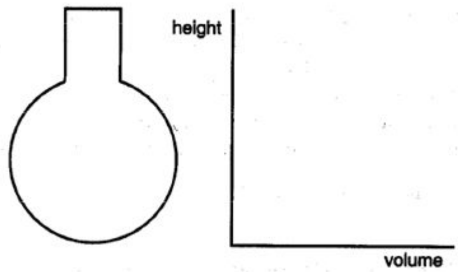
For instance, twenty students who had completed their second semester calculus with a course grade ‘A’ participated in their study and responded to five items that involved an analysis of covariant aspects of dynamic events. The five-item instrument was completed by each of twenty subjects within a week of completing the final exam. Written items were then scored using carefully developed and tested rubrics (Carlson, 1998), and the percentage of the students who provided each response type for each item was determined. The six interviews were conducted within two days of the student’s completion of the five-item written instrument. During the interview, the participant initially read each question aloud and made general reference to his or her written response and was subsequently prompted to describe and justify the solution verbally. After the student summarized the written response, the researcher made general inquiries, using prompts such as "explain" or "clarify,"and continued to ask more specific questions until the student appeared to have communicated all relevant knowledge. Through the study, they observed weakness in students' ability to interpret and represent rate-of-change information (MA4; cf. Table 1). Along with the use of kinesthetic enactment, these students were more often able to observe patterns in the changing magnitude of the output variable(MA3), as well as patterns in the changing nature of the instantaneous rate (MA5). Nonetheless, their difficulty in viewing an instantaneous rate by imaging smaller and smaller refinements of the average rate of change appeared to persist (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Nevertheless, Carlson et al.'s study revealed some limitation. That is, although they provided very detailed descriptions about the status of students' covariational reasoning under the various dynamic situations based on their covariation framework, they seem to have failed to capture the moment of development from a lower level to a higher one and described the process through such levels.

Methods of inquiry

At the end of the semester in 2007 spring, a pair of junior undergraduate students (preservice teachers) under mathematics education program were interviewed. Lora and Amy were intentionally chosen for the clinical interview because rich verbal data on what the students were doing could be expected by their prior experience of working together in class. Those students already had passed a basic differential and integral calculus, and a linear algebra course as prerequisites. Since the course they were taking was being conducted in technology-enriched environment where Geometer's Sketchpad (GSP) was frequently used, two students would not have difficulty using GSP simulations designed for four interview tasks. The interview was video-recorded with two cameras: one for the full feature of working students and one for the detailed hands-on activities and GSP simulations on their computer screen. It was conducted once for about an hour in a small conference room and interview questions were flexibly asked relying on the interviewee's responses to the task. The main role of interviewers was to investigate the interviewee's notion of covariational reasoning and their spontaneous development rather than to guide them intentionally in a certain way. All collected data were transcirbed and analyzed based on the five mental actions of the covariational framework.

The interview tasks were initiated by the realization of restraints of Carlson et al.'s study indicated above, and designed for the possibility of capturing the moment of development under their five mental actions of the covariation framework. The tasks consisted of six problems asking the students to sketch a function between two varying factors in particular dynamic situations. The first problem and the last (the sixth) problems were actually same each other (the bottle problem) which originated from Carlson et al.'s five items. The bottle problem (see Figure 1) prompted students to construct a graph of a dynamic situation with a continuously changing rate and with an instance of the rate changing from decreasing to increasing.

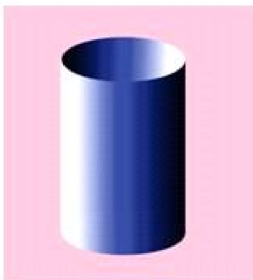
> Imagine the bottle filling with water at a constant rate. Construct a graph of the height as a function of volume of water for the given bottle. Explain the reasoning you used when constructing your graph.



[Figure 1] The bottle problem (from Carlson, Oehrtman and Thompson, 2007, p. 164)

The reason to put the same question onto the first and the last place in the interview process was to easily see their change of the location among the covariational reasoning levels as a result of the interview. In other words, the students were expected to rethink of the same situation and reflect an influence of the whole interview tasks. In order to help the students develop their covariational reasoning in a dynamic situation, four questions and four supportive GSP simulations, which might encourage students to get the clear picture of the water filling situation, were inserted in the middle of the two bottle problems.

> Imagine this cylinder filling with water at a constant rate. You need to construct a graph of the height as a function of volume for the given cylinder.



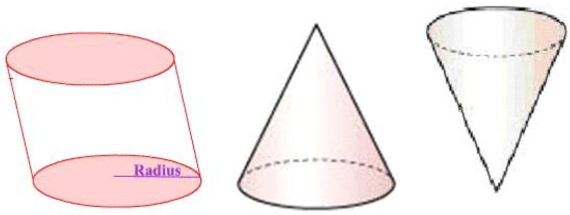
2.1. What are the independent variable and the dependent variable in this dynamic situation? Why?

2.2. Using established construction in GSP, simulate the above dynamic situation. For the simulation of water filling, refer to the given gsp animation on "Cylinder" tab. Can you guess the relationship between the amount of volume and the height?

2.3. Construct a graph of the height as a function of volume for the given cylinder and explain the reasoning you used when constructing your graph. (Cooperative work)

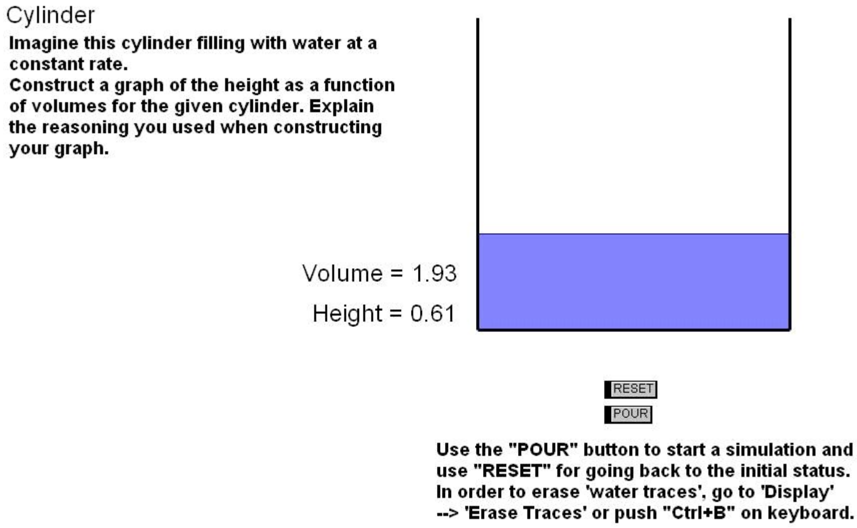
[Figure 2] Cylinder problem

Although the ultimate expected outcomes of four questions were basically same as the bottle filling problem, the shapes of the water-holders were different one another: cylinder, slant cylinder, circular cone, flipping cone (cf. Figure 2, 3).



[Figure 3] Slant cylinder, circular cone and flipping cone

Additionally, two sub-questions were built up in the four middle problems for support of the students' mathematical thinking. The first sub-question asked students to find out an independent variable and a dependent variable in a particular dynamic situation. The reason to ask the question was that Carlson et al's covariational framework did not explicitly state 'causality' between variables although they described the distinction as one characteristic of MA1 stage. The second question was to encourage the students to simulate a given dynamic situation using a GSP construction, which was built on the basis of the algebraic equation between two variables in the situation. By clicking "pour" button, students can see the water-holder filling with water at a constant rate and changing numerical values of two variables. While students' repeating of the simulation, we anticipated that the simulation could help students develop their covariational reasoning in the situation (see Figure 4).



[Figure 4] GSP simulation for cylinder problem

Analysis

The following analysis for the interview with Lora and Amy is organized into three chronological phases:

- Phase 1: Coordinating the direction of change of one variable with changes in the other variable (Problem 1, 2, 3).
- Phase 2: Coordinating the average rate-of-change of the function with uniform increments of change in the input variable (Problem 4, 5).
- Phase 3: Coordinating the instantaneous rate of the function with continuous changes for the entire domain of the function (Problem 6).

At the beginning of the interview, Amy and Lora had some difficulty in understanding the situation of the first problem. The verbal expression of 'the bottle with water at a constant rate' made them confused for imagining what the water was doing in that situation. Thus, only after they seemed to understand the situation exactly by clarifying the dynamic situation as the bottle filling situation with water at a constant rate, they were allowed to solve the interview tasks.

Phase 1: Coordinating the direction of change of one variable with changes in the other variable (Problem 1, 2, 3).

Working on the first problem (the bottle problem), they showed specific behaviors and verbal expressions which seemed to correspond to MA2.

// … // Denotes concurrent talk.

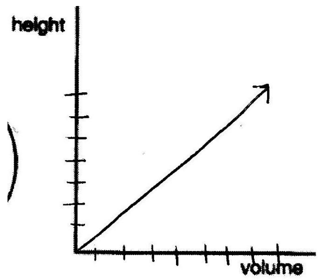
(inaudible) Denotes a time when it was impossible to understand what a student said.

(…) Denote a comment inserted while transcribing.

'Int' stands for the interviewer

Int: So your graph, you two are similar or a little different. What do you think about that?
Lora: Ya,
Amy: (inaudible) because we had the same thought about the…
Int: Same thought?
Amy: Ya, the water increasing and height and then the volume increasing.
Int: Uh-huh
Amy: So goes up

According to Carlson et al., MA2 can be identified by observing students construct an increasing straight line, or verbalize that as more water is added, the height of the water in the bottle increases. As being observed in the above excerpt, Amy and Lora were definitely aware of the direction (increasing) of the change of volume while considering changes in the height. In addition, the graphs for the first problem on their worksheet supported that they were at least at the position of the Direction Level (L2) since both drew a straight line as a possible graph for that dynamic situation (see Figure 5).



[Figure 5] Lora's graph for problem 1 (Bottle)

When the interviewer asked if there was any specific reason that the graph would be a straightline rather than a curve, Amy did not provide a clear justification for her drawing and she admitted the possibility of a curve-shaped graph. In other words, she did not seem to be the level of Quantitative Coordination (L3), coordinating the amount of changes between two variables. Another interesting aspect in the first problem was that Lora and Amy actually exchanged the role of the independent variable (height) and the dependent variable (volume) and disregarded the volume on x-axis and the height on y-axis on their given worksheets. However, such a response was not surprising because Carlson et al. (2002) already reported that some students treated a y-coordinate as the independent variable. Although it appeared problematic in terms of correct description of a function under such a particular situation, it was hard to locate the behavior in Carlson's covariation framework because her framework did not explicitly illuminate the *causality* between variables for a function concept. At that time, the interviewer expected that Amy and Lora would have become to distinguish the independent variable and the dependent variable through solving the subsequent problems asking to find the independent and the dependent variable in each situation, but it turned out to be his wrong anticipation.

As Amy and Lora went on the second (cylinder) problem, they still showed a sign of MA2, like the verbal expression that as one increases, the other increases. However, when the interviewer asked the difference between the bottle problem and the cylinder problem, Amy used an interesting word, 'more consistent'

Int: In first problem, you said as one increasing, the other increasing. Right?
Amy: Uh-huh. Same thing.
Int: Same thing. Can you see the difference or is that basically the same?
Amy: The difference in the cylinder and the bottle?
Int: Uh-huh
Amy: Well this one will move more constant because the other one was rounded and then went up.
Int: The shape is, the shape is different
Amy: Ya,

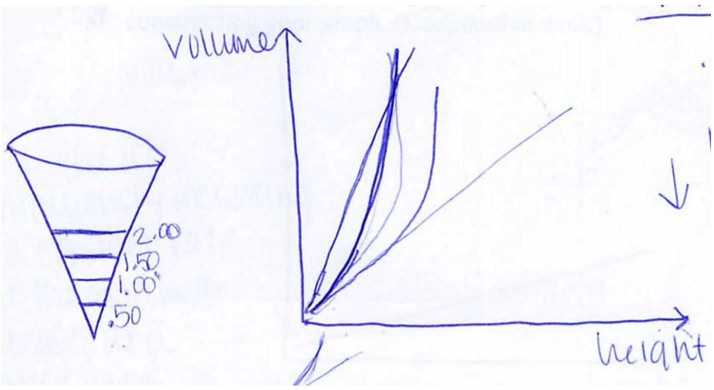
Amy explained the difference between the two problems in the sense that the water in the cylinder would move more constantly than one in the bottle. Although the interviewer should have investigated the exact meaning of 'move more constant', one possibility is that she might feel vaguely the difference of the amount of increasing volume or height between two different shapes as the other variable changes at a constant rate, which implies Quantitative Coordination (L3). However, it would not be a sufficient evidence of MA3. In the third (slant cylinder) problem, both agreed that the situation was same as the previous two situations and the graph would be a straight line. The GSP simulation for the problem seemed to strengthen the confidence of their conjecture. Overall, the stage of phase 1, Amy and Lora were at L2 although Amy's comment 'move more constant' remains unclear. In some sense, their staying at L2 through the problem 2 and 3 should not be surprising because there was no change of rate of volume in the situation of the problem 2 and 3 whose intention was to design tasks along with gradual progresses of the covariation framework.

Phase 2: Coordinating the average rate-of-change of the function with uniform increments of change in the input variable (Problem 4, 5)

The most conspicuous aspect at this phase was to begin to use the verbal expression of comparative degree (e.g. go faster, get bigger). In addition, from this phase they seemed to rely more on the GSP simulation of problem 4 (circular cone) and problem 5 (flipping cone) than the previous problems (cylinder, slant cylinder).

Lora: Well the height... As the water's added at a constant rate, the height at the beginning isn't gonna…
Amy: The height is same. Is it? It's
Lora: It's not increasing at a constant rate though because
Amy: Ya, it is.
Lora: The bottom is wider than the top and the height here
Amy: But that's the height, that's the height.
Lora: Don't draw on the computer.
Amy: I'm sorry. That's the height.
Lora: Okay, but so the height is gonna start out slow and then get taller really quick like increase really quickly the height of the water. Not height of this one (inaudible) of the cone
Amy: Ya, ya.
Lora: So the height is gonna start out slow and then it's gonna get the amount of water taller faster but what is the volume doing?

Lora seemed to realize that the height did not increase constantly as the water was added at a constant rate in the problem 4 (circular cone). According to Carlson et al.(2002), MA4 can be identified in students by observing their construction of contiguous line segments on the graph, with the slope of each segment adjusted to reflect the (relative) rate for the specified amount of water; or by hearing remarks that express their awareness of the rate of change of the height with respect to the volume while they consider equal amounts of water. While solving the problem 4 and 5, Amy and Lora definitely described their awareness of the rate of change and even further their constructions of graph for the problem 4 and 5 on their worksheet showed an indication for MA5, that is, construction of a smooth curve (see Figure 6). Nonetheless, it is unlikely that they were going through L5, or even getting at L4 because they were still struggling with the distinction between the two variables (height and volume).



[Figure 6] Lora's graph for problem 4 (Circular cone)

The typical remarks for MA5 would be the expression of the instantaneous changes in the rate of change of the height with respect to the volume while they consider the entire range of the water volume. However, as seen in the last comment of Lora in the above interview excerpt, they still had difficulty in finding out the role of height and volume in the dynamic situation. In the problem 5 (flipping cone), they still considered the height of poured water as the independent variable although they explicitly expressed the rate of change in terms of two variables.

Int: Then what about flipping cone?
Amy: Do what? The flipped cone?
Int: Inverse cone, ya.
Amy: That was the volume increase faster than the height right? No, no the height increases faster than the volume a lot faster. So then this one... The height,
Lora: So it will be more down right? So the slanted cone will be more up. Cause like the height's continuously increasing for the volume is increasing slower.
Amy: Ya.
Lora: whereas the other cone increase faster.
Amy: and is the height still our independent?
Lora: Um-huh.

With the covariation framework, the detailed explanations for this kind of confusion, related to the causality of a function notion, do not seem feasible. Although Carlson et al. (2002) noted that some students had been observed switching the roles of the independent (volume) and the dependent (height) variable several times in the context of discussing the thinking that they had used to construct the graph for this type of tasks, they did not explain such a phenomenon in terms of the covariation framework.

With the interviewer's intention of helping the students make sense of the situation, Amy and Lora were asked to make a numeral table for the values of two variables for the problem 4 (circular cone) using a GSP option. However, even with this effort, their expression was under the framework of MA3, that is, the expression of the awareness of how the height changed while they considered increases in the amount of water and they were still switching the role of the independent (volume) and the dependent (height).

Amy: You stop that at the same time.
Lora: I know. Wow, see I can't even stop it passing that for the 1.5. 2 and 2.09. So the height increases a lot here but the volume doesn't increase much here.
Int: Uh-huh
Amy: That was at 1.67 and the volume is still 2.09. If that means anything,
Lora: Okay,
Int: Which variables are... which variables, which variable did increase constantly?
Amy: The height
Lora: The height
Int: Height?
Lora: But the volume increases slower and then really fast
Int: The volume increase
Amy: As always the height increases slow and then fast
Lora: No, because we didn't even variables we did .25 .5 1.6 and then 2. I guess it increases at a steady rate the volume increased slowly but at the end it was like from 1.8 to 2.9 and three of these.
Amy: Okay.
Int: So //which increase fast?
Lora: So as a height increases// at a steady rate the volume increases steady or slower and then really fast.

When Lora interpreted the collected data for making a numeral table between two variables, she correctly described the situation in terms of two variables as 'the height increases a lot here but the volume doesn't increase much here.' However, she finally changed the role of two variables and stated that 'the volume increases steady or slower and then really fast as a height increases.'

Their difficulty might arise from the conflict between their prior knowledge and their obtaining knowledge from the interview tasks. In other words, the relationship between height and volume in the formula that they learned before played a more dominant and more consistent role in determining the independent and the dependent variable so that they could not realized 'added water' as a volume. Actually, Lora's definition of independent and dependent variable was exactly aligned with normative mathematics. The dependent variable depends on the independent variable and the dependent variable will change according to the independent variable. Nonetheless, they even drew out three variables from the situation: added water, height, and volume and after all they identified the added water with the height, not with the volume! Since the volume should always be subject to the height in their formula, the volume could not control the height and the added water, which was distinguished from the volume. Instead, the added water controled the height. The next interview excerpt corroborates the indicated discordance between their prior knowledge and their gaining knowledge from the observation in the GSP simulation.

Lora: I mean where the water is what's been added. So how much water is added
Amy: Which is also the height of the water
Lora: is gonna control..
Amy: Height.
Lora: but it's the volume too how much water's added. //The water
Amy: but we,// but our only variables are the height and the volume. The water is the height.
Lora: (laughing)
Int: Let me phrase this in a different way. Which determine which.
Lora: I don't know. I mean the height is the part of the volume equation.
Int: Uh-huh.
Lora: So, you would think that the height determines the volume.

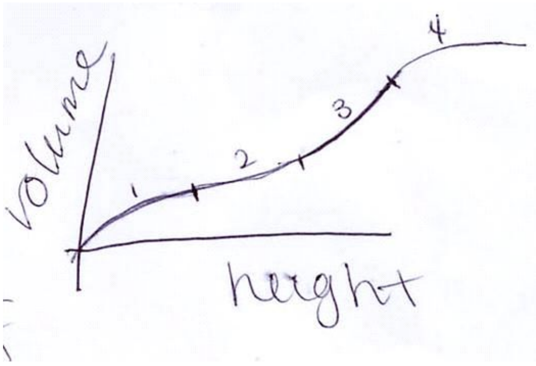
Phase 3: Coordinating the instantaneous rate of the function with continuous changes for the entire domain of the function (Problem 6).

MA5 has been identified in students (1) by observing the construction of a smooth curve that was concave down then concave up, then linear and (2) by hearing remarks that suggested an understanding that the smooth curve resulted from considering the changing nature of the rate while imaging the water changing continuously (Carlson et al., 2002). In phase 3, Amy's and Lora's verbal expressions and their constructions of graph on their worksheet revealed that they finally arrived at the Instantaneous Rate Level (L5). They could follow the trace of the change of volume (actually it should be height) as water was being added even though they identified the change of height as the change of volume.

Int: So both of you, in your work, have some bending points, right? Can you explain where, what's the bending point? What does it mean by that?
Amy: It means that well because this increases slowly and then we said that it was a cylinder then it increases at a constant rate so when it comes around this bend it kind of slows down a little bit and then when it comes back to our cone type shape it goes faster which creates the curve, maybe not that drastic of curve but some of a curve.
Lora: Just because it was like a point right here well it's gonna the height is gonna increase at a same, constant rate as the volume.
Int: Uh-huh
Lora: But like at the top of the bottle and the bottom of the volume, bottle they're

not gonna increase at a constant rate.
Amy: //they are the same rate as this
Lora: The height will always increase// but the volume will increases faster and slower depending on what curve

The shape of their graphs for the last bottle problem also showed that they were developing their covariational reasoning toward the L5. They drew S-type curves which correctly reflected the trace of the rate of change. These constructions clearly contrasted to their previous works for the same bottle problem at the beginning, which were straight lines (cf. Figure 6).



[Figure 6] Amy's graph for problem 6 (Bottle)

Conclusion

Past research emphasized on the instances of transformational reasoning, but they seemed to fail to investigate how the transformation of reasoning progressed. On the other hand, the aim of this study was to provide some useful information about the process of developing covariation reasoning through *intentionally designed* several dynamic situations. Through the retrospective analysis of the interview data with the covariation framework, it might be asserted that the purpose of this project was achieved quite well in the sense that the interview with Amy and Lora demonstrated that their covariational level could be transformed from the Direction Level (L2) to the Instantaneous Rate (L5). It should not be neglected that the carefully designed tasks (cylinder, slant cylinder, circular cone, and flipping cone) with the help of GSP simulation for each dynamic situation played an important role in the development of their mathematical thinking. Thus, this study casts some useful information regarding the effectiveness of certain curricular interventions for developing of students' ability to apply covariational reasoning when solving the problems that involve real-world dynamic situations.

However, it has to be pointed out that there is something to be revisited for the further study in students' development of a function conception. One of the interesting features in the observation of those two students was that their judgment for the independent (volume) variable and the dependent (height) variable in each situation stayed misinterpreted to the last moment when they revisited the first problem at the end of the interview although they showed gradual progresses along with the levels of covariation framework while working through the interview tasks. In some sense, we could argue that all the evidence of Amy and Lora could be matched up with the covariational framework except that they changed the role of the independent and the dependent. In other words, it can be claimed that the causality of a function notion has nothing to do with the covariational reasoning because being able to compare the simultaneously varying amount of two quantities is the essential point of covariational reasoning, rather to find out the causal relationship between two variables. However, since (1) the conceptions of *covariation* and *causality* are too important to be separated for students' development of a function concept and (2) the collected data from Amy and Lora showed that the two conceptions were entangled each other in the real problem situations, it does not seem reasonable to discuss such phenomenon under different categories. Although the data in this study might not be enough to explicate the reciprocal interactions in details between their covariational reasoning and their realization of causality, the examination of the influences between those two components may well be considered seriously for future research in students' development of a function notion.

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모의실험을 통한 두 예비교사의 공변추론 이해에 관한 연구

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초 록

본 연구는 예비교사들이 어떻게 공변추론(covariational reasoning)의 개념을 이해하는지를 정성연구방법을 통하여 연구하였다. Geometer's Sketchpad 를 이용해 만들어진, 문제상황들을 위한 모의실험을 통해 두 학생들은 공변 추론의 단계가 '방향'수준에서 '순간비율' 수준으로 발전하였음이 연구분석 결과로 나타났다. 하지만, 이 연구를 통해 함수 학습을 위한 중요한 개념 중 하나인 '인과성'이 공변추론 양식틀에서 빠져있음을 알 수 있었고, 따라서 앞으로 학생들의 함수개념 발달의 연구를 위해서 공변추론과 인과성이 서로 연계되어 이루어 져야 할 필요성이 제시되었다.

주요용어 : 함수, 공변추론, GSP 모의실험

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