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CHUNG-TYPE LAW OF THE ITERATED LOGARITHM OF l^∞-VALUED GAUSSIAN PROCESSES

Yong-Kab Choi, Zhenyan Lin, and Wensheng Wang

Abstract. In this paper, by estimating small ball probabilities of l^∞-valued Gaussian processes, we investigate Chung-type law of the iterated logarithm of l^∞-valued Gaussian processes. As an application, the Chung-type law of the iterated logarithm of l^∞-valued fractional Brownian motion is established.

1. Introduction and main results

Let $\{Y(t), t \geq 0\} = \{X_k(t), t \geq 0\}_{k=1}^{\infty}$ be a sequence of independent centered Gaussian processes with stationary increments $\sigma_k^2(h) = E(X_k(t+h) - X_k(t))^2$, where $\sigma_k(h)$ is assumed to be a non-decreasing function in h for each $k \geq 1$. Put

$$\sigma^*(h) = \max_{k \geq 1} \sigma_k(h). \tag{1.1}$$

There has been a lot of papers to study the limit behavior of $Y(\cdot)$. For path properties, Csáki and Csörgő [1] investigated the moduli of continuity for $Y(\cdot) \in l^p$, $1 \leq p \leq 2$. Applying a general Fernique type inequality and well-known Borell inequality, Csörgő and Shao [4] studied the increments of $Y(\cdot) \in l^p$ for every $1 \leq p < \infty$. Wang and Zhang [14] studied Chung-type law of the iterated logarithm (LIL) of $Y(\cdot) \in l^p$ for every $1 \leq p < \infty$. When $Y(\cdot)$ is an l^∞-valued process, Csörgő et al. [2] and Lin and Qin [9] studied the moduli of continuity and large increment theorem of $Y(\cdot)$ respectively. Up to now, as far as we know, little is known on limit inferior behavior of $Y(\cdot) \in l^\infty$. In this paper, by estimating small ball probabilities of $Y(\cdot) \in l^\infty$, we investigate Chung-type LIL of l^∞-valued Gaussian processes. As an immediate consequence of our results, we establish Chung-type law of the iterated logarithm of l^∞-valued fractional Brownian motion.

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Hoffmann-Jørgensen et al. [5] studied the lower tail probability of l^∞-valued Gaussian processes. They studied the behavior of

$$P(\max_{k\geq 1}|X_k(1)-X_k(0)|\leq\varepsilon)\quad\text{as } \varepsilon\to 0^+.$$

But up to now, as far as we know, little is known for the behavior of

$$P(\sup_{0\leq t\leq 1}\max_{k\geq 1}|X_k(t)-X_k(0)|\leq\varepsilon)\quad\text{as } \varepsilon\to 0^+. \tag{1.2}$$

One of the purposes of this present paper is to estimate small ball probabilities of l^∞-valued Gaussian processes. We obtain a sharp bound for (1.2) in this paper.

It is well known that by the Borel-Cantelli lemma, one can easily obtain a lower bound of limit inferior provided that an upper bound of small ball probability is available. However, deducing an upper bound of limit inferior from the small ball probability need certain independence and it is not so obvious. In this paper, we use the spectral representation of Gaussian processes (as [11] or [13] did) to get the necessary independence.

Our main results read as follows.

Theorem 1.1. *Assume that the following conditions are satisfied*:

(i) $X_k(0)=0$ *with probability one for every* $k\geq 1$;

(ii) $\sigma^*(h)/h^\alpha$ *is quasi-increasing on* $(0,1)$ *for some* $\alpha>0$;

(iii) *there exists* $0<\tau<2$ *such that*

$$\sigma^*(2h)\leq\tau\sigma^*(h),\quad\forall\ 0<h<1/2;$$

(iv) $\max_{k\geq 1}\max_{i\geq 3}E\{(X_k(2x)-X_k(x))(X_k(ix)-X_k((i-1)x))\}\leq 0$;

(v) *there exist positive constants* A_1 *and* θ_0*, which are independent of* k*, such that*

$$\inf_{0<s\leq 1/2}\frac{\sigma^*(s)}{\sigma_k(s)}\geq A_1\sigma_k^{-1}(\theta_0)\quad\textit{for every } k\geq 1;$$

(vi) *there exist* $A_2>0$, $p\geq 1$ *and* $r>1$ *such that for any* $n\geq 1$

$$\sum_{k=n+1}^{\infty}\sigma_k(\theta_0)^p\leq A_2n^{-rp},$$

where θ_0 *is as in* (v)*. Then there exists a positive constant* A_3 *such that*

$$1/A_3\leq\liminf_{h\to 0}\sup_{0\leq t\leq h}\max_{k\geq 1}\frac{|X_k(t)|}{\sigma^*(h/\log\log(1/h))}\leq A_3\quad a.s.. \tag{1.3}$$

Remark 1.2. If condition (iv) is replaced by the following condition:

(iv)$'$ there exists $\tau\in(0,4)$ such that $\sigma_k^2(2x)\leq\tau\sigma_k^2(x)$ for $0\leq x\leq 1/2$ and each $k\geq 1$; moreover,

$$\max_{k\geq 1}\max_{i\geq 2}E\Big\{(X_k(3x)+X_k(x)-2X_k(2x))(X_k((2i+1)x)+X_k((2i-1)x)-2X_k(2ix))\Big\}\leq 0,$$

then (1.3) remains true.

In order to prove Theorem 1.1, we need to estimate small ball probabilities of l^∞-valued Gaussian processes, which will be given in Section 2. The proof of Theorem 1.1 will be given in Section 3. Applications to l^∞-valued fractional Brownian motion are discussed in Section 4.

In what follows, we will use K to denote unspecified positive and finite constants whose value may be different in each occurrence. Constants that are referred to in the sequel will be denoted by $A_1, A_2, \ldots$.

2. Small ball probabilities of l^∞-valued Gaussian processes

In this section we concern small ball probabilities of l^∞-valued Gaussian processes. Theorems 2.1 and 2.2 below give a sharp bound for (1.2).

Theorem 2.1. *We have that*

(a) *if condition* (iv) *is satisfied, then there exists a positive constant* A_4 *such that for any* $0 < x \le 1/4$

$$P\Big(\sup_{0\le t\le 1}\max_{k\ge 1}|X_k(t)-X_k(0)| \le \sigma^*(x)\Big) \le e^{-A_4/x}; \tag{2.1}$$

(b) *if condition* (iv)$'$ *is satisfied, then* (2.1) *remains true.*

Theorem 2.2. *If conditions* (ii), (v), *and* (vi) *are satisfied, then there exists a positive constant* A_5 *such that for any* $0 < x \le 1$

$$P\Big(\sup_{0\le t\le 1}\max_{k\ge 1}|X_k(t)-X_k(0)| \le A_5\sigma^*(x)\Big) \ge e^{-A_5/x}. \tag{2.2}$$

The proof of Theorem 2.2 needs a Khatri-Sdák type lemma.

Lemma 2.3. *Let* $\{Y(t), t\in T\} = \{X_k(t), t\in T\}_{k=1}^\infty$ *be a sequence of independent centered separable Gaussian processes,* $\{\lambda(t), t\in T\}$ *be a positive real function, where* $T\subset\mathbb{R}$. *Then for any* $t_0\in T$

$$\begin{aligned} &P\Big(\sup_{t\in T}\max_{k\ge 1}\frac{|X_k(t)-X_k(0)|}{\lambda(t)} \le 1\Big)\\ &\ge P\Big(\sup_{t\in T/\{t_0\}}\max_{k\ge 1}\frac{|X_k(t)-X_k(0)|}{\lambda(t)} \le 1\Big)P\Big(\max_{k\ge 1}\frac{|X_k(t_0)-X_k(0)|}{\lambda(t_0)} \le 1\Big).\end{aligned}$$

It is an immediately consequence of Proposition 1.2.2 in [10].

Proof of Theorem 2.1. (a) Take $v = v(x)$ such that $\sigma_v(x) = \sigma^*(x)$. Clearly

$$\max_{k\ge 1}\frac{|X_k(ix)-X_k((i-1)x)|}{\sigma^*(x)} \ge \frac{X_v(ix)-X_v((i-1)x)}{\sigma_v(x)} =: \xi(x,i). \tag{2.3}$$

Then, by Slepian's inequality (cf., e.g., [8]),

$$\begin{aligned}
&P\Big(\sup_{0\le t\le 1}\max_{k\ge 1}|X_k(t)-X_k(0)|\le \sigma^*(x)\Big)\\
&\le P\Big(\max_{1\le i\le [1/x]}\max_{k\ge 1}|X_k(ix)-X_k((i-1)x)|\le 2\sigma^*(x)\Big)\\
&\le P\Big(\max_{2\le i\le [1/x]}\xi(x;i)\le 2\Big)\\
&\le (\Phi(2))^{[1/x]-1}\le \exp\Big\{\frac{\log\Phi(2)}{2x}\Big\},
\end{aligned}$$

which implies the desired result. Here, and in the sequel, $[\cdot]$ is the greatest integer function.

(b) Let

$$\eta(x;i)=\xi(x;2i)-\xi(x;2i-1),\quad 2\le i\le [1/(2x)],$$

where $\xi(x;i)$ is as above. Then, a direct calculation shows that

$$E\{\eta^2(x;i)\}=4\sigma_v^2(x)-\sigma_v^2(2x)\ge (4-\tau)\sigma_v^2(x),\quad 2\le i\le [1/(2x)].$$

Hence, by Slepian's lemma again, we obtain

$$\begin{aligned}
&P\Big(\sup_{0\le t\le 1}\max_{k\ge 1}|X_k(t)-X_k(0)|\le \sigma^*(x)\Big)\\
&\le P\Big(\max_{1\le i\le [1/x]}\max_{k\ge 1}|X_k(ix)-X_k((i-1)x)|\le 2\sigma^*(x)\Big)\\
&\le P\Big(\max_{1\le i\le [1/x]}\xi(x;i)\le 2\Big)\\
&\le P\Big(\max_{2\le i\le [1/(2x)]}\eta(x,i)\le 4\Big)\\
&\le \prod_{i=2}^{[1/(2x)]}\Phi(4/\sqrt{4-\tau})\le \exp\Big\{\frac{\log\Phi(4/\sqrt{4-\tau})}{4x}\Big\}.
\end{aligned}$$

Hence (2.1) is true. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. For each $0\le t\le 1$ we can write $t=\sum_{l=1}^{\infty}\varepsilon_l 2^{-l}$, where $\varepsilon_l=0$ or 1. Hence

$$\sup_{0\le t\le 1}\max_{k\ge 1}|X_k(t)-X_k(0)|\le \sum_{l=1}^{\infty}\max_{1\le i\le 2^l}\max_{k\ge 1}|X_k(i2^{-l})-X_k((i-1)2^{-l})|. \tag{2.4}$$

Let n_0 be a positive integer such that

$$1/x\le 2^{n_0}\le 2/x.$$

By condition (ii), fix c such that

$$\sigma^*(x)/x^\alpha\le c\sigma^*(y)/y^\alpha\quad\text{for all } 0<x<y<1.$$

For $2 < \theta < 2\sqrt{2}$ with $\log_\theta 2 > 1/r$, where r is given in condition (vi), define

$$x_l = \sigma^*((\theta/2)^{-|l-n_0|}x)(1-2^{-\alpha/2})/(2c), \quad l = 1, 2, \ldots.$$

Since $\sigma^*(x)/x^\alpha$ is quasi-increasing, we have for $0 < a < 1$ that $\sigma^*(ax) \le c a^\alpha \sigma^*(x)$. Hence

$$\begin{aligned}\sum_{l=1}^\infty x_l &\le \sum_{l=1}^\infty (\theta/2)^{-|l-n_0|\alpha}\sigma^*(x)(1-2^{-\alpha/2})/2 \\ &\le \sum_{l=0}^\infty (\theta/2)^{-l\alpha}\sigma^*(x)(1-2^{-\alpha/2}) \\ &= \sigma^*(x)(1-2^{-\alpha/2})/(1-(2/\theta)^\alpha) \\ &\le \sigma^*(x)\end{aligned}$$

since $2 < \theta < 2\sqrt{2}$.

Let c_0 be a constant satisfying

$$c_0 \ge 2cA_1^{-1}(4A_2)^{1/p}(1-2^{-\alpha/2})^{-1}(E|N(0,1)|^p)^{1/p},$$

where A_1 and A_2 are as in conditions (v) and (vi) respectively. By Lemma 2.3 and (2.4) we have

$$\begin{aligned}B &:= P\Big(\sup_{0\le t\le 1}\max_{k\ge 1}|X_k(t)-X_k(0)| \le c_0\sigma^*(x)\Big) \\ &\ge \prod_{l=1}^\infty \prod_{1\le i\le 2^l} P\Big(\max_{k\ge 1}|X_k(i2^{-l}) - X_k((i-1)2^{-l})| \le c_0 x_l\Big) \\ &= \prod_{l=1}^\infty \Big(P\Big(\max_{k\ge 1}\frac{|X_k(2^{-l})-X_k(0)|}{\sigma^*(2^{-l})} \le c_0 x_l/\sigma^*(2^{-l})\Big)\Big)^{2^l}.\end{aligned}$$

Let r_0 be a large positive number, which is independent of n_0 and x, and will be specified later on. Put

$$Z_l := Z(l) = \max_{k\ge 1}\frac{|X_k(2^{-l})-X_k(0)|}{\sigma^*(2^{-l})},$$

$$D_{1l} := D_1(n_0,l) = \frac{1}{2c}\Big(\frac{1}{\theta}\Big)^{n_0-l}(1-2^{-\alpha/2})$$

and

$$D_{2l} := D_2(n_0,l) = \frac{1}{2c^2}(1-2^{-\alpha/2})(4/\theta)^{\alpha(l-n_0)}.$$

Then by rewriting B we obtain

$$B \ge B_1 \times B_2 \times B_3,$$

where

$$B_1 := \prod_{l=1}^{n_0}\Big(P\Big(Z_l \le c_0 x_l/\sigma^*(2^{-l})\Big)\Big)^{2^l},$$

$$B_2 := \prod_{l=n_0+1}^{n_0+r_0} \Big(P\Big(Z_l \le c_0 x_l/\sigma^*(2^{-l})\Big)\Big)^{2^l}$$

and

$$B_3 := \prod_{l=n_0+r_0+1}^{\infty} \Big(P\Big(Z_l \le c_0 x_l/\sigma^*(2^{-l})\Big)\Big)^{2^l}.$$

Since $2^{-n_0} \le x$, $\sigma^*(\cdot)$ is non-decreasing,

$$\theta^{n_0-l}\sigma^*((\theta/2)^{-(n_0-l)}2^{-n_0}) \ge \sigma^*(2^{-l})$$

by Minkowski's inequality and $X_k(\cdot),\ k=1,2,\dots$ are independent, we have

$$\begin{aligned} B_1 &\ge \prod_{l=1}^{n_0}(P(Z_l \le c_0 D_{1l}))^{2^l} \\ &= \prod_{l=1}^{n_0} P\Big(\max_{1\le k\le n} |X_k(2^{-l}) - X_k(0)| \le c_0 D_{1l}\sigma^*(2^{-l})\Big) \\ &\quad \times P\Big(\max_{k\ge n+1} |X_k(2^{-l}) - X_k(0)| \le c_0 D_{1l}\sigma^*(2^{-l})\Big) \end{aligned} \tag{2.5}$$

for any positive integer n.

Let $0<\beta<1$ such that $\beta \ge \frac{1}{r\log_\theta 2}$. From conditions (v) and (vi), it is easy to see that

$$\begin{aligned} \sup_{l\ge 1} \sum_{k=[2^{\beta(n_0-l)}]+1}^{\infty} \frac{\sigma_k(2^{-l})^p}{\sigma^*(2^{-l})^p} &\le A_1^{-p} \sum_{k=[2^{\beta(n_0-l)}]+1}^{\infty} \sigma_k(\theta_0)^p \\ &\le A_1^{-p}A_2 2^{-\beta(n_0-l)rp} \\ &\le 4^{-1}(2c)^{-p}(1-2^{-\alpha/2})^p c_0^p \delta_p^{-p}\theta^{-pn}. \end{aligned} \tag{2.6}$$

Choosing $n=[2^{\beta(n_0-l)}]$, we have by (2.6)

$$\begin{aligned} &P\Big(\max_{k\ge n+1} |X_k(2^{-l}) - X_k(0)| \le c_0 D_{1l}\sigma^*(2^{-l})\Big) \\ &\ge 1 - c_0^{-p}D_{1l}^{-p}(\sigma^*(2^{-l}))^{-p} \sum_{k=n+1}^{\infty} E|X_k(2^{-l}) - X_k(0)|^p \\ &\ge 1 - 2c_0^{-p}D_{1l}^{-p}(\sigma^*(2^{-l}))^{-p}\delta_p^p \sum_{k=n+1}^{\infty} \sigma_k(2^{-l})^p \\ &\ge 1/2. \end{aligned} \tag{2.7}$$

Next, noting that

$$P(|N(0,1)| \le y) \ge 2\exp(-y_0^2/2)y/\sqrt{2\pi} \quad \text{for } 0 \le y \le y_0 \text{ with } y_0 > 0,$$

and $X_k(\cdot), k = 1, 2, \ldots$ are independent, we have

$$\begin{aligned}
&P\Big(\max_{1\le k\le n} |X_k(2^{-l}) - X_k(0)| \le c_0 D_{1l}\sigma^*(2^{-l})\Big)\\
&\ge \prod_{k=1}^{n} P\Big(|X_k(2^{-l}) - X_k(0)| \le c_0 D_{1l}\sigma_k(2^{-l})\Big)\\
&\ge (P(|N(0,1)| \le c_0 D_{1l})^n\\
&\ge (c_0 \exp(-c_0^2/2) D_{1l}/(\sqrt{2\pi}))^n.
\end{aligned} \tag{2.8}$$

Combining these estimates we arrive at

$$P(Z_l \le c_0 D_{1l}) \ge \Big(c_0 \exp(-c_0^2/2) D_{1l}/(\sqrt{2\pi})\Big)^n/2. \tag{2.9}$$

Therefore (note that we have taken $n = [2^{\beta(n_0-l)}]$ above),

(2.10)

$$\begin{aligned}
B_1 &\ge \prod_{l=1}^{n_0} (P(Z_l \le c_0 D_{1l}))^{2^l}\\
&\ge \prod_{l=1}^{n_0} \exp\Big\{2^l (n \log(c_0 \exp(-c_0^2/2)) + n \log D_{1l} - n \log(\sqrt{2\pi}) - \log 2)\Big\}\\
&= \exp\Big\{-\sum_{l=1}^{n_0} 2^l n(-\log(c_0 \exp(-c_0^2/2)) + \log(2c/(1-2^{-\alpha/2}))\\
&\qquad + (n_0 - l)\log\theta + \log 6)\Big\}\\
&\ge \exp\Big\{-2^{n_0} \sum_{l=1}^{n_0} 2^{-(1-\beta)(n_0-l)}(|\log(c_0 \exp(-c_0^2/2))| + \log(2c/(1-2^{-\alpha/2}))\\
&\qquad + (n_0 - l)\log\theta + \log 6)\Big\}\\
&\ge \exp\{-c_1 2^{n_0}\},
\end{aligned}$$

where $c_1 > 0$ is a constant.

Now $\sigma^*(h)/h^\alpha$ quasi-increasing and $x2^{n_0} \ge 1$, together imply

$$\begin{aligned}
B_2 \text{ (or } B_3) &\ge \prod_{l=n_0+1}^{n_0+r_0} \Big(\text{or} \prod_{l=n_0+r_0+1}^{\infty}\Big)\Big(P\Big(Z_l \le \frac{1}{2c^2} c_0 (1-2^{-\alpha/2})(4/\theta)^{\alpha(l-n_0)}\Big)\Big)^{2^l}\\
&=: \prod_{l=n_0+1}^{n_0+r_0} \Big(\text{or} \prod_{l=n_0+r_0+1}^{\infty}\Big)(P(Z_l \le c_0 D_{2l}))^{2^l}.
\end{aligned}$$

Following the same lines of the estimation for B_1 (but choosing $n = [2^{\beta(l-n_0)}]$ instead of $n = [2^{\beta(n_0-l)}]$), we have that there exists a constant $c_2 > 0$ such that

$$B_2 \ge \exp\{-c_2 2^{n_0}\}. \tag{2.11}$$

Now consider B_3. Note that

$$P(|N(0,1)| \le y) \ge \exp(-3e^{-y^2/2}) \quad \text{for } y > 1. \tag{2.12}$$

Choose $r_0 = r_0(\alpha, \theta)$ large enough such that $\frac{1}{4}(1-2^{-\alpha/2})(4/\theta)^{\alpha r_0} c_0 \ge 1$. Then, we have

$$\begin{aligned} & P\Big(\max_{k\ge 1} |X_k(2^{-l}) - X_k(0)| \le c_0 D_{2l} \sigma^*(2^{-l})\Big) \\ = & \prod_{k=1}^{\infty} P\Big(|N(0,1)| \le c_0 D_{2l} \frac{\sigma^*(2^{-l})}{\sigma_k(2^{-l})}\Big) \\ \ge & \exp\Big\{ -3 \sum_{k=1}^{\infty} e^{-c_0^2 D_{2l}^2 \frac{(\sigma^*(2^{-l}))^2}{2\sigma_k^2(2^{-l})}} \Big\}. \end{aligned} \tag{2.13}$$

Combining these estimates we arrive at

$$B_3 \ge \exp\Big\{-3 \sum_{l=n_0+r_0+1}^{\infty} \sum_{k=1}^{\infty} 2^l \exp\Big\{ -c_0^2 (2c^2)^{-2} (1-2^{-\alpha/2})^2 (4/\theta)^{2\alpha(l-n_0)} \frac{(\sigma^*(2^{-l}))^2}{2\sigma_k^2(2^{-l})} \Big\}\Big\}.$$

From condition (vi), it is easy to see that

$$\sum_{k=1}^{\infty} d^{-\sigma_k^{-2}(\theta_0)} < \infty \quad \text{for } d > 1. \tag{2.14}$$

Thus, by condition (v) and (2.14) we have

$$\begin{aligned} B_3 &\ge \exp\Big\{-2^{n_0} \times 3 \sum_{v=r_0+1}^{\infty} \sum_{k=1}^{\infty} 2^v \exp\{-c_0^2(2c^2)^{-2}(1-2^{-\alpha/2})^2 (4/\theta)^{2\alpha v} \frac{(\sigma^*(2^{-v-n_0}))^2}{2\sigma_k^2(2^{-v-n_0})}\}\Big\} \\ &\ge \exp\Big\{-2^{n_0} \times 3 \sum_{v=r_0+1}^{\infty} \sum_{k=1}^{\infty} \exp\{-c_0^2(2c^2)^{-2}(1-2^{-\alpha/2})^2 (4/\theta)^{2\alpha v} \frac{(\sigma^*(2^{-v-n_0}))^2}{4\sigma_k^2(2^{-v-n_0})}\}\Big\} \\ &\ge \exp\Big\{-2^{n_0} \times 3 \sum_{v=r_0+1}^{\infty} \exp\{-c_0^2(2c^2)^{-2}(1-2^{-\alpha/2})^2 (4/\theta)^{2\alpha v}\} \\ &\qquad \times \sum_{k=1}^{\infty} \exp\{-c_0^2(2c^2)^{-2}(1-2^{-\alpha/2})^2 (4/\theta)^{2\alpha r_0} \times A_2^2 \frac{1}{4\sigma_k^2(\theta_0)}\}\Big\} \\ &\ge \exp\{-c_3 2^{n_0}\}, \end{aligned}$$

where $c_3 > 0$ is a constant. Hence, with $c_4 = c_1 + c_2 + c_3$, we have

$$B \ge \exp\{-c_4 2^{n_0}\} \ge \exp\{-2c_4/x\},$$

which implies (2.2) with $A_5 = \max\{c_0, 2c_4\}$ immediately. □

3. Proof of Theorem 1.1

We first prove the following two general theorems, which may be of independent interest.

Theorem 3.1. *Assume that condition* (ii) *is satisfied, and that there exist two positive constants* x_0 *and* A_6 *such that for any* $0 < x \le x_0$,

$$P\Big(\sup_{0\le t\le 1}\max_{k\ge 1}|X_k(t)-X_k(0)| \le \sigma^*(x)\Big) \le e^{-A_6/x}. \tag{3.1}$$

Then

$$\liminf_{h\to 0}\sup_{0\le t\le h}\max_{k\ge 1}\frac{|X_k(t)-X_k(0)|}{\sigma^*(A_6h/\log\log(1/h))} \ge 1 \quad a.s., \tag{3.2}$$

where A_6 *is the constant in* (3.1)*.*

Proof. Using (3.1) and the standard argument, one can obtain (3.2) easily. □

Theorem 3.2. *Assume that conditions* (i)*,* (ii)*,* (iii)*,* (v)*, and* (vi) *are satisfied, and that there exists a positive constant* A_7 *such that for any* $0 < x \le 1$,

$$P\Big(\sup_{0\le t\le 1}\max_{k\ge 1}|X_k(t)| \le A_7\sigma^*(x)\Big) \ge e^{-A_7/x}. \tag{3.3}$$

Then we have

$$\liminf_{h\to 0}\sup_{0\le t\le h}\max_{k\ge 1}\frac{|X_k(t)|}{A_7\sigma^*(A_7h/\log\log(1/h))} \le 1 \quad a.s., \tag{3.4}$$

where A_7 *is the constant in* (3.3)*.*

Remark 3.3. If condition (vi) in Theorem 3.2 is weakened by the following condition:

(vi)$'$ there exists $d > 1$ such that

$$\sum_{k=1}^{\infty} d^{-\sigma_k^{-2}(\theta_0)} < \infty, \tag{3.5}$$

where $\theta_0 > 0$ is a constant given as in condition (v), then, (3.4) remains true.

Proof of Theorem 3.2. Let

$$M(h) = \sup_{0\le t\le h}\max_{k\ge 1}|X_k(t)|.$$

For $0 < \varepsilon < 1$, put

$$s_n = \exp(-n^{1+\varepsilon}), \quad d_n = \exp(n^{1+\varepsilon}+n^{\varepsilon}), \quad \sigma_{(n)} = A_7\sigma^*(A_7 s_n/\log\log(1/s_n)).$$

It suffices to show that

$$\liminf_{n\to\infty} M(s_n)/\sigma_{(n)} \le 1 \quad \text{a.s..} \tag{3.6}$$

To prove (3.6), we use the spectral representation of $X_k(\cdot)$, as Shao and Wang [13] (or, as [11]) did. It is well-known that for each $k \ge 1$, $E\{X_k(s)X_k(t)\}$ has a unique Fourier representation of the form

$$E\{X_k(s)X_k(t)\} = \int_{\mathbb{R}}\Big(e^{is\lambda}-1\Big)\Big(e^{-it\lambda}-1\Big)\Delta_k(d\lambda) + B_k st. \tag{3.7}$$

Here B_k is some positive number and $\Delta_k(d\lambda)$ is a nonnegative measure on $\mathbb{R} - \{0\}$ satisfying

$$\int_{\mathbb{R}} \frac{\lambda^2}{1+\lambda^2} \Delta_k(d\lambda) < \infty.$$

Moreover, there exist a centered, complex-valued Gaussian random measure $W_k(d\lambda)$ and a Gaussian random variable V_k which is independent of W_k such that

$$X_k(t) = \int_{\mathbb{R}} \left(e^{it\lambda} - 1\right) W_k(d\lambda) + V_k t. \tag{3.8}$$

The measures W_k and Δ_k are related by the identity $E\{W_k(A)\overline{W_k(B)}\} = \Delta_k(A \cap B)$ for all Borel sets A and B in $\mathbb{R}$. Furthermore, $W_k(-A) = \overline{W_k(A)}$. It follows from (3.8) that for $0 < h < 1$,

$$\sigma_k^2(h) = 2\int_{\mathbb{R}} (1 - \cos(h\lambda))\Delta_k(d\lambda) + h^2 B_k \geq 2\int_{\mathbb{R}} (1 - \cos(h\lambda))\Delta_k(d\lambda). \tag{3.9}$$

We have

$$\begin{aligned} \int_{|\lambda|\geq 1/h} \Delta_k(d\lambda) &\leq \frac{1}{1 - \sin 1} \int_{|\lambda|\geq 1/h} \left(1 - \frac{\sin(h\lambda)}{h\lambda}\right) \Delta_k(d\lambda) \\ &= \frac{1}{(1 - \sin 1)h} \int_{|\lambda|\geq 1/h} \int_0^h (1 - \cos(u\lambda)) du \Delta_k(d\lambda) \\ &\leq \frac{1}{(1 - \sin 1)h} \int_0^h \int_{\mathbb{R}} (1 - \cos(u\lambda))\Delta_k(d\lambda) du \\ &\leq 4\sigma_k^2(h). \end{aligned} \tag{3.10}$$

Similarly, by (3.9)

$$\begin{aligned} \int_{|\lambda|\leq 1/h} |\lambda|^2 \Delta_k(d\lambda) &\leq 4h^{-2} \int_{|\lambda|\leq 1/h} (1 - \cos(h\lambda))\Delta_k(d\lambda) \\ &\leq 4h^{-2}\sigma_k^2(h). \end{aligned} \tag{3.11}$$

Define for $n = 1, 2, \ldots$ and $0 \leq t \leq 1$,

$$X_k^{(n)}(t) = \int_{|\lambda|\in(d_{n-1}, d_n]} (e^{it\lambda} - 1) W_k(d\lambda),$$

$$\tilde{X}_k^{(n)}(t) = \int_{|\lambda|\notin(d_{n-1}, d_n]} (e^{it\lambda} - 1) W_k(d\lambda).$$

Clearly,

$$X_k(t) = X_k^{(n)}(t) + \tilde{X}_k^{(n)}(t) + V_k t. \tag{3.12}$$

By (3.12), we have

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{M(s_n)}{\sigma_{(n)}} \\
&\le \liminf_{n\to\infty} \sup_{0\le t\le s_n} \max_{k\ge1} \frac{|X_k^{(n)}(t)|}{\sigma_{(n)}} + \limsup_{n\to\infty} \sup_{0\le t\le s_n} \max_{k\ge1} \frac{|\tilde{X}_k^{(n)}(t)|}{\sigma_{(n)}} \\
&\quad + \limsup_{n\to\infty} \frac{s_n \max_{k\ge1} |V_k|}{\sigma_{(n)}} \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{3.13}$$

From condition (iii) it follows that there is $0 < \delta < 1$ such that

$$\sigma^*(lh) \le 2l^{1-\delta}\sigma^*(h) \tag{3.14}$$

for every $0 < h < 1$ and integers l with $1 \le l \le 1/h$. It follows that

$$\frac{s_n}{\sigma_{(n)}} \le K s_n(\log\log(1/s_n)/s_n)^{1-\delta} = K s_n^\delta(\log\log(1/s_n))^{1-\delta}. \tag{3.15}$$

For $0 < \eta < 1$ and $n \ge 1$, define

$$\begin{aligned}
E(n,\eta) &:= \Big\{ \sup_{0\le t\le s_n} \max_{k\ge1} |X_k(t)| \le (1+\eta)\sigma_{(n)} \Big\}, \\
F(n,\eta) &:= \Big\{ \sup_{0\le t\le s_n} \max_{k\ge1} |X_k^{(n)}(t)| \le (1+\eta)\sigma_{(n)} \Big\}, \\
G(n,\eta) &:= \Big\{ \sup_{0\le t\le s_n} \max_{k\ge1} |\tilde{X}_k^{(n)}(t)| \ge \eta\sigma_{(n)} \Big\}, \\
H(n,\eta) &:= \Big\{ s_n \max_{k\ge1} |V_k| \ge \eta\sigma_{(n)} \Big\}.
\end{aligned}$$

Clearly,

$$P(F(n,3\eta)) \ge P(E(n,\eta)) - P(G(n,\eta)) - P(H(n,\eta)). \tag{3.16}$$

It is easy to see that $EV_k^2 \le \sigma_k^2(\theta_0)$ for every $k \ge 1$. Then, by (3.15) and the inequality

$$\Big(\frac{2}{\pi}\Big)^{1/2}(1+t)^{-1}e^{-t^2/2} \le P(|N(0,1)| \ge t) \le \frac{4}{3}\Big(\frac{2}{\pi}\Big)^{1/2}(1+t)^{-1}e^{-t^2/2}, \quad \forall t \ge 0, \tag{3.17}$$

we have

$$\begin{aligned}
P(H(n,\eta)) &\le \sum_{k=1}^\infty \exp\Big\{ -\frac{\eta^2\sigma_{(n)}^2}{3s_n^2\sigma_k^2(\theta_0)} \Big\} \\
&\le \sum_{k=1}^\infty \exp\Big\{ -K s_n^{-\delta}(\log\log(1/s_n))^{-(1-\delta)}\sigma_k^{-2}(\theta_0) \Big\} \\
&\le \sum_{k=1}^\infty \exp\{-n\sigma_k^{-2}(\theta_0)\}
\end{aligned}$$

for large n by recalling the definition of s_n. From condition (vi), it follows that (3.5) holds. Hence, by (3.5),

(3.18)
$$\begin{aligned}\sum_{n=1}^{\infty} P(H(n,\eta)) \quad &\le n_0 + \sum_{n=n_0+1}^{\infty}\sum_{k=1}^{\infty}\exp\{-n\sigma_k^{-2}(\theta_0)\}\\ &\le n_0 + \sum_{n=n_0+1}^{\infty}\exp\{-n\sigma_k^{-2}(\theta_0)/2\}\sum_{k=1}^{\infty}\exp\{-n\sigma_k^{-2}(\theta_0)/2\}\\ &\le n_0 + \sum_{n=n_0+1}^{\infty}\exp\{-Kn\}\sum_{k=1}^{\infty} d^{-\sigma_k^{-2}(\theta_0)} < \infty,\end{aligned}$$

where $n_0 = n_0(d)$ is a finite number such that $e^{n_0/2} \ge d$. Thus, by the Borel-Cantelli lemma we get

$$I_3 = 0 \quad \text{a.s.} \tag{3.19}$$

by the arbitrariness of η.

From (3.10), (3.11), and (3.14), we obtain that for $0 \le t \le s_n$ and each $k \ge 1$,

$$\begin{aligned}\mathrm{Var}(\tilde{X}_k^{(n)}(t)) &= 2\int_{|\lambda|\notin(d_{n-1},d_n]}(1-\cos(t\lambda))\Delta_k(d\lambda)\\ &\le \int_{|\lambda|\le d_{n-1}} t^2|\lambda|^2\Delta_k(d\lambda) + 4\int_{|\lambda|\ge d_n}\Delta_k(d\lambda)\\ &\le 4s_n^2 d_{n-1}^2\sigma_k^2(s_n/(s_n d_{n-1})) + 4\sigma_k^2(s_n/(s_n d_n))\\ &\le 4(s_n d_{n-1})^{2\delta}\sigma_k^2(s_n) + 4\sigma_k^2(s_n/(s_n d_n))\\ &\le \max\{8\exp(-\varepsilon n^{\varepsilon})\sigma_k^2(s_n), 8\sigma_k^2(s_n/(s_n d_n))\}.\end{aligned}$$

Therefore, for every $0 \le s,\ t \le s_n,\ |s-t| \le h \le s_n$,

$$\mathrm{Var}(\tilde{X}_k^{(n)}(s) - \tilde{X}_k^{(n)}(t)) \le \tilde{\sigma}_k^{(n)}(h)^2,$$

where $\tilde{\sigma}_k^{(n)}(h)^2 := \min\Big\{\sigma_k^2(h), \max\{16\exp(-\varepsilon n^{\varepsilon})\sigma_k^2(s_n), 16\sigma_k^2(s_n/(s_n d_n))\}\Big\}$.

Let

$$\sigma^*(n,h) = \max_{k\ge 1}\tilde{\sigma}_k^{(n)}(h).$$

Since $\sigma^*(h)/h^{\alpha}$ is quasi-increasing on $(0,1)$ for some $\alpha > 0$, we have easily

$$\sigma^*(n,h) \le 4\exp\{-\varepsilon n^{\varepsilon}/2\}\sigma^*(s_n). \tag{3.20}$$

Following the same lines of the proof of Lemma 1.1.1 in [3], for any $\varepsilon > 0$, $0 \le T \le t_0$, $0 < h \le h_0$, and $y \ge y_0$, with some $t_0 > 0$, $h_0 > 0$ and $y_0 > 0$, there exist $K = K(\varepsilon) > 0$ such that

$$\begin{aligned} &P\Big(\sup_{0\le t\le T}\sup_{0\le s\le h}\max_{k\ge 1}|X_k(t+s)-X_k(t)| \ge (1+\varepsilon)y\sigma^*(h)\Big) \\ &\le K\frac{T}{h}\sum_{k=1}^{\infty} e^{-y^2(\sigma^*(h))^2/2\sigma_k^2(h)}. \end{aligned} \tag{3.21}$$

From condition (v), (3.5), and (3.21) (with $\tilde{X}_k^{(n)}(\cdot)$ and $\sigma^*(n,h)$ instead of $X_k(\cdot)$ and $\sigma^*(h)$ therein respectively) yield

$$\begin{aligned} P(G(n,\eta)) &\le K\sum_{k=1}^{\infty}\exp\Big\{-\frac{(\eta\sigma_{(n)})^2}{Kn\exp(-\varepsilon n^{\varepsilon})\sigma_k^2(s_n)}\Big\} \\ &\le K\sum_{k=1}^{\infty}\exp\Big\{-K\frac{\exp(\varepsilon n^{\varepsilon})}{n}\times\frac{1}{\sigma_k^2(\theta_0)}\Big\} \end{aligned}$$

for every $\eta > 0$. Thus, similarly to (3.18), we have $\sum_n P(G(n,\eta)) < \infty$, which, by the Borel-Cantelli lemma and the arbitrariness of η, implies that

$$I_2 = 0 \quad \text{a.s..} \tag{3.22}$$

Since $\sigma^*(\cdot)$ is non-decreasing, there is $0 < \eta' \le \eta$ such that $(1+\eta)\sigma^*(x) \ge \sigma^*((1+\eta')x)$. Then, by (3.3), we have

$$P(E(n,\eta)) \ge \exp(-(1+\eta')^{-1}\log\log(1/s_n)) \ge n^{-1}$$

if we choose $0 < \varepsilon \le \eta'$. It follows that $\sum_n P(E(n,\eta)) = \infty$. Therefore, by (3.16) and combining the above estimates, we arrive at

$$\sum_n P(F(n,\eta)) = \infty$$

for every $\eta > 0$. Since $F(n,\eta), n = 1, 2, \ldots$ are independent, by the Borel-Cantelli lemma, we get

$$I_1 \le 1 \quad \text{a.s.} \tag{3.23}$$

since we can take $\eta > 0$ arbitrarily small.

Combining (3.13), (3.19), (3.22), and (3.23), we get (3.4) and complete the proof. □

Proof of Theorem 1.1. From Theorems 2.1(a), 2.2, 3.1, and 3.2, Theorem 1.1 follows immediately. □

4. Applications

In this section we concern applications of Theorem 1.1 to l^∞-fractional Brownian motion. For a fixed constant $0 < H < 1$, a fractional Brownian motion with index H is a centered Gaussian process $X^H = \{X^H(t), t \ge 0\}$ with values in $\mathbb{R}$ and covariance function given by

$$E\Big[X^H(s)X^H(t)\Big] = \frac{1}{2}\Big(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\Big), \quad \forall s,t \ge 0. \tag{4.1}$$

When $H = 1/2$, X^H is the real-valued Brownian motion. Fractional Brownian motion is naturally related to long range dependence which makes it important for modeling phenomena with self-similar and/or long memory properties, see Kahane [6] and Samorodnitsky and Taqqu [12] for more historical information, probabilistic and statistical properties of fractional Brownian motion. For laws of the iterated logarithm, the increments, as well as the Lévy moduli of continuity for fractional Brownian motion, we refer to Monrad and Rootzén [11], Csörgő and Shao [4] and many others (cf. e.g., the references cited therein). In this section, as an application of Theorem 1.1, we investigate Chung-type law of the iterated logarithm of l^∞-valued fractional Brownian motion.

Theorem 4.1. *Let $\{X_k(t), 0 \le t \le 1\}_{k=1}^\infty$ be a sequence of independent standard fractional Brownian motion of index H, $0 < H < 1$. Let $\gamma > 1$ and $c_k = k^{-\gamma}$, $k \ge 1$. Then, there exists a positive constant A_8 such that*

$$1/A_8 \le \liminf_{h\to 0} \sup_{0\le s\le h} \frac{(\log\log(1/h))^H}{h^H} \max_{k\ge 1} |c_k X_k(s)| \le A_8 \quad a.s..$$

Proof. Let $\{Y(t), 0 \le t \le 1\} = \{c_k X_k(t), 0 \le t \le 1\}_{k=1}^\infty$. By (4.1) and a direct calculation, we have that for $0 < H \le 1/2$, Condition (iv) is satisfied and that for $1/2 < H < 1$, Condition (iv)$'$ is satisfied (see e.g., Kuelbs et al. [7] for detailed calculation). We can also verify easily that the Conditions (i)-(iii) and (v)-(vi) of Theorem 1.1 are satisfied. Hence, from Theorem 1.1 and Remark 1.1, we get the desired result immediately. □

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Yong-Kab Choi
Department of Mathematics
Gyeongsang National University
Chinju 660-701, Korea
E-mail address: mathykc@nongae.gsnu.ac.kr

Zhenyan Lin
Department of Mathematics
Zhejiang University
Hangzhou 310027, China
E-mail address: lin@zju.edu.cn

Wensheng Wang
Department of Statistics
East China Normal University
Shanghai 200241, China
and
Department of Mathematics
Hangzhou Normal University
Hangzhou 310036, China
E-mail address: wswang@stat.ecnu.edu.cn