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GENERALIZED FOURIER-WIENER FUNCTION SPACE TRANSFORMS

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Abstract. In this paper, we define generalized Fourier-Hermite functionals on a function space $C_{a,b}[0,T]$ to obtain a complete orthonormal set in $L_2(C_{a,b}[0,T])$ where $C_{a,b}[0,T]$ is a very general function space. We then proceed to give a necessary and sufficient condition that a functional F in $L_2(C_{a,b}[0,T])$ has a generalized Fourier-Wiener function space transform $\mathcal{F}_{\sqrt{2},i}(F)$ also belonging to $L_2(C_{a,b}[0,T])$.

1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space, that is the space of real-valued continuous function x on $[0,T]$ with $x(0) = 0$. The concept of the Fourier-Wiener transforms was introduced by Cameron and Martin in [2]. In [3], the authors defined a modified Fourier-Wiener transform and gave various relationships for the modified Fourier-Wiener transform of functionals in $L_2(C_0[0,T])$. For these works, in [4], using the Wiener measure on $C_0[0,T]$ and completeness properties of the Hermite polynomials on $\mathbb{R}$, they introduced a complete orthonormal set in $L_2(C_0[0,T])$ and gave a Fourier development for functionals in $L_2(C_0[0,T])$ which converges in the $L_2(C_0[0,T])$.

The function space $C_{a,b}[0,T]$ induced by generalized Brownian motion was introduced by J. Yea in [16] and was used extensively by Chang and Chung [7]. In this paper, we extend the results of [1-4] to a very general function space $C_{a,b}[0,T]$ rather than the Wiener space $C_0[0,T]$. The Wiener process used in [1-6, 10] is stationary in time and is free of drift while the stochastic process used in this paper as well as in [7-9], in nonstationary in time, is subject to a drift $a(t)$, and can be used to explain the position of the Ornstein-Uhlenbeck process in an external force field [14]. However, when $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, the general function space $C_{a,b}[0,T]$ reduces to the Wiener space $C_0[0,T]$.

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2. Definitions and preliminaries

Let $D=[0,T]$ and let $(\mathbf{\Omega},\mathcal{B},P)$ be a probability measure space. A real-valued stochastic process Y on $(\mathbf{\Omega},\mathcal{B},P)$ and D is called *a generalized Brownian motion process* if $Y(0,\omega)=0$ almost everywhere and for $0=t_0<t_1<\cdots<t_n\le T$, the n-dimensional random vector $(Y(t_1,\omega),\ldots,Y(t_n,\omega))$ is normally distributed with density function

$$
\tag{2.1}
\begin{aligned}
W_n(\vec{t},\vec{\eta}) = {}& \Big((2\pi)^n\prod_{j=1}^{n}(b(t_j)-b(t_{j-1}))\Big)^{-1/2}\\
&\cdot\ \exp\bigg\{-\frac{1}{2}\sum_{j=1}^{n}\frac{((\eta_j-a(t_j))-(\eta_{j-1}-a(t_{j-1})))^2}{b(t_j)-b(t_{j-1})}\bigg\},
\end{aligned}
$$

where $\vec{\eta}=(\eta_1,\ldots,\eta_n)$, $\eta_0=0$, $\vec{t}=(t_1,\ldots,t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0)=0$, $a'(t)\in L_2[0,T]$ and $b(t)$ is a strictly increasing continuously differentiable real-valued function with $b(t)>0$ and $b'(t)>0$ for each $t\in[0,T]$.

As explained in [15, pp. 18–20], Y induces a probability measure μ on the measurable space $(\mathbb{R}^D,\mathcal{B}^D)$, where $\mathbb{R}^D$ is the space of all real valued functions $x(t)$, $t\in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x)=x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D,\mathcal{B}^D,\mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t)=\min\{b(s),b(t)\}$. By Theorem 14.2 [15, p. 187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0)=0$ under the sup norm). Hence $(C_{a,b}[0,T],\mathcal{B}(C_{a,b}[0,T]),\mu)$ is the function space induced by Y, where $\mathcal{B}(C_{a,b}[0,T])$ is the Boreal σ-algebra of $C_{a,b}[0,T]$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided ρB is $\mathcal{B}(C_{a,b}[0,T])$-measurable for all $\rho>0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided $\mu(\rho N)=0$ for all $\rho>0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere(s-a.e.). If two functionals F and G are equal scale-invariant almost everywhere, we write $F\approx G$.

Let $L^2_{a,b}[0,T]$ be the Hilbert space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$
\tag{2.2}
L^2_{a,b}[0,T]=\left\{v:\int_0^T v^2(s)db(s)<\infty \text{ and } \int_0^T v^2(s)d|a|(s)<\infty\right\},
$$

where $|a|(t)$ denotes the total variation of the function a on the interval $[0,t]$.

For $u, v \in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b} = \int_0^T u(t)v(t)d[b(t) + |a|(t)]. \tag{2.3}$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular, note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space. Note that all functions of bounded variation on $[0,T]$ are elements of $L^2_{a,b}[0,T]$. Also note that if $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, then $L^2_{a,b}[0,T] = L^2[0,T]$. In fact,

$$(L^2_{a,b}[0,T], ||\cdot||_{a,b}) \subset (L^2_{0,b}[0,T], ||\cdot||_{0,b}) = (L^2[0,T], ||\cdot||_2)$$

since the two norms $||\cdot||_{0,b}$ and $||\cdot||_2$ are equivalent.

Let $\{\phi_j\}_{j=1}^\infty$ be a complete orthonormal set of real-valued functions of bounded variation on $[0,T]$ such that

$$(\phi_j, \phi_k)_{a,b} = \begin{cases} 0, & j \neq k \\ 1, & j = k, \end{cases}$$

and for each $v \in L^2_{a,b}[0,T]$, let

$$v_n(t) = \sum_{j=1}^n (v, \phi_j)_{a,b}\phi_j(t) \tag{2.4}$$

for $n = 1, 2, \ldots$. Then for each $v \in L^2_{a,b}[0,T]$, the Palely-Wiener-Zygmund (PWZ) stochastic integral $\langle v, x\rangle$ is defined by the formula

$$\langle v, x\rangle = \lim_{n\to\infty} \int_0^T v_n(t)dx(t) \tag{2.5}$$

for all $x \in C_{a,b}[0,T]$ for which the limit exists; one can show that for each $v \in L^2_{a,b}[0,T]$, the PWZ integral $\langle v, x\rangle$ exists for μ-a.e. $x \in C_{a,b}[0,T]$.

Followings are some facts about the PWZ stochastic integral.

(1) For each $v \in L^2_{a,b}[0,T]$, the PWZ integral $\langle v, x\rangle$ exists for μ-a.e. $x \in C_{a,b}[0,T]$.
(2) The PWZ integral $\langle v, x\rangle$ is essentially independent of the complete orthonormal set $\{\phi_j\}_{j=1}^\infty$.
(3) If v is of bounded variation on $[0,T]$, then the PWZ integral $\langle v, x\rangle$ equals the Riemann-Stieltjes integral $\int_0^T v(t)dx(t)$ for s-a.e. $x \in C_{a,b}[0,T]$.
(4) The PWZ integral has the expected linearity properties.
(5) For all $v \in L^2_{a,b}[0,T]$, $\langle v, x\rangle$ is a Gaussian random variable with mean $\int_0^T v(s)da(s)$ and variance $\int_0^T v^2(s)db(s)$.

We denote the function space integral of a $\mathcal{B}(C_{a,b}[0,T])$-measurable functional F by

$$E[F]=\int_{C_{a,b}[0,T]}F(x)d\mu(x) \tag{2.6}$$

whenever the integral exists.

3. Complete orthonormal sets in $L_2(\mathbb{R})$ and $L_2(\mathbb{R}^n)$

In this section, we define generalized normalizing Hermite functions. We use them to obtain a complete orthonormal set in $L_2(\mathbb{R})$ and $L_2(\mathbb{R}^n)$.

Definition 3.1. For each $m=0,1,\dots$ and $t\in[0,T]$, we define a generalized Hermite polynomial in $\frac{u-a(t)}{\sqrt{b(t)}}$ of degree m by

(3.1)

$$H_m(u;t)\equiv(-1)^m\big(b(t)\big)^{\frac{m}{2}}\exp\bigg\{\frac{\big(u-a(t)\big)^2}{2b(t)}\bigg\}\frac{d^m}{du^m}\bigg(\exp\bigg\{-\frac{\big(u-a(t)\big)^2}{2b(t)}\bigg\}\bigg).$$

For examples, we see that

$$H_0(u;t)=1, H_1(u;t)=\frac{u-a(t)}{\sqrt{b(t)}} \text{ and } H_2(u;t)=-1+\bigg(\frac{u-a(t)}{\sqrt{b(t)}}\bigg)^2.$$

We note that for each $m=0,1,\dots$ and $t\in[0,T]$,

$$H_{m+1}(u;t)=\bigg(\frac{u-a(t)}{\sqrt{b(t)}}\bigg)H_m(u;t)-\big(b(t)\big)^{\frac{1}{2}}H_m'(u;t),$$

where $'$ means $\frac{d}{du}$. And for each $m=1,2,\dots$ and $t\in[0,T]$,

$$H_{m+1}(u;t)-\bigg(\frac{u-a(t)}{\sqrt{b(t)}}\bigg)H_m(u;t)+mH_{m-1}(u;t)=0$$

and so

$$H_m'(u;t)=\frac{m}{\sqrt{b(t)}}H_{m-1}(u;t). \tag{3.2}$$

Lemma 3.2. *For any nonnegative integers m and k,*

$$\begin{aligned} I&\equiv\int_{\mathbb{R}}\exp\bigg\{-\frac{\big(u-a(t)\big)^2}{2b(t)}\bigg\}H_k(u;t)H_m(u;t)du\\ &=\begin{cases}0 & \text{if} \quad k\neq m\\ k!\sqrt{2\pi b(t)} & \text{if} \quad k=m.\end{cases}\end{aligned} \tag{3.3}$$

Proof. Assume that $m\le k$. Let

$$\varphi(u;t)=\exp\bigg\{-\frac{\big(u-a(t)\big)^2}{2b(t)}\bigg\}.$$

Then

$$\exp\left\{-\frac{\big(u-a(t)\big)^2}{2b(t)}\right\}H_k(u;t) \ = (-1)^k\big(b(t)\big)^{\frac{k}{2}}\varphi^{(k)}(u;t).$$

By using this above, we obtain that

$$I \equiv (-1)^k\big(b(t)\big)^{\frac{k}{2}}\int_{\mathbb{R}}\varphi^{(k)}(u;t)H_m(u;t)du.$$

By using integration by parts formulas, we obtain that

$$\begin{aligned} I &\equiv (-1)^k\big(b(t)\big)^{\frac{k}{2}}\left[\varphi^{(k-1)}(u;t)H_m(u;t)\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}}\varphi^{(k-1)}(u;t)H_m'(u;t)du\right] \\ &= (-1)^{k+1}\big(b(t)\big)^{\frac{k}{2}}\int_{\mathbb{R}}\varphi^{(k-1)}(u;t)H_m'(u;t)du. \end{aligned}$$

Continuing on this manner, we obtain that

$$I = (-1)^{k+m}\big(b(t)\big)^{\frac{k}{2}}\int_{\mathbb{R}}\varphi^{(k-m)}(u;t)H_m^{(m)}(u;t)du.$$

But if $m < k$, then $I = 0$. If $k = m$, then by using equation (3.2) above,

$$\begin{aligned} I &= (-1)^{2k}\big(b(t)\big)^{\frac{k}{2}}\int_{\mathbb{R}}\frac{k!}{b(t)^{\frac{k}{2}}}\exp\left\{-\frac{\big(u-a(t)\big)^2}{2b(t)}\right\}du \\ &= k!\int_{\mathbb{R}}\exp\left\{-\frac{\big(u-a(t)\big)^2}{2b(t)}\right\}du \\ &= k!\sqrt{2\pi b(t)} \end{aligned}$$

which completes the proof of Lemma 3.2. □

We are ready to define generalized Hermite functions and a sequence of generalized normalizing Hermite functions $\{K_m\}_{m=0}^{\infty}$ which is a complete orthonormal set in $L_2(\mathbb{R})$.

Definition 3.3. For each $m = 0, 1, \ldots$ and $t \in [0, T]$, we define a generalized Hermite function of degree m by

$$h_m(u;t) \equiv H_m(u;t)\exp\left\{-\frac{\big(u-a(t)\big)^2}{4b(t)}\right\}$$

and we define a generalized normalizing Hermite function of degree m by

$$K_m(u;t) \equiv \big(m!\sqrt{2\pi b(t)}\big)^{-\frac{1}{2}}h_m(u;t). \tag{3.4}$$

Now, we are ready to obtain a complete orthonormal set in $L_2(\mathbb{R})$.

Theorem 3.4. *The set of functions* $\{K_m\}_{m=0}^{\infty}$ *defined by* (3.4) *is an orthonormal set in* $L_2(\mathbb{R})$.

Proof. By using equation (3.3) above, it immediately follows that for all non-negative integers k and m,

$$\begin{aligned}&\int_{\mathbb{R}} K_k(u;t)K_m(u;t)du\\&= \int_{\mathbb{R}} \big(k!\sqrt{2\pi b(t)}\big)^{-\frac{1}{2}} h_k(u;t)\big(m!\sqrt{2\pi b(t)}\big)^{-\frac{1}{2}} h_m(u;t)du\\&= \big(k!\sqrt{2\pi b(t)}\big)^{-\frac{1}{2}}\big(m!\sqrt{2\pi b(t)}\big)^{-\frac{1}{2}} \int_{\mathbb{R}} H_k(u;t)H_m(u;t)\exp\bigg\{-\frac{\big(u-a(t)\big)^2}{2b(t)}\bigg\}du\\&= \delta_{k,m}.\end{aligned}$$

□

Theorem 3.5. *The set* $\{K_m\}_{m=0}^{\infty}$ *is a complete orthonormal set in* $L_2(\mathbb{R})$.

Proof. For each $m = 0, 1, \ldots$ and $f \in L_2(\mathbb{R})$, assume that

$$\int_{\mathbb{R}} K_m(u;t)f(u)du = 0.$$

It suffices to show that $f = 0$ a.e.. Let

$$g(u) = \exp\bigg\{-\frac{\big(u-a(t)\big)^2}{4b(t)}\bigg\}f(u)$$

for $u \in \mathbb{R}$. Then $g \in L_1(\mathbb{R})$ and so the Fourier transform

$$F(z) \equiv \int_{\mathbb{R}} \exp\{izu\}g(u)du$$

exists for all $z \in \mathbb{C}$. Also $\frac{1}{\sqrt{2\pi}}F|_{\mathbb{R}}$ is the inverse Fourier transform of g. Thus if $F(z) \equiv 0$, then by uniqueness of the inverse Fourier transform, $g = 0$ and so $f = 0$ a.e. on $\mathbb{R}$.

Since F is an entire function, we can write

$$F(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where $b_n = \frac{F^{(n)}(0)}{n!}$. But

$$F^{(n)}(z) = \int_{\mathbb{R}} (iu)^n \exp\bigg\{izu - \frac{\big(u-a(t)\big)^2}{4b(t)}\bigg\}f(u)du$$

and $F^{(n)}(0) = 0$ and hence $b_n = 0$ for all n and so $F(z) = 0$. □

Definition 3.6. For each $j = 1, 2, \ldots$, let m_j be a nonnegative integer. For $(u_1, \ldots, u_n) \in \mathbb{R}^n$, let

$$K_{(m_1,\ldots,m_n)}(u_1,\ldots,u_n;t) \equiv \prod_{j=1}^{n} K_{m_j}(u_j;t), \tag{3.5}$$

where $K_{m_j}(u_j;t)$ is given by equation (3.4) above.

In our next theorem, we also give a complete orthonormal set in $L_2(\mathbb{R}^n)$.

Theorem 3.7. *The set of generalized normalizing Hermite functions*

$$\{K_{(m_1,\ldots,m_n)}\}_{m_1,\ldots,m_n=0}^{\infty}$$

is a complete orthonormal set in $L_2(\mathbb{R}^n)$.

Proof. By using Theorem 3.4 and the fact that the set of functions of the form $\{f_1 f_2 \cdots f_n : f_j \in L_2(\mathbb{R}), j = 1, \ldots, n\}$ is dense in $L_2(\mathbb{R}^n)$, we obtain the desired result. □

Definition 3.8. For each $g \in L_2(\mathbb{R}^n)$, the generalized Hermite coefficient of g with respect to the complete orthonormal set $\{K_{(m_1,\ldots,m_n)}\}_{m_1,\ldots,m_n=0}^{\infty}$ is defined by the formula

$$\begin{aligned} a^g_{(m_1,\ldots,m_n)} &\equiv \int_{\mathbb{R}^n} g(\vec{u}) K_{(m_1,\ldots,m_n)}(\vec{u};t)d\vec{u} \\ &= \int_{\mathbb{R}^n} g(\vec{u}) \bigg(\prod_{j=1}^{n} K_{m_j}(u_j;t) \bigg) d\vec{u}. \end{aligned} \tag{3.6}$$

Remark 3.9. By Theorem 3.7, it follows that

$$g(u_1,\ldots,u_n) = \lim_{N\to\infty} \sum_{m_1,\ldots,m_n=0}^{N} a^g_{(m_1,\ldots,m_n)} \prod_{j=1}^{n} K_{m_j}(u_j;t);$$

that is to say that

$$\int_{\mathbb{R}^n} \bigg| g(\vec{u}) - \sum_{m_1,\ldots,m_n=0}^{N} a^g_{(m_1,\ldots,m_n)} \prod_{j=1}^{n} K_{m_j}(u_j;t) \bigg|^2 d\vec{u}$$

goes to zero as $N \to \infty$.

4. A complete orthonormal set in $L_2(C_{a,b}[0,T])$

In this section, we define the generalized Fourier-Hermite coefficient and the generalized Fourier-Hermite functionals. We then obtain a complete orthonormal set in $L_2(C_{a,b}[0,T])$.

The following notations are used throughout this paper:

$$A_j \equiv \int_0^T \alpha_j(t)da(t) \tag{4.1}$$

and

$$B_j \equiv \int_0^T \alpha_j^2(t)db(t), \tag{4.2}$$

where $\{\alpha_j\}$ is a complete orthonormal set in $L^2_{a,b}[0,T]$. We note that for each $j = 1, 2, \ldots$,

$$0 < B_j = \int_0^T \alpha_j^2(t)db(t) \le \int_0^T \alpha_j^2(t)d[b(t) + |a|(t)] = ||\alpha_j||^2_{a,b} = 1,$$

while A_j may be positive, negative or zero. We also note that if $a(t) \equiv 0$ on $[0,T]$, then $A_j = 0$ and $B_j = 1$ for each $j = 1, 2, \ldots$.

The following integration formula used several times throughout this paper:

Let $\{\alpha_1, \ldots, \alpha_n\}$ be an orthonormal set of functions from $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$. Let $h : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable and let $H(x) = h(\langle\alpha_1, x\rangle, \ldots, \langle\alpha_n, x\rangle)$. Then

$$\begin{aligned} \int_{C_{a,b}[0,T]} H(x)d\mu(x) &= \Big(\prod_{j=1}^n 2\pi B_j\Big)^{-\frac{1}{2}} \int_{\mathbb{R}^n} h(u_1, \ldots, u_n) \\ &\quad \cdot \exp\Big\{ - \sum_{j=1}^n \frac{(u_j - A_j)^2}{2B_j} \Big\} du_1 \cdots du_n \end{aligned} \tag{4.3}$$

in the sense that if either side exists, both sides exist and equality holds.

Using formula (4.3) we observe that $E[\langle\alpha_j, x\rangle] = A_j, E[\langle\alpha_j, x\rangle^2] = B_j + A_j^2$ and that $\mathrm{Var}(\langle\alpha_j, x\rangle) = B_j$ for each $j = 1, \ldots, n$.

Also note that the complete orthonormal set $\{\alpha_1, \alpha_2, \ldots\}$ in $L^2_{a,b}[0,T]$ is completely at our disposal. For example, we could choose the α_j's to the continuous and of bounded variation on $[0,T]$, or we could choose the α_j's to the Haar functions on $[0,T]$, etc.

To obtain a complete orthonormal set in $L_2(C_{a,b}[0,T])$, we define a generalized Hermite polynomial and a generalized normalizing Hermite function as below.

Definition 4.1. For each $m = 0, 1, \ldots$ and for each $j = 1, 2, \ldots$, we define a generalized Hermite polynomial by

$$\tilde{H}^j_m(u) \equiv (-1)^m (m!)^{-\frac{1}{2}} (B_j)^{\frac{m}{2}} \exp\Big\{ \frac{(u - A_j)^2}{2B_j} \Big\} \frac{d^m}{du^m} \Big(\exp\Big\{ -\frac{(u - A_j)^2}{2B_j} \Big\} \Big)$$

and we define a generalized normalizing Hermite polynomial by

$$\tilde{K}^j_m(u) \equiv (2\pi B_j)^{-\frac{1}{4}} \tilde{H}^j_m(u) \exp\Big\{ -\frac{(u - A_j)^2}{4B_j} \Big\},$$

where A_j and B_j are as in equations (4.1) and (4.2) above respectively.

Remark 4.2. (1) If $a(t) \equiv 0$ on $[0,T]$, then for all $j = 1, 2, \ldots$, $A_j = 0$ and $B_j = 1$. In case, $\tilde{H}^j_m$ and $\tilde{K}^j_m$ are defined independently of j's.

(2) We can easily see that for each $j = 1, 2, \dots$, $\{\tilde{K}_m^j\}_{m=0}^\infty$ is a complete orthonormal set in $L_2(\mathbb{R})$. Furthermore, for all $j = 1, \dots, n$, let m_j be a nonnegative integer. Then the set $\{\tilde{K}_{(m_1,\dots,m_n)}^{(1,\dots,n)}\}_{m_1,\dots,m_n=0}^\infty$ is a complete orthonormal set in $L_2(\mathbb{R}^n)$, where $\tilde{K}_{(m_1,\dots,m_n)}^{(1,\dots,n)}(\vec{u}) = \prod_{j=1}^n \tilde{K}_{m_j}^j(u_j)$.

(3) For each positive integer k and nonnegative integer m, let

$$\varphi_{(m,k)}(x) \equiv \tilde{H}_m^k(\langle \alpha_k, x\rangle).$$

For each $j = 1, \dots, k$, let m_j be a nonnegative integer. Let

$$\begin{aligned} \Phi_{(m_1,\dots,m_k)}(x) &\equiv \varphi_{(m_1,1)}(x)\varphi_{(m_2,2)}(x)\cdots\varphi_{(m_k,k)}(x) \\ &= \prod_{j=1}^k \tilde{H}_{m_j}^j(\langle \alpha_k, x\rangle). \end{aligned} \tag{4.4}$$

The functionals in equation (4.4) are called the generalized Fourier-Hermite functionals on $C_{a,b}[0,T]$.

The three assertions in the following statements follow by easily from Remark 4.2:

(1) $\varphi_{(0,k)}(x) \equiv 1$ for all positive integer k.

(2) $\Phi_{(m_1,\dots,m_k)}(x) = \Phi_{(m_1,\dots,m_k,0,\dots,0)}(x)$ for all positive integer k.

(3) Inserting zeroes to the left of a non-zero entry in Φ change the values of Φ.

Theorem 4.3. *The set of functionals $\mathcal{M} \equiv \{\Phi_{(m_1,\dots,m_k)}\}_{k=1}^\infty$ is an orthonormal set in $L_2(C_{a,b}[0,T])$.*

Proof. By using formula (4.3) above, it follows that

$$\begin{aligned} &\int_{C_{a,b}[0,T]} \Phi_{(m_1,\dots,m_k)}(x)\Phi_{(n_1,\dots,n_k)}(x)d\mu(x) \\ &= \int_{C_{a,b}[0,T]} \left[\prod_{j=1}^k \tilde{H}_{m_j}^j(\langle\alpha_j, x\rangle)\right]\left[\prod_{j=1}^k \tilde{H}_{n_j}^j(\langle\alpha_j, x\rangle)\right] d\mu(x) \\ &= \left(\prod_{j=1}^k 2\pi B_j\right)^{-\frac{1}{2}} \int_{\mathbb{R}^k} \left[\prod_{j=1}^k \tilde{H}_{m_j}^j(u_j)\right]\left[\prod_{j=1}^k \tilde{H}_{n_j}^j(u_j)\right] \exp\left\{-\sum_{j=1}^k \frac{(u_j - A_j)^2}{2B_j}\right\} du_1\cdots du_k \\ &= \int_{\mathbb{R}^k} \left[\prod_{j=1}^k \tilde{K}_{m_j}^j(u_j)\right]\left[\prod_{j=1}^k \tilde{K}_{n_j}^j(u_j)\right] du_1\cdots du_k \\ &= \delta_{m_1,n_1}\delta_{m_2,n_2}\cdots\delta_{m_k,n_k}. \end{aligned}$$

□

In order to show that the set of functionals $\mathcal{M}$ in Theorem 4.3 is complete we will show that every functional F in $L_2(C_{a,b}[0,T])$ has a generalized Fourier-Hermite series expansion in terms of the functionals in $\mathcal{M}$.

Definition 4.4. For each $F \in L_2(C_{a,b}[0,T])$ and $\Phi_{(m_1,\dots,m_k)} \in \mathcal{M}$, the generalized Fourier-Hermite coefficient of F is defined by

$$A^F_{(m_1,\dots,m_k)} \equiv \int_{C_{a,b}[0,T]} F(x)\Phi_{(m_1,\dots,m_k)}(x)d\mu(x). \tag{4.5}$$

Lemma 4.5. *For $\vec{u} = (u_1,\dots,u_n) \in \mathbb{R}^n$, let $f(\vec{u})$ be a measurable function such that*

$$f(\vec{u})\exp\bigg\{-\sum_{j=1}^n \frac{\big(u_j - A_j\big)^2}{4B_j}\bigg\} \in L_2(\mathbb{R}^n).$$

Let $F : C_{a,b}[0,T] \to \mathbb{R}$ be a functional defined by

$$F(x) = f\big(\langle\alpha_1,x\rangle,\dots,\langle\alpha_n,x\rangle\big), \tag{4.6}$$

where $\{\alpha_j\}$ is a complete orthonormal set in $L^2_{a,b}[0,T]$. Then

$$I \equiv \int_{C_{a,b}[0,T]} \bigg| F(x) - \sum_{m_1,\dots,m_n=0}^{N} A^F_{(m_1,\dots,m_n)}\Phi_{(m_1,\dots,m_n)}(x)\bigg|^2 d\mu(x) \to 0 \tag{4.7}$$

as $N \to \infty$.

Theorem 4.6. *Let $\mathcal{L}$ be the set of all functionals F which has the form* (4.6) *above. Then $\mathcal{L}$ is dense in $L_2(C_{a,b}[0,T])$.*

Proof. By the usual Lebesgue argument, it suffices to show that the characteristic functional can be represented by linear combination of elements of $\mathcal{L}$. For $0 = t_0 < t_1 < \cdots < t_n \le T$, let $I = \{x \in C_{a,b}[0,T] : \alpha_j \le x(t_j) \le \beta_j\}$. Then

$$\chi_I(x) = \prod_{j=1}^n \chi_{[\alpha_j,\beta_j]}(x(t_j)).$$

For given $\varepsilon > 0$ and $j = 1,\dots,n$, define a trapezoid function $Z_{j,\epsilon}$ by

$$Z_{j,\varepsilon}(s) = \begin{cases} 0 & \text{if } s \in (-\infty, \alpha_j - \varepsilon) \\ \frac{1}{\varepsilon}(s-\alpha_j)+1 & \text{if } s \in [\alpha_j - \varepsilon, \alpha_j) \\ 1 & \text{if } s \in [\alpha_j, \beta_j] \\ -\frac{1}{\varepsilon}(s-\beta_j)+1 & \text{if } s \in (\beta_j, \beta_j + \varepsilon] \\ 0 & \text{if } s \in (\beta_j + \varepsilon, \infty). \end{cases}$$

Let $C_\varepsilon(x) \equiv \prod_{j=1}^n Z_{j,\varepsilon}(x(t_j))$. Then $C_\varepsilon(x) \to \chi_I(x)$ as $\varepsilon \downarrow 0$. Next, let

$$g_j(t) \equiv \chi_{[0,t_j]}(t).$$

Then we write that $g_j(t) \equiv \sum\limits_{l=1}^{\infty} b_{j,l}\alpha_l(t)$. Let $G_{j,m}(t) \equiv \sum\limits_{l=1}^{m} b_{j,l}\alpha_l(t)$. Then

$$||g_j - G_{j,m}||_{a,b} \to 0 \text{ as } m \to \infty.$$

Thus we obtain that

$$\langle G_{j,m}, x\rangle \to \langle g_j, x\rangle$$

in the $L_2(C_{a,b}[0,T])$ sense as $m \to \infty$. Thus there exists a subsequence $\{G_{j,m_k}\}$ of $\{G_{j,m}\}$ such that

$$\langle G_{j,m_k}, x\rangle \to \langle g_j, x\rangle$$

as $k \to \infty$ for a.e. $x \in C_{a,b}[0,T]$. Let $S_{\varepsilon,k}(x) \equiv \prod_{j=1}^{n} Z_{j,\varepsilon}(\langle G_{j,m_k}, x\rangle)$. Then

$$S_{\varepsilon,k}(x) \to \prod_{j=1}^{n} Z_{j,\varepsilon}(\langle g_j, x\rangle)$$

as $k \to \infty$ for a.e. $x \in C_{a,b}[0,T]$. Thus $S_{\varepsilon,k}(x) \to C_\varepsilon(x)$ in the $L_2(C_{a,b}[0,T])$ sense as $k \to \infty$. But $S_{\varepsilon,k}(x)$ is an element of $\mathcal{L}$. Hence we complete the proof as desired. □

Remark 4.7. Note, we have shown above for F given by equation (4.6), that $A_{(m_1,\dots,m_k)}$ given by (4.5) is zero for $k > n$ and $m_k > 0$. Hence the number of subscripts on Φ and A^F in (4.7) may be increased beyond n without changing the sum. Hence we may take $N > n$ subscripts in (4.7) and thus for any F given by (4.6),

$$\int_{C_{a,b}[0,T]} \left| F(x) - \sum_{m_1,\dots,m_N=0}^{N} A^F_{(m_1,\dots,m_N)} \Phi_{(m_1,\dots,m_N)}(x) \right|^2 d\mu(x) \to 0$$

as $N \to \infty$. That is to say,

$$F(x) = \lim_{N\to\infty} \sum_{m_1,\dots,m_N=0}^{N} A^F_{(m_1,\dots,m_N)} \Phi_{(m_1,\dots,m_N)}(x), \tag{4.8}$$

and the right-hand side of (4.8) is called the generalized Fourier-Hermite series of F.

Theorem 4.6 above, together with Remark 4.7, tell us that every $F \in L_2(C_{a,b}[0,T])$ has a convergent generalized Fourier-Hermite series expansion. This observation plays a key role in Section 5 below.

Corollary 4.8. *Let F be an any functional on $C_{a,b}[0,T]$ with*

$$\int_{C_{a,b}[0,T]} |F(x)|^2 d\mu(x) < \infty,$$

and for $N = 1, 2, \dots,$ let

$$F_N(x) = \sum_{m_1,\dots,m_N=0}^{N} A^F_{(m_1,\dots,m_N)} \Phi_{(m_1,\dots,m_N)}(x). \tag{4.9}$$

Then

$$\int_{C_{a,b}[0,T]} |F_N(x) - F(x)|^2 d\mu(x) \to 0 \tag{4.10}$$

as $N \to \infty$ *and*

$$
\begin{aligned}
F(x) &= \lim_{N\to\infty} F_N(x) \\
&= \lim_{N\to\infty} \sum_{m_1,\ldots,m_N=0}^{N} A^F_{(m_1,\ldots,m_N)} \Phi_{(m_1,\ldots,m_N)}(x)
\end{aligned} \tag{4.11}
$$

is called the generalized Fourier-Hermite series expansion of F.

5. The generalized Fourier-Wiener function space transforms

In this section we obtain a formula for the generalized Fourier-Wiener function space transform of the generalized Fourier-Hermite function. We then proceed to obtain formula for the generalized Fourier-Wiener function space transform of a functional F in $L_2(C_{a,b}[0,T])$.

Let $K_{a,b}[0,T]$ be the set of all complex-valued continuous functions $x(t)$ defined on $[0,T]$ which vanish at $t=0$ and whose real and imaginary parts are elements of $C_{a,b}[0,T]$; namely,

$$K_{a,b}[0,T] = \{x : [0,T] \to \mathbb{C} \mid x(0) = 0,\ \mathrm{Re}(x), \mathrm{Im}(x) \in C_{a,b}[0,T]\}.$$

Then $C_{a,b}[0,T]$ is a subspace of all real-valued functions in $K_{a,b}[0,T]$.

Definition 5.1. Let F be a functional defined on $K_{a,b}[0,T]$. The generalized Fourier-Wiener function space transform $\mathcal{F}_{\sqrt{2},i}(F)$ of F is defined by

$$\mathcal{F}_{\sqrt{2},i}(F)(y) = \int_{C_{a,b}[0,T]} F(\sqrt{2}x + iy)d\mu(x), \qquad y \in K_{a,b}[0,T] \tag{5.1}$$

if it exists.

Remark 5.2. When $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, the generalized Fourier-Wiener function space transform is the modified Fourier-Wiener transform introduced by Cameron and Martin in [3].

Throughout this paper, in order to ensure that various integrals exist, we will assume that $\beta = c + id$ is a nonzero complex number satisfying the inequality

$$\mathrm{Re}(1-\beta^2) = 1 + d^2 - c^2 > 0. \tag{5.2}$$

We note that $\mathrm{Re}(1-\beta^2) = 1 + d^2 - c^2 > 0$ if and only if the point $(c,d) \in \mathbb{R}^2$ lies in the open region determined by the hyperbola $c^2 - d^2 = 1$ containing the d-axis. Hence, for all $|\beta| \le 1, \beta \ne \pm 1$ and $\mathrm{Re}(1-\beta^2) > 0$. Let $\gamma = \sqrt{1-\beta^2}$ with $-\frac{\pi}{4} < \arg(\gamma) < \frac{\pi}{4}$ so that $\gamma^2 + \beta^2 = 1$ and $\mathrm{Re}(\gamma^2) = \mathrm{Re}(1-\beta^2) > 0$.

The following lemma plays a key role in finding the generalized integral transform of functionals given by equation (4.4).

Lemma 5.3. *Let* $\beta = c + id$ *be a nonzero complex number with* $\beta \neq \pm 1$, *satisfying inequality* (5.2) *and let* $\gamma = \sqrt{1-\beta^2}$. *Then for* $\lambda \in \mathbb{R}$, $n = 0, 1, 2, \ldots$, *and* $j = 1, 2, \ldots$,

$$\begin{aligned} &\int_{\mathbb{R}} \tilde{H}_n^j(u) \exp\Big\{-\frac{1}{2\gamma^2 B_j}\big(u - (\lambda\beta + \gamma A_j)\big)^2\Big\} du \\ &= \beta^n \sqrt{2\pi\gamma^2 B_j}\, \tilde{H}_n^j\Big(\lambda + \frac{\gamma + \beta - 1}{\beta} A_j\Big). \end{aligned} \tag{5.3}$$

Proof. Since $\tilde{H}_n^j(u)$ is a polynomial of degree n, $B_j > 0$ and $\mathrm{Re}(\gamma^2) > 0$, the integral in (5.3) exists and

$$\begin{aligned} I_n &\equiv \int_{\mathbb{R}} \tilde{H}_n^j(u) \exp\Big\{-\frac{1}{2\gamma^2 B_j}\big(u - (\lambda\beta + \gamma A_j)\big)^2\Big\} du \\ &= (-1)^n (n!)^{-\frac{1}{2}} B_j^{\frac{n}{2}} \int_{\mathbb{R}} \exp\Big\{\frac{\big(u - A_j\big)^2}{2B_j}\Big\} \frac{d^n}{du^n}\Big(\exp\Big\{-\frac{\big(u - A_j\big)^2}{2B_j}\Big\}\Big) \\ &\qquad\qquad \cdot \exp\Big\{-\frac{1}{2\gamma^2 B_j}\big(u - (\lambda\beta + \gamma A_j)\big)^2\Big\} du \\ &= (-1)^n (n!)^{-\frac{1}{2}} B_j^{\frac{n}{2}} \exp\Big\{\frac{1}{2\beta^2 B_j}\big((\beta\lambda + \gamma A_j) - A_j\big)^2\Big\} \\ &\quad \cdot \int_{\mathbb{R}} \exp\Big\{-\frac{\beta^2}{2\gamma^2 B_j}\Big(u - \frac{(\beta\lambda + \gamma A_j) - \gamma^2 A_j}{\beta^2}\Big)^2\Big\} \frac{d^n}{du^n}\Big(\exp\Big\{-\frac{\big(u - A_j\big)^2}{2B_j}\Big\}\Big) du. \end{aligned}$$

Next, integrating by parts n times, we obtain that

$$\begin{aligned} I_n &= (n!)^{-\frac{1}{2}} B_j^{\frac{n}{2}} \exp\Big\{\frac{1}{2\beta^2 B_j}\big((\beta\lambda + \gamma A_j) - A_j\big)^2\Big\} \\ &\quad \cdot \int_{\mathbb{R}} \frac{d^n}{du^n}\Big(\exp\Big\{-\frac{\beta^2}{2\gamma^2 B_j}\Big(u - \frac{(\beta\lambda + \gamma A_j) - \gamma^2 A_j}{\beta^2}\Big)^2\Big\}\Big) \exp\Big\{-\frac{\big(u - A_j\big)^2}{2B_j}\Big\} du \\ &= (n!)^{-\frac{1}{2}} B_j^{\frac{n}{2}} (-\beta)^n \exp\Big\{\frac{1}{2\beta^2 B_j}\big((\beta\lambda + \gamma A_j) - A_j\big)^2\Big\} \\ &\quad \cdot \int_{\mathbb{R}} \frac{d^n}{d\lambda^n}\Big(\exp\Big\{-\frac{\beta^2}{2\gamma^2 B_j}\Big(u - \frac{(\beta\lambda + \gamma A_j) - \gamma^2 A_j}{\beta^2}\Big)^2\Big\}\Big) \exp\Big\{-\frac{\big(u - A_j\big)^2}{2B_j}\Big\} du \\ &= (n!)^{-\frac{1}{2}} B_j^{\frac{n}{2}} (-\beta)^n \exp\Big\{\frac{1}{2\beta^2 B_j}\big((\beta\lambda + \gamma A_j) - A_j\big)^2\Big\} \\ &\quad \cdot \frac{d^n}{d\lambda^n}\Big(\int_{\mathbb{R}} \exp\Big\{-\frac{\beta^2}{2\gamma^2 B_j}\Big(u - \frac{(\beta\lambda + \gamma A_j) - \gamma^2 A_j}{\beta^2}\Big)^2\Big\} \exp\Big\{-\frac{\big(u - A_j\big)^2}{2B_j}\Big\} du\Big) \\ &= (n!)^{-\frac{1}{2}} B_j^{\frac{n}{2}} (-\beta)^n \exp\Big\{\frac{1}{2\beta^2 B_j}\big((\beta\lambda + \gamma A_j) - A_j\big)^2\Big\} \\ &\quad \cdot \frac{d^n}{d\lambda^n}\Big(\int_{\mathbb{R}} \exp\Big\{-\frac{1}{2\gamma^2 B_j}\Big[u - (\beta\lambda + \gamma A_j)\Big]^2\Big\} du \exp\Big\{-\frac{1}{2\beta^2 B_j}\Big[(\beta\lambda + \gamma A_j) - A_j\Big]^2\Big\}\Big) \end{aligned}$$

$$
\begin{aligned}
&= (n!)^{-\frac{1}{2}} B_j^{\frac{n}{2}} (-\beta)^n \exp\bigg\{ \frac{1}{2\beta^2 B_j} \big((\beta\lambda + \gamma A_j) - A_j\big)^2 \bigg\} \\
&\quad \cdot \frac{d^n}{d\lambda^n} \bigg(\gamma\sqrt{2\pi B_j} \exp\bigg\{ -\frac{1}{2\beta^2 B_j} \Big[(\beta\lambda + \gamma A_j) - A_j \Big]^2 \bigg\} \bigg) \\
&= (n!)^{-\frac{1}{2}} B_j^{\frac{n}{2}} (-\beta)^n \exp\bigg\{ \frac{1}{2\beta^2 B_j} \big((\beta\lambda + \gamma A_j) - A_j\big)^2 \bigg\} \\
&\quad \cdot \gamma\sqrt{2\pi B_j} \frac{d^n}{d\lambda^n} \bigg(\exp\bigg\{ -\frac{1}{2\beta^2 B_j} \Big[(\beta\lambda + \gamma A_j) - A_j \Big]^2 \bigg) \\
&= (n!)^{-\frac{1}{2}} B_j^{\frac{n}{2}} (-\beta)^n \exp\bigg\{ \frac{1}{2B_j} \big((\lambda + \frac{\gamma}{\beta} A_j) - \frac{1}{\beta} A_j\big)^2 \bigg\} \\
&\quad \cdot \gamma\sqrt{2\pi B_j} \frac{d^n}{d\lambda^n} \bigg(\exp\bigg\{ -\frac{1}{2B_j} \Big[(\lambda + \frac{\gamma}{\beta} A_j) - \frac{1}{\beta} A_j \Big]^2 \bigg) \\
&= \beta^n \sqrt{2\pi\gamma^2 B_j} \tilde{H}_n^j \bigg(\lambda + \frac{\gamma + \beta - 1}{\beta} A_j \bigg)
\end{aligned}
$$

which is complete the proof of Lemma 5.3. □

Remark 5.4. Equation (5.3) actually holds for each $\lambda \in \mathbb{C}$ since $\tilde{H}_{m_j}^j(u)$ is a polynomial of degree m_j in $\frac{u-A_j}{\sqrt{B_j}}$ and thus both sides of equation (5.3) are analytic functions of λ throughout $\mathbb{C}$.

Lemma 5.5. *Let $\beta = i$ and $\gamma = \sqrt{2}$. Then β and γ satisfying the hypothesis of Lemma* 5.3. *For each positive integer j, let m_j be a nonnegative integer, and let $\varphi_{(m_j,j)}(x) = \tilde{H}_{m_j}^j(\langle \alpha_j, x\rangle)$. Then for each $y \in K_{a,b}[0,T]$, the generalized Fourier-Wiener function space transform $\mathcal{F}_{\sqrt{2},i}(\varphi_{(m_j,j)})(y)$ exists and is given by the formula*

$$
\begin{aligned}
\mathcal{F}_{\sqrt{2},i}(\varphi_{(m_j,j)})(y) &= \mathcal{F}_{\sqrt{2},i}(\tilde{H}_{m_j}^j)(\langle \alpha_j, y\rangle) \\
&= i^{m_j} \tilde{H}_{m_j}^j \bigg(\langle \alpha_j, y\rangle + \frac{\sqrt{2} + i - 1}{i} A_j \bigg) \\
&= i^{m_j} \tilde{H}_{m_j}^j \bigg(\langle \alpha_j, y\rangle - i(\sqrt{2} + i - 1) A_j \bigg).
\end{aligned} \tag{5.4}
$$

Proof. Using equation (5.1), formula (4.3), equation (5.3) with $\lambda = \langle \alpha_j, y\rangle$ and $n = m_j$, it follows that

$$
\begin{aligned}
&\mathcal{F}_{\sqrt{2},i}(\varphi_{(m_j,j)})(y) = \mathcal{F}_{\sqrt{2},i}(\tilde{H}_{m_j}^j)(\langle \alpha_j, y\rangle) \\
&= \int_{C_{a,b}[0,T]} \tilde{H}_{m_j}^j \big(\sqrt{2}\langle \alpha_j, x\rangle + i\langle \alpha_j, y\rangle \big) d\mu(x) \\
&= \big(2\pi B_j\big)^{-\frac{1}{2}} \int_{\mathbb{R}} \tilde{H}_{m_j}^j (\sqrt{2} u_j + i\langle \alpha_j, y\rangle) \exp\bigg\{ -\frac{(u_j - A_j)^2}{2B_j} \bigg\} du_j
\end{aligned}
$$

$$= \big(2\pi B_j\big)^{-\frac{1}{2}} \int_{\mathbb{R}} \tilde{H}^j_{m_j}(\gamma u_j + \beta\langle\alpha_j, y\rangle) \exp\bigg\{-\frac{(u_j - A_j)^2}{2B_j}\bigg\} du_j$$
$$= \big(2\pi\gamma^2 B_j\big)^{-\frac{1}{2}} \int_{\mathbb{R}} \tilde{H}^j_{m_j}(v) \exp\bigg\{-\frac{\big(v - (\beta\langle\alpha_j, y\rangle + \gamma A_j)\big)^2}{2\gamma^2 B_j}\bigg\} dv$$
$$= \frac{\beta^{m_j}\sqrt{2\pi\gamma^2 B_j}}{\sqrt{2\pi\gamma^2 B_j}} \tilde{H}^j_{m_j}\bigg(\langle\alpha_j, y\rangle + \frac{\gamma+\beta-1}{\beta} A_j\bigg)$$
$$= \beta^{m_j} \tilde{H}^j_{m_j}\bigg(\langle\alpha_j, y\rangle + \frac{\gamma+\beta-1}{\beta} A_j\bigg)$$
$$= i^{m_j} \tilde{H}^j_{m_j}\bigg(\langle\alpha_j, y\rangle + \frac{\sqrt{2}+i-1}{i} A_j\bigg)$$
$$= i^{m_j} \tilde{H}^j_{m_j}\bigg(\langle\alpha_j, y\rangle - i(\sqrt{2}+i-1) A_j\bigg). \qquad \square$$

Next, applying Lemma 5.5 N times, we obtain a formula for generalized Fourier-Wiener function space transform of the generalized Fourier-Hermite functionals $\Phi_{(m_1,\dots,m_N)}$.

Theorem 5.6. *Let N be a positive integer, for each $j = 1, \dots, N$, let m_j be a nonnegative integer, and let*

$$\Phi_{(m_1,\dots,m_N)}(x) \equiv \prod_{j=1}^{N} \varphi_{(m_j,j)}(x) = \prod_{j=1}^{N} \tilde{H}^j_{m_j}(\langle\alpha_j, x\rangle). \tag{5.5}$$

Then for each $y \in K_{a,b}[0,T]$, the generalized Fourier-Wiener function space transform $\mathcal{F}_{\sqrt{2},i}(\Phi_{(m_1,\dots,m_N)})(y)$ exists and is given by the formula

(5.6)

$$\mathcal{F}_{\sqrt{2},i}(\Phi_{(m_1,\dots,m_N)})(y) = \mathcal{F}_{\sqrt{2},i}\Big(\prod_{j=1}^{N} \tilde{H}^j_{m_j}\Big)(\langle\alpha_j, y\rangle)$$
$$= i^{m_1+\cdots+m_N} \prod_{j=1}^{N} \tilde{H}^j_{m_j}\bigg(\langle\alpha_j, y\rangle - i(\sqrt{2}+i-1)A_j\bigg)$$
$$= \prod_{j=1}^{N} \mathcal{F}_{\sqrt{2},i}(\tilde{H}^j_{m_j})(\langle\alpha_j, y\rangle).$$

In the special cases $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, the function space $C_{a,b}[0,T]$ reduces to the Wiener space $C_0[0,T]$ because $A_j = 0$ and $B_j = 1$ for all $j = 1, 2, \dots$. Hence many of the results in [1-4] are corollaries of the results in this paper. In particular:

Corollary 5.7. *Let $\Phi_{(m_1,\dots,m_N)}$ be given by* (5.5)*. Then*

$$\mathcal{F}_{\sqrt{2},i}(\Phi_{(m_1,\dots,m_N)})(y) = i^{m_1+\cdots+m_N} \Phi_{(m_1,\dots,m_N)}(y), \quad y \in K_0[0,T]$$

and

$$\|\mathcal{F}_{\sqrt{2},i}(\Phi_{(m_1,\dots,m_N)})\|_2 = 1.$$

Definition 5.8. Let $F \in L_2(C_{a,b}[0,T])$ be given and let (4.11) be its Fourier-Hermite series expansion with F_N defined by (4.9) satisfying condition (4.10). Then we define the generalized Fourier-Wiener function space transform $\mathcal{F}_{\sqrt{2},i}(F)$ of F to be

$$\mathcal{F}_{\sqrt{2},i}(F)(x) = \lim_{N\to\infty} \mathcal{F}_{\sqrt{2},i}(F_N)(x), \qquad x \in C_{a,b}[0,T] \tag{5.7}$$

if it exists; that is to say if

$$\lim_{N\to\infty} \int_{C_{a,b}[0,T]} |\mathcal{F}_{\sqrt{2},i}(F)(x) - \mathcal{F}_{\sqrt{2},i}(F_N)(x)|^2 d\mu(x) = 0. \tag{5.8}$$

The following theorem is one of our main results.

Theorem 5.9. *Let $F \in L_2(C_{a,b}[0,T])$ be given by equation (4.11) with the generalized Fourier-Hermite coefficients of F given by equation (4.5). Then the generalized Fourier-Wiener function space transform $\mathcal{F}_{\sqrt{2},i}(F)$ of F exists and is an element of $L_2(C_{a,b}[0,T])$ if and only if*

$$\begin{aligned} &\lim_{N\to\infty} \sum_{m_1,\dots,m_N=0}^{N} |A^F_{(m_1,\dots,m_N)}|^2 \\ &\quad \cdot \prod_{j=1}^{N} \int_{C_{a,b}[0,T]} \bigg| \tilde{H}^j_{m_j}\Big(\langle \alpha_j, x\rangle - i(\sqrt{2}+i-1)A_j\Big)\bigg|^2 d\mu(x) < \infty. \end{aligned} \tag{5.9}$$

Furthermore if (5.9) holds, then the generalized Fourier-Hermite series expansion of $\mathcal{F}_{\sqrt{2},i}(F)$ is given by

$$\mathcal{F}_{\sqrt{2},i}(F)(x) = \lim_{N\to\infty} \mathcal{F}_{\sqrt{2},i}(F_N)(x)$$

with

$$\begin{aligned} \mathcal{F}_{\sqrt{2},i}(F_N)(x) &= \sum_{m_1,\dots,m_N=0}^{N} A^F_{(m_1,\dots,m_N)} i^{m_1+\cdots+m_N} \\ &\qquad \cdot \prod_{j=1}^{N} \tilde{H}^j_{m_j}\Big(\langle \alpha_j, x\rangle - i(\sqrt{2}+i-1)A_j\Big) \\ &= \sum_{m_1,\dots,m_N=0}^{N} A^F_{(m_1,\dots,m_N)} \prod_{j=1}^{N} \mathcal{F}_{\sqrt{2},i}(\tilde{H}^j_{m_j})(\langle \alpha_j, x\rangle) \end{aligned}$$

for each $x \in C_{a,b}[0,T]$ and each $N = 1, 2, \dots$.

Proof. First assume that $\mathcal{F}_{\sqrt{2},i}(F)$ exists and is an element of $L_2(C_{a,b}[0,T])$. For any $\epsilon > 0$ we have that

$$\int_{C_{a,b}[0,T]} |\mathcal{F}_{\sqrt{2},i}(F)(x) - \mathcal{F}_{\sqrt{2},i}(F_N)(x)|^2 d\mu(x) < \epsilon$$

for all sufficiently large N. Hence for N sufficiently large,

$$\begin{aligned}
&\bigg(\sum_{m_1,\ldots,m_N=0}^{N} |A^F_{(m_1,\ldots,m_N)}|^2 \prod_{j=1}^{N} \int_{C_{a,b}[0,T]} \bigg| \tilde{H}^j_{m_j}\Big(\langle \alpha_j, x\rangle - i(\sqrt{2}+i-1)A_j \Big) \bigg|^2 d\mu(x) \bigg)^{\frac{1}{2}} \\
&= \|\mathcal{F}_{\sqrt{2},i}(F_N)\|_2 \\
&\le \|\mathcal{F}_{\sqrt{2},i}(F)\|_2 + \|\mathcal{F}_{\sqrt{2},i}(F) - \mathcal{F}_{\sqrt{2},i}(F_N)\|_2 \\
&\le \|\mathcal{F}_{\sqrt{2},i}(F)\|_2 + \epsilon
\end{aligned}$$

and thus by condition (5.9) holds.

To proof the converse, suppose that condition (5.9) holds. For integers M and N with $M > N \ge 1$, let

$$I_M = \{(m_1, \ldots, m_M) : m_1, \ldots, m_M = 0, 1, \ldots, M\}$$

and let

$$I_N = \{(m_1, \ldots, m_M) : m_1, \ldots, m_N = 0, 1, \ldots, N \text{ and } m_{N+1} = \cdots = m_M = 0\}.$$

Then it follows that

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} |\mathcal{F}_{\sqrt{2},i}(F_M)(x) - \mathcal{F}_{\sqrt{2},i}(F_N)(x)|^2 d\mu(x) \\
&= \int_{C_{a,b}[0,T]} \bigg| \sum_{I_M - I_N} A^F_{(m_1,\ldots,m_M)} i^{m_1+\cdots+m_M} \\
&\qquad\qquad \cdot \prod_{j=1}^{M} \tilde{H}^j_{m_j}\Big(\langle \alpha_j, x\rangle - i(\sqrt{2}+i-1)A_j \Big) \bigg|^2 d\mu(x) \\
&= \sum_{I_M - I_N} \Big| A^F_{(m_1,\ldots,m_M)} i^{m_1+\cdots+m_M} \Big|^2 \\
&\qquad\qquad \cdot \prod_{j=1}^{M} \int_{C_{a,b}[0,T]} \bigg| \tilde{H}^j_{m_j}\Big(\langle \alpha_j, x\rangle - i(\sqrt{2}+i-1)A_j \Big) \bigg|^2 d\mu(x) \\
&= \sum_{m_1,\ldots,m_M=0}^{M} \Big| A^F_{(m_1,\ldots,m_M)} i^{m_1+\cdots+m_M} \Big|^2 \\
&\qquad\qquad \cdot \prod_{j=1}^{M} \int_{C_{a,b}[0,T]} \bigg| \tilde{H}^j_{m_j}\Big(\langle \alpha_j, x\rangle - i(\sqrt{2}+i-1)A_j \Big) \bigg|^2 d\mu(x)
\end{aligned}$$

$$- \sum_{m_1,\ldots,m_N=0}^{N} \left| A^F_{(m_1,\ldots,m_N)} i^{m_1+\cdots+m_N} \right|^2 \cdot \prod_{j=1}^{N} \int_{C_{a,b}[0,T]} \left| \tilde{H}^j_{m_j} \left(\langle \alpha_j, x \rangle - i(\sqrt{2}+i-1)A_j \right) \right|^2 d\mu(x)$$

which goes to 0 as $M, N \to \infty$. Hence $\{\mathcal{F}_{\sqrt{2},i}(F_N)\}_{N=1}^{\infty}$ is a Cauchy sequence in $L_2(C_{a,b}[0,T])$ and since $L_2(C_{a,b}[0,T])$ is complete,

$$\mathcal{F}_{\sqrt{2},i}(F)(x) = \lim_{N\to\infty} \mathcal{F}_{\sqrt{2},i}(F_N)(x), \qquad x \in C_{a,b}[0,T]$$

exists and is an element of $L_2(C_{a,b}[0,T])$. □

Remark 5.10. The main result in [10], namely Theorem 6 on page 1385, as well as the main results of [3], namely Theorem 1 on page 103, follows immediately from Theorem 5.9 above by choosing $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, and $\beta = i$ and $\gamma = \sqrt{2}$.

Corollary 5.11. *Suppose that condition* (5.9) *holds. If* $a(t) \equiv 0$ *on* $[0,T]$*, then*

$$\|\mathcal{F}_{\sqrt{2},i}(F)\|_2^2 = \lim_{N\to\infty} \sum_{m_1,\ldots,m_N=0}^{N} |A^F_{(m_1,\ldots,m_N)}|^2 = \|F\|_2^2.$$

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