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ZERO-DIVISOR GRAPHS WITH RESPECT TO PRIMAL AND WEAKLY PRIMAL IDEALS

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Abstract. We consider zero-divisor graphs with respect to primal, non-primal, weakly prime and weakly primal ideals of a commutative ring R with non-zero identity. We investigate the interplay between the ring-theoretic properties of R and the graph-theoretic properties of $\Gamma_I(R)$ for some ideal I of R. Also we show that the zero-divisor graph with respect to primal ideals commutes by localization.

1. Introduction

The idea of associating a graph with the zero-divisors of a commutative ring was introduced by Beck in 1988, where the author talked about the colorings of such graphs. By the definition he gave, every element of the ring R was a vertex in the graph, and two vertices x, y were adjacent if and only if $xy = 0$ ([4]). We adopt the approach used by D. F. Anderson and P. S. Livingston ([2]) and consider only non-zero zero-divisors as vertices of the graph. The zero-divisor graph of a commutative ring has been studied extensively by several authors (see, for example, [2, 4, 5, 10, 11, 12]).

Redmond [13] introduced the definition of the zero-divisor graph with respect to an ideal. Let I be an ideal of a ring R. The zero-divisor graph of R with respect to I is an undirected graph, denoted by $\Gamma_I(R)$, with vertices $\{x \in R - I : xy \in I \text{ for some } y \in R - I\}$ where distinct vertices x and y are adjacent if and only if $xy \in I$. Therefore, if $I = 0$ then $\Gamma_I(R) = \Gamma(R)$, and I is a non-zero prime ideal if and only if $\Gamma_I(R) = \emptyset$ ([13]). The graphs $\Gamma_I(R)$ and $\Gamma(R/I)$ are different graphs. If fact for $x, y \in R\backslash I$, if $x + I$ is adjacent to $y + I$ in $\Gamma(R/I)$, then x and y are adjacent in $\Gamma_I(R)$; while the converse is true only when $x+I \neq y+I$ (see [13, Theorem 2.5]). Hence the study of the graph $\Gamma_I(R)$ is worthy of study. There are many basic open questions concerning the zero-divisor graph with respect to an ideal. One of the essential questions is whether a zero-divisor graph with respect to an ideal commutes with localization, and in this case, what are the relations between the diameters (resp. girths) of such graphs. We give a condition giving an affirmative answer to these questions.

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For the sake of completeness, we state some definitions and notations used throughout. We will use R to denote a commutative ring with identity. We use $Z(R)$ to denote the set of zero-divisors of R; we use $Z(R)^*$ to denote the set of non-zero zero-divisors of R. By the zero-divisor graph of R, denoted $\Gamma(R)$, we mean the graph whose vertices are the non-zero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, there is an edge connecting x and y if and only if $xy = 0$. A graph is said to be connected if there exists a path between any two distinct vertices. For two distinct vertices a and b in a graph G, the distance between a and b, denoted $\mathrm{d}(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, $\mathrm{d}(a, b) = \infty$. The diameter of a connected graph is the supremum of the distances between vertices. We will use the notation $\mathrm{diam}(G)$ to denote the diameter of the graph of G. A graph is complete if it is connected with diameter 0. A bipartite graph is a graph G whose vertex set V can be partitioned into two non-empty sets V_1 and V_2 in such a way that every edge of G joins a vertex in V_1 to a vertex in V_2. The girth of a graph G, denoted $\mathrm{gr}(G)$, is the length of a shortest cycle in G, provided G contains a cycle; otherwise, $\mathrm{gr}(G) = \infty$.

An ideal I of R is called a radical ideal if $I = \sqrt{I}$. A ring R is called reduced if it contains no non-zero nilpotent elements. It is easy to see that I is a radical ideal of R if and only if R/I is a reduced ring. Denote by $\mathrm{Min}(I)$ the set of minimal prime ideals of R containing I.

Let R be a commutative ring. A proper ideal P of R is said to be weakly prime if $0 \neq ab \in P$ implies that $a \in P$ or $b \in P$ ([3]). However, since 0 is always weakly prime (by definition), a weakly prime ideal need not be prime. We recall from [8] and [7], that an element $a \in R$ is called prime (resp. weakly prime) to an ideal I of R if $ra \in I$ (resp. $0 \neq ra \in I$) (where $r \in R$) implies that $r \in I$. Denote by $S(I)$ (resp. $w(I)$) the set of elements of R that are not prime (resp. are not weakly prime) to I. A proper ideal I of R is said to be primal if $S(I)$ forms an ideal (so 0 is not necessarily primal); this ideal is always a prime ideal, called the adjoint ideal P of I. In this case we also say that I is a P-primal ideal of R ([8]). Not that if $r \in R$ and $a \in S(I)$, then clearly $ra \in S(I)$. So what we require for I being primal is that if a and b are not prime to I, then their difference is also not prime to I. For example assume that $R = \mathbb{Z}$ the ring of integers and let $I = 6\mathbb{Z}$. Then 2 and 3 are not prime to I but $3 - 2$ is prime to I, so I is not primal. A ring R is said to be primal if the zero ideal is a primal ideal of R. Also, a proper ideal I of R is called weakly primal if the set $P = w(I) \cup \{0\}$ forms an ideal; this ideal is always a weakly prime ideal ([7, Proposition 4]). In this case we also say that I is a P-weakly primal ideal. If R is not an integral domain, then 0 is a 0-weakly primal ideal of R (by definition), so a weakly primal ideal need not be primal.

Assume that S is a multiplicatively closed subset of a ring R, X a non-empty subset of R and I an ideal of R. Set $S^{-1}X = \{a/s | a \in X, s \in S\} \subseteq S^{-1}R$. We say that a zero-divisor graph with respect to I commutes with localization if $S^{-1}(\Gamma_I(R)) = \Gamma_{S^{-1}I}(S^{-1}R)$. The main goal of this paper is to show that if I is primal (resp. weakly primal), then $\Gamma_I(R)$ commutes with localization. Surely, there is much more work to be done.

Here is a brief summary of our paper. In Section 2, it is shown that (Theorem 2.5), if I and J are P-primal ideals of R, then $\Gamma_I(R) = \Gamma_J(R)$ if and only if $I = J$. It is proved that (Theorem 2.8) if I is a primal ideal of a Noetherian ring R, then $\mathrm{diam}(\Gamma(R/I)) \leq 2$. In Theorems 2.16 and 2.18 (resp. Theorems 4.7 and 4.8), it is shown that, if S is a multiplicatively closed subset of R which consists of regular elements of R and I is a P-primal (resp. P-weakly primal) ideal of R with $P \cap S = \emptyset$, then $\mathrm{diam}(\Gamma_I(R)) = \mathrm{diam}(\Gamma_{S^{-1}I}(S^{-1}R))$ and $\mathrm{gr}(\Gamma_I(R)) = \mathrm{gr}(\Gamma_{S^{-1}I}(S^{-1}R))$.

In Section 3, it is proved that, if $I = \sqrt{I}$ is not a primal ideal of R with $|\mathrm{Min}(I)| \geq 3$, then $\mathrm{diam}(\Gamma_I(R)) = 3$ (Theorem 3.3). Also, it is shown that, if $I \neq \sqrt{I}$ is not a primal ideal of R, then $\mathrm{diam}(\Gamma_I(R)) = 3$ (Theorem 3.7).

In Section 4, we study the zero-divisor graph with respect to a weakly primal ideal. We put $Z_I(R) = \{r \in R - I : ra = 0 \text{ for some } a \in R - I\}$ where I is an ideal of R. It is proved that (Theorem 4.3), if I is an ideal of a ring R and P is a weakly prime ideal with $w(I) \subseteq P$ and $(P - I) \cap Z_I(R) = \emptyset$, then $\Gamma_I(R) = (P - I) \cup Z_I(R)$ if and only if I is a P-weakly primal ideal of R. It is shown that (Theorem 4.4) if I is a weakly prime ideal of R, then $\Gamma_I(R) = Z_I(R)$. In particular, $\Gamma_I(R)$ is a subgraph of $\Gamma(R)$.

2. Primal ideals

In this section, we will investigate the ideal-based zero-divisor graph with respect to primal ideals. The class of primal ideals is a large class. For example all primary ideals and irreducible ideals are primal. So the structure of zero-divisor graphs with respect to primal ideals is worthy of study. Our starting point is the following lemma:

Lemma 2.1. *Let I be a proper ideal of a ring R. Then the following hold*:

(i) $I \subseteq S(I)$.

(ii) $\Gamma_I(R) = S(I) - I$. *In particular,* $\Gamma_I(R) \cup I = S(I)$.

(iii) *If I is a radical ideal of R, then* $S(I) = \bigcup_{P \in \mathrm{Min}(I)} P$.

Proof. (i) Let $x \in I$. As $x.1_R \in I$ with $1_R \notin I$, we must have x is not prime to I; hence $I \subseteq S(I)$.

(ii) Let $r \in \Gamma_I(R)$. Then $r \notin I$ and $rx \in I$ for some $x \notin I$, so r is not prime to I; hence $r \in S(I) - I$. Thus $\Gamma_I(R) \subseteq S(I) - I$. For the other containment, assume that $a \in S(I) - I$. As a is not prime to I, there exists $y \notin I$ such that $ay \in I$. Then $a \in \Gamma_I(R)$, so we have equality.

(iii) Let $x \in S(I)$. Thus we may assume that $x \notin I$, so $x \in \Gamma_I(R)$. Then $xy \in I$ for some $y \in R - I$, so $(0 :_{R/I} x + I) \neq 0$; hence $x + I \in P/I$ for some minimal prime ideal P/I of R/I by [9, Corollary 2.3]. Therefore, $P \in \mathrm{Min}(I)$ and $x \in P$ gives $S(I) \subseteq \bigcup_{P \in \mathrm{Min}(I)} P$. Conversely, assume that $x \in P$ for some minimal prime ideal P of I. If $x \in I$, then $x \in S(I)$ by (i). So we can assume that $x \notin I$. By [9, Theorem 2.1], there exist $y \notin P$ and a positive integer n such that $yx^n \in I$ but $yx^{n-1} \notin I$. This implies that $x \in \Gamma_I(R)$, so we have equality. □

Proposition 2.2. *Let I and P be ideals of a ring R with $I \subseteq P$. Then I is a P-primal ideal of R if and only if $\Gamma_I(R) = P - I$.*

Proof. If I is a P-primal ideal of R, then $\Gamma_I(R) = S(I) - I = P - I$ by Lemma 2.1. Conversely, assume that $\Gamma_I(R) = P - I$. It suffices to show that P is exactly the set of elements of R that are not prime to I. First, suppose that $c \in P$. Since every element of I is not prime to I, we can assume that $c \in P - I = \Gamma_I(R)$. Then there exists $z \notin I$ such that $cz \in I$, so c is not prime to I. Next suppose that s is not prime to I. If $s \in I$, then $s \in P$. If $s \notin I$, then there is an element $t \notin I$ such that $st \in I$, so $s \in \Gamma_I(R) = P - I \subseteq P$. Thus I is a P-primal ideal of R. □

Theorem 2.3. *Let I be an ideal of a ring R. Then I is a primal ideal of R if and only if $\Gamma_I(R) \cup I$ is an (prime) ideal of R.*

Proof. This follows from Proposition 2.2. □

Examples 2.4. (1) Let $R = \mathbb{Z}_{16}$, $I = (\bar{4})$ and $P = (\bar{2})$. It is easy to check that I is a P-primal ideal of R. Then $\Gamma_I(R) = P - I = \{\bar{2}, \bar{6}, \bar{10}, \bar{14}\}$ by Proposition 2.2.

(2) Let $R = \mathbb{Z}$, $I = 9\mathbb{Z}$ and $P = 3\mathbb{Z}$. Then, I is a P-primary and hence a P-primal ideal of R by [7, Lemma 19]. Hence $\Gamma_I(R) = 3\mathbb{Z} - 9\mathbb{Z}$ by Proposition 2.2.

(3) Let $R = Q[x, y]$ the ring of polynomials in x and y with rational numbers for their coefficients. Then $I = (x^2, xy)$ is a primal ideal with adjoint ideal $P = (x, y)$ ([8]). Hence $\Gamma_I(R) = (x, y) - (x^2, xy)$ by Proposition 2.2.

Let I, J be ideals of a ring R. It is natural to ask whether $\Gamma_I(R) = \Gamma_J(R)$ implies $I = J$? In this case, we have the following theorem:

Theorem 2.5. *Let I and J be P-primal ideals of a ring R. Then $\Gamma_I(R) = \Gamma_J(R)$ if and only if $I = J$.*

Proof. By Lemma 2.1, $I \subseteq P$ and $J \subseteq P$. It then follows from Proposition 2.2 that $\Gamma_I(R) = \Gamma_J(R)$ if and only if $P - I = P - J$; and this holds if and only if $I = J$. □

Theorem 2.6. *Let I be an ideal of a ring R. Then I is primary if and only if $\Gamma_I(R) = \sqrt{I} - I$.*

Proof. If I is primary, then I is a $\sqrt{I}$-primal ideal of R by [7, Lemma 19], so Proposition 2.2 gives $\Gamma_I(R) = \sqrt{I} - I$. Conversely, suppose that $a, b \in R$ are such that $ab \in I$ but $a \notin I$ and $b \notin \sqrt{I}$ (so $b \notin I$). Then $b \in \Gamma_I(R) = \sqrt{I} - I$, which is a contradiction. Thus I is primary. □

Theorem 2.7. *Let I be an ideal of a commutative ring R. Then:*

(1) 0 *is a primal ideal of R if and only $Z(R)$ is an (prime) ideal of R.*
(2) *I is a primal ideal of R if and only if $Z(R/I)$ is an ideal of R/I.*

Proof. (1) It is easy to show that $S(0) = Z(R)$. Now the result follows from the definition.

(2) It follows from (1). □

Let I be an ideal of R. It is shown in [2, Theorem 2.3] and [13, Theorem 2.4] that $\operatorname{diam}(\Gamma(R/I)) \leq 3$ and $\operatorname{diam}(\Gamma_I(R)) \leq 3$, but for a primal ideal we have the following results:

Theorem 2.8. *Let R be a Noetherian ring. If I is a primal ideal of R, then* $\operatorname{diam}(\Gamma(R/I)) \leq 2$.

Proof. Let I be a P-primal ideal of R. Then $\Gamma(R/I) \cup \{0+I\} = P/I$ is a prime ideal of R/I by Theorem 2.7 and by [14, Corollary 9.36], $P/I = \bigcup_{\bar{Q} \in \operatorname{Ass}(R/I)} \bar{Q}$; hence $P/I \in \operatorname{Ass}(R/I)$. Therefore, $P/I = (0 :_{R/I} \bar{a})$ for some $\bar{a} \in \Gamma(R/I)$. It follows that $\operatorname{diam}(\Gamma(R/I)) \leq 2$. □

Let R be a principal ideal domain and let p be an irreducible element of R. Then Rp^t is a primary ideal of R for every positive integer t. So $Nil(R/Rp^t) = Z(R/Rp^t)$. Therefore $\operatorname{diam}(\Gamma(R/Rp^t)) \leq 2$ by [10, Lemma 2.3]. Using Theorem 2.8, we give another proof for this result.

Corollary 2.9. *Let R be a principal ideal domain, and let p be an irreducible element of R. Then for every positive integer t,* $\operatorname{diam}(\Gamma(R/Rp^t)) \leq 2$. *In particular,* $\operatorname{diam}(\Gamma(\mathbb{Z}_{p^t})) \leq 2$ *for every prime number p.*

Proof. Since $I = Rp^t$ is primary, we must have I is a primal ideal of R by [7, Lemma 19]. Now the assertion follows from Theorem 2.8. □

Example 2.10. Let $R = \mathbb{Z}$ and $I = 18\mathbb{Z}$. Then I is not a primal ideal of R since 2 and 3 are not prime to I, but $3 - 2 = 1$ is prime to I. Consider the elements $\overline{2}$ and $\overline{3}$ in R/I. As $\overline{2}.\overline{3} \neq 0$ we have $d(\overline{2}, \overline{3}) \neq 1$. If there is a vertex $\overline{a}$ in $\Gamma(R/I)$ such the $\overline{2} - \overline{a} - \overline{3}$ is a path, then $\overline{a} = 0$ which is a contradiction. Hence $d(\overline{2}, \overline{3}) \neq 2$. Thus $\operatorname{diam}(\Gamma(R/I)) = 3$. Therefore, the condition "I is a primal ideal of R" is not superficial in the Theorem 2.8.

Let R be a Noetherian ring and assume that $Q(R)$, the total quotient ring of R, is local. In this case, $Z(Q(R)) = ann(x)$ for some $0 \neq x \in Q(R)$. This implies that $\operatorname{diam}(\Gamma(Q(R)) \leq 2$. On the other hand, $\Gamma(Q(R))$ is isomorphic to $\Gamma(R)$ by [1, Theorem 2.2]. So $\operatorname{diam}(\Gamma(R)) \leq 2$. However this fact may be proved as the following theorem.

Theorem 2.11. *Assume that R is a Noetherian ring let $Q(R)$, the total quotient ring of R a local ring. Then* $\operatorname{diam}(\Gamma(R)) \leq 2$.

Proof. Assume that S is the set of non-zero-divisors of R/I and let M be the unique maximal ideal of $Q(R)$. Then there exists a prime ideal P of R such that $P \cap S = \emptyset$ and $M = S^{-1}P$. First we show that $P = Z(R)$. As $P \cap S = \emptyset$, we have $S \subseteq R - P$. For every $a \in R - P$, we have $a/1 \notin S^{-1}P = M$. Hence $a/1$ is a unit in $Q(R)$. Thus $a \notin Z(R)$, that is $R - P \subseteq S$. Therefore $P = Z(R)$. So 0 is a P-primal ideal of R by Theorem 2.7. Now the result follows from Theorem 2.8. □

Lemma 2.12. *Let I be a radical ideal of a ring R. Then* $\mathrm{diam}(\Gamma(R/I)) = \mathrm{diam}(\Gamma_I(R))$.

Proof. This follows immediately from [5, Lemma 2.1 and Proposition 2.2]. □

Corollary 2.13. *Let I be a primal radical ideal of a Noetherian ring R. Then* $\mathrm{diam}(\Gamma_I(R)) \leq 2$.

Proof. This follows from Theorem 2.8 and Lemma 2.12. □

We shall require the following proposition, and its proof is a modification of those in [6, Lemma 2.11 and Proposition 2.14], but we give the details for convenience.

Proposition 2.14. *Assume that S is a multiplicatively closed subset of a ring R and let I be a P-primal ideal of R with $P \cap S = \emptyset$. Then the following hold*:

(i) *If $a/s \in S^{-1}I$, then $a \in I$.*
(ii) *$S^{-1}I$ is a $S^{-1}P$-primal ideal of $S^{-1}R$.*

Proof. (i) Suppose that $a/s \in S^{-1}I$, but $a \notin I$. Then there are elements $a' \in I$ and $t \in S$ such that $a/s = a'/t$, so $uta = usa' \in I$ for some $u \in S$. It follows that ut is not prime to I; hence $ut \in P \cap S$ which is a contradiction, as needed.

(ii) Clearly, $S^{-1}P$ is a prime ideal of $S^{-1}R$. It is enough to show that $S^{-1}P$ is exactly the set of elements of $S^{-1}R$ which are not prime to $S^{-1}I$. Let $r/s \in S^{-1}P$. Then r is not prime to I, so there exists $c \in R - I$ with $rc \in I$. Since $P \cap S = \emptyset$, we get $sc \notin I$; hence $(sc)/1 \notin S^{-1}I$ by (i). As $(r/s)(sc)/1 \in S^{-1}I$, we must have r/s is not prime to $S^{-1}I$. Now assume that r/s is not prime to $S^{-1}I$. Then there exists $d/t \notin S^{-1}I$ with $(r/s)(d/t) \in S^{-1}I$; hence $rd \in I$ by (i). Since $d \notin I$, it follows that r is not prime to I. Thus $r \in P$, and hence $r/s \in S^{-1}P$, as required. □

Proposition 2.15. *Assume that S is a multiplicatively closed subset of a ring R and let I be a P-primal ideal of R with $P \cap S = \emptyset$. Then $S^{-1}(\Gamma_I(R)) = \Gamma_{S^{-1}I}(S^{-1}R)$.*

Proof. By Proposition 2.2 and Proposition 2.14, we must have $\Gamma_{S^{-1}I}(S^{-1}R) = S^{-1}P - S^{-1}I$. It suffices to show that $S^{-1}(P - I) = S^{-1}P - S^{-1}I$. First, suppose that $a/s \in S^{-1}P - S^{-1}I$. Then $a/s \notin S^{-1}I$ (so $a \notin I$) and $(a/s)(b/t) = ab/st \in S^{-1}I$ for some $b/t \notin S^{-1}I$ (so $b \notin I$), so $ab \in I$ by Proposition 2.14(i); hence a is not prime to I. It follows that $a/s \in S^{-1}(P - I)$. Thus $S^{-1}P -$

$S^{-1}I \subseteq S^{-1}(P-I)$. Next, assume that $a/s \in S^{-1}(P-I)$. Then $a \in P-I$ implies that $ab \in I$ for some $b \notin I$ by Proposition 2.2, so by Proposition 2.14(i), $b/1 \notin S^{-1}I$. Now $(a/s)(b/1) = ab/s \in S^{-1}I$ gives a/s is not prime to $S^{-1}I$, so $a/s \in S^{-1}P - S^{-1}I$; hence we have equality. □

Theorem 2.16. *Assume that S is a multiplicatively closed subset of a ring R which consists of regular elements of R and let I be a P-primal ideal of R with $P \cap S = \emptyset$. Then* $\mathrm{diam}(\Gamma_I(R)) = \mathrm{diam}(\Gamma_{S^{-1}I}(S^{-1}R))$.

Proof. Suppose that $\mathrm{diam}(\Gamma_I(R)) = 1$. For every distinct vertices $a/s, b/t$ of $\Gamma_{S^{-1}I}(S^{-1}R))$, Proposition 2.15 gives a and b are distinct elements of $\Gamma_I(R)$, so $ab \in I$; hence $(a/s)(b/t) \in S^{-1}I$. Thus $\mathrm{diam}(\Gamma_{S^{-1}I}(S^{-1}R)) = 1$. If $\mathrm{diam}(\Gamma_{S^{-1}I}(S^{-1}R)) = 1$, then for every distinct vertices a, b of $\Gamma_I(R) = P - I$, we have the distinct vertices $a/1, b/1 \in S^{-1}(P-I) = \Gamma_{S^{-1}I}(S^{-1}R)$ by Proposition 2.15 (since if $a/1 = b/1$, then $ta = tb$ for some $t \in S$; this implies that $a = b$ which is a contradiction), so $(a/1)(b/1) \in S^{-1}I$. It follows from Proposition 2.14(i) that $ab \in I$. Thus $\mathrm{diam}(\Gamma_I(R)) = 1$.

Now assume that $\mathrm{diam}(\Gamma_I(R)) = 2$. Let $a/s, b/t \in \Gamma_{S^{-1}I}(S^{-1}R))$. If $(a/s)(b/t) \notin S^{-1}I$, then $ab \notin I$, so there exists $c \in \Gamma_I(R)$ such that $ac \in I$ and $bc \in I$, so $c/1 \in \Gamma_{S^{-1}I}(S^{-1}R)$ by Proposition 2.15. As $(a/s)(c/1) \in S^{-1}I$ and $(c/1)(b/t) \in S^{-1}I$, we must have $\mathrm{diam}(\Gamma_{S^{-1}I}(S^{-1}R)) = 2$. Conversely, assume that $\mathrm{diam}(\Gamma_{S^{-1}I}(S^{-1}R)) = 2$. Let $a, b \in \Gamma_I(R)$ with $a \neq b$. If $ab \notin I$, then $ab/1 \notin S^{-1}I$ by Proposition 2.14(i), so there is an element c/s of $\Gamma_{S^{-1}I}(S^{-1}R)$ with $(a/1)(c/s) \in S^{-1}I$ and $(c/s)(b/1) \in S^{-1}I$. In this case, by Proposition 2.15, we must have $c \in \Gamma_I(R)$. Moreover, Proposition 2.14(i) gives $ac \in I$ and $cb \in I$; hence $\mathrm{diam}(\Gamma_I(R)) = 2$. Since, in general, the diameter of every zero-divisor graph with respect to an ideal is at most 3, we have proved the result. □

Example 2.17. Let $R = \mathbb{Z}$ and $I = 6\mathbb{Z}$. Then I is not a primal ideal of R. Since 2 and 3 are not prime to I, but $3 - 2 = 1$ is prime to I. since $I = 2\mathbb{Z} \cap 3\mathbb{Z}$, $\Gamma_I(R)$ is a complete bipartite with the parts $2\mathbb{Z} - 3\mathbb{Z}$ and $3\mathbb{Z} - 2\mathbb{Z}$ by [11, Theorem 3.1]. Thus $\mathrm{diam}(\Gamma_I(R)) = 2$. Set $S = \{3^n : n \text{ is a non-negative integer}\}$. Then S is a multiplicatively closed subset of R whose elements are regular and $S^{-1}I$ is a prime ideal of $S^{-1}R$; hence $\Gamma_{S^{-1}I}(S^{-1}R) = \emptyset$. This example shows that the condition "I is primal" in Theorem 2.16 is not superficial.

Theorem 2.18. *Assume that S is a multiplicatively closed subset of a ring R which consists of regular elements of R and let I be a P-primal ideal of R with $P \cap S = \emptyset$. Then* $\mathrm{gr}(\Gamma_I(R)) = \mathrm{gr}(\Gamma_{S^{-1}I}(S^{-1}R))$.

Proof. First assume that $\mathrm{gr}(\Gamma_I(R)) = \infty$. If $\mathrm{gr}(\Gamma_{S^{-1}I}(S^{-1}R)) = n$, then there is a cycle $a_1/s_1 - a_2/s_2 - \cdots - a_n/s_n$ in $\Gamma_{S^{-1}I}(S^{-1}R)$. In this case $a_1 - a_2 - \cdots - a_n$ forms a cycle in $\Gamma_I(R)$ by Proposition 2.14(i) which is a contradiction. So $\mathrm{gr}(\Gamma_{S^{-1}I}(S^{-1}R)) = \infty$. If $\mathrm{gr}(\Gamma_{S^{-1}I}(S^{-1}R)) = \infty$, then since $\Gamma_I(R)$ is a subgraph of $\Gamma_{S^{-1}I}(S^{-1}R)$, we must have $\mathrm{gr}(\Gamma_I(R)) = \infty$. By [13, Lemma 5.1], the girth of every ideal-based zero-divisor graph of a commutative ring, when finite, is either 3 or 4. Assume that $\mathrm{gr}(\Gamma_{S^{-1}I}(S^{-1}R)) = 3$. So there exist distinct vertices $a/s, b/t, c/u$ in $\Gamma_{S^{-1}I}(S^{-1}R)$ such that $(a/s)(b/t)$, $(b/t)(c/u)$ and $(c/u)(a/s)$ are elements of $S^{-1}I$, so Proposition 2.15 gives a, b, c are distinct vertices of $\Gamma_I(R)$; hence $ab, bc, ca \in I$ by Proposition 2.14(i). It follows that $\mathrm{gr}(\Gamma_I(R)) = 3$. Conversely, assume that $\mathrm{gr}(\Gamma_I(R)) = 3$. Since the canonical homomorphism $R \to S^{-1}R$ is injective, we can assume that R is a subring of $S^{-1}R$, so in this case, $\Gamma_I(R)$ is a subgraph of $\Gamma_{S^{-1}I}(S^{-1}R)$ (for if $a \in \Gamma_I(R)$, then $a/1 \in S^{-1}(\Gamma_I(R)) = \Gamma_{S^{-1}I}(S^{-1}R)$) by Proposition 2.15); hence $\mathrm{gr}(\Gamma_{S^{-1}I}(S^{-1}R)) \leq \mathrm{gr}(\Gamma_I(R)) = 3$. Since the girth of a graph is at least 3, we must have $\mathrm{gr}(\Gamma_{S^{-1}I}(S^{-1}R)) = 3$. Now it is clear that $\mathrm{gr}(\Gamma_{S^{-1}I}(S^{-1}R)) = 4$ if and only if $\mathrm{gr}(\Gamma_I(R)) = 4$. □

3. Non-primal ideals

In this section we study the diameter of $\Gamma_I(R)$ where I is not a primal ideal. First, we will give the following definition.

Definition. Let I be an ideal of a ring R. An ideal J of R is called prime to I if $(I :_R J) = I$.

Proposition 3.1. *Let I be an ideal of a ring R. If there are nonadjacent elements $a, b \in \Gamma_I(R)$ such that the ideal $\langle a, b\rangle$ is prime to I, then* $\mathrm{diam}(\Gamma_I(R)) = 3$.

Proof. Since a and b are nonadjacent, we must have $\mathrm{d}(a, b) \neq 1$. If $\mathrm{d}(a, b) = 2$, then there is an element $c \in R - I$ such that $ac, cb \in I$, so $c \in (I : \langle a, b\rangle)$ which is a contradiction. Thus $\mathrm{d}(a, b) \neq 2$, as required. □

Proposition 3.2. *Let I be an ideal of a ring R. If I is not primal, there exist elements a and b of $\Gamma_I(R)$ such that the ideal $\langle a, b\rangle$ is prime to I.*

Proof. Suppose that I is not primal. Then by Lemma 2.1, $\Gamma_I(R) \cup I = S(I)$ is not an ideal of R, so there exist $a, b \in S(I)$ with $a - b \notin S(I)$. If $a, b \in I$, then $a - b \in I \subseteq S(I)$ by Lemma 2.1 which is a contradiction. So suppose that $a \in I$ but $b \notin I$. Then $b \in \Gamma_I(R)$ and $bc \in I$ for some $c \in R - I$, so $(a - b)c \in I$; hence $a - b$ is not prime I which is a contradiction. Similarly, for $a \notin I$ and $b \in I$, we get a contradiction. Thus, we must have $a, b \in R - I$, so $a, b \in \Gamma_I(R)$. It suffices to show that $(I : \langle a, b\rangle) \subseteq I$. If $r \in (I : \langle a, b\rangle)$, then $r(a - b) \in I$, so $r \in I$ since $a - b$ is prime to I. Thus $\langle a, b\rangle$ is prime to I. □

Theorem 3.3. *Let I be a radical ideal of a ring R and suppose that I is not a primal ideal of R and $|\mathrm{Min}(I)| \geq 3$. Then* $\mathrm{diam}(\Gamma_I(R)) = 3$.

Proof. By Proposition 3.2, there exist $a, b \in \Gamma_I(R)$ such that the ideal $\langle a, b\rangle$ is prime to I, so the ideal $\langle a + I, b + I\rangle$ of R/I has no non-zero annihilator. As R/I is a reduced ring and $|\mathrm{Min}(R/I)| \geq 3$, it follows from ([10, Theorem 2.1]) that $\mathrm{diam}(\Gamma(R/I)) = 3$. Now the assertion follows from Lemma 2.12. □

Theorem 3.4. *Assume that I is a radical ideal of a ring R and I is not a primal ideal of R. Then* $\mathrm{diam}(\Gamma_I(R)) \leq 2$ *if and only if* $|\mathrm{Min}(I)| = 2$.

Proof. First, assume that $\mathrm{diam}(\Gamma_I(R)) \leq 2$. By assumption, $S(I)$ is not an ideal of R, so there exist $a, b \in S(I)$ such that $a + b \notin S(I)$. If there is an element r of $(I : \langle a, b\rangle)$ with $r \notin I$, then $a + b \in \Gamma_I(R) \subseteq S(I)$ which is a contradiction, so we must have $\langle a, b\rangle$ is prime to I. Therefore, I is a radical ideal gives I has at least two minimal prime ideals by Lemma 2.1(iii). If I has more than two minimal primes, then $\mathrm{diam}(\Gamma_I(R)) = 3$ by Theorem 3.3; hence I must have exactly two minimal prime ideals. Next, assume that $|\mathrm{Min}(I)| = 2$. If P_1 and P_2 are the only minimal prime ideals of I, then $S(I) = P_1 \cup P_2$ by Lemma 2.1(iii) and we may assume $a \in P_1 - P_2$ and $b \in P_2 - P_1$. Clearly, $ab \in P_1 \cap P_2 = I$. Consider two distinct vertices x and y in $\Gamma_I(R)$. If $xy \in I$, then $\mathrm{d}(x, y) = 1$. On the other hand, if $xy \notin I$, then either $\langle x, y\rangle \subseteq P_1$ or $\langle x, y\rangle \subseteq P_2$ since $S(I) = P_1 \cup P_2$. If $\langle x, y\rangle \subseteq P_1$, then $x - b - y$ is a path in $\Gamma_I(R)$; hence $\mathrm{d}(x, y) = 2$. A similar argument shows that if $\langle x, y\rangle \subseteq P_2$, then $\mathrm{d}(x, y) = 2$. It follows that $\mathrm{diam}(\Gamma_I(R)) \leq 2$. □

Lemma 3.5. *Let I be an ideal of a ring R which is not a radical ideal, and let J be an ideal of R which is not prime to I. If $z \in \sqrt{I}$, then the ideal $Rz + J$ is not prime to I.*

Proof. By hypothesis, $S = R/I$ is a non-reduced ring and

$$(0 :_{R/I} (I + J)/I) \neq 0.$$

If $z \in \sqrt{I}$, then $z+I$ is a nilpotent element of S. It follows from [10, Lemma 2.3] that there exists a non-zero element $r + I$ in the annihilator of the ideal $S(z + I) + (I + J)/I$ of S; hence $r \in (I :_R Rz + J) - I$, as required. □

Proposition 3.6. *Assume that I is not a radical ideal of a ring R. If $z \in \sqrt{I}$ and $a \in \Gamma_I(R)$, then $a + z \in \Gamma_I(R)$ and the ideal $\langle a, z\rangle$ is not prime to I.*

Proof. This follows from Lemma 3.5. □

Theorem 3.7. *Let I be an ideal of a ring R which is not a radical ideal and suppose that I is not a primal ideal of R. Then* $\mathrm{diam}(\Gamma_I(R)) = 3$.

Proof. By Proposition 3.2, there are elements $a, b \in \Gamma_I(R)$ such that the ideal $\langle a, b\rangle$ is prime to I, so $\mathrm{d}(a, b) \neq 2$. By Proposition 3.6, neither a nor b can be elements of $\sqrt{I}$. If $ab \notin I$, then $\mathrm{d}(a, b) \neq 1$, so $\mathrm{d}(a, b) = 3$; hence $\mathrm{diam}(\Gamma_I(R)) = 3$. So we can assume that $ab \in I$. Then

$$(I : \langle a^2, b^2\rangle) = (I : \langle a^2, ab, b^2\rangle) = (I : \langle a, b\rangle^2) = (I : \langle a, b\rangle) = I.$$

Therefore, there is an element $z \in \sqrt{I}$ such that $z \notin (I : \langle a^2, b^2\rangle)$. Without loss of generality we may assume that $zb^2 \notin I$. By assumption and Proposition 3.6, we must have $a + bz \in \Gamma_I(R)$. Since $(I : \langle a + bz, b\rangle) = (I : \langle a, b\rangle) = I$, we get $\mathrm{d}(a + bz, b) \neq 2$. But $(a + bz)b = ab + b^2z \notin I$, so $\mathrm{d}(a + bz, b) \neq 1$. Thus $\mathrm{d}(a + bz, b) = 3$ and $\mathrm{diam}(\Gamma_I(R)) = 3$. □

4. Weakly primal ideals

In this section we study the ideal-based zero-divisor graph with respect to weakly prime and weakly primal ideals.

Theorem 4.1. *Let R be a finite local ring with unique maximal ideal M. Then $\Gamma(R)$ is a complete graph if and only if every proper ideal of R is weakly prime.*

Proof. Assume that $\Gamma(R)$ is a complete graph. Then $\Gamma(R) = M - \{0\}$. It then follows from [2, Theorem 2.8] that $xy = 0$ for all $x, y \in \Gamma(R)$, so $M^2 = 0$; hence the result follows from [3, Theorem 8]. Conversely, assume that every proper ideal of R is weakly prime. Then by [3, Theorem 8], we must have $M^2 = 0$. Now the assertion follows from [2, Theorem 2.8]. □

Let I be an ideal of a ring R. Set

$$Z_I(R) = \{r \in R - I : ra = 0 \text{ for some } a \in R - I\}.$$

Lemma 4.2. *Let I be a P-weakly primal ideal of a ring R. Then $\Gamma_I(R) = (P - I) \cup Z_I(R)$.*

Proof. Assume that I is a P-weakly primal ideal of R and let $r \in \Gamma_I(R)$. Then there is an element $a \in R-I$ with $ra \in I$. If $ra \neq 0$, then r is not weakly prime to I, and so $r \in P-I$. If $ra = 0$, then $r \in Z_I(R)$. So $\Gamma_I(R) \subseteq (P-I) \cup Z_I(R)$. For the reverse containment, assume that $s \in (P-I) \cup Z_I(R)$. If $s \in P-I$, then s is not weakly prime to I, so $0 \neq sb \in I$ for some $b \in R - I$; hence $s \in \Gamma_I(R)$. If $s \in Z_I(R)$, then there is an element $c \in R - I$ such that $sc = 0 \in I$; hence $s \in \Gamma_I(R)$, so we have equality. □

Theorem 4.3. *Let I be a an ideal of a ring R and let P be a weakly prime ideal of R with $w(I) \subseteq P$ and $(P - I) \cap Z_I(R) = \emptyset$. Then $\Gamma_I(R) = (P - I) \cup Z_I(R)$ if and only if I is a P-weakly primal ideal of R.*

Proof. By Lemma 4.2, it suffices to show that if $\Gamma_I(R) = (P - I) \cup Z_I(R)$, then I is a P-weakly primal ideal of R. We show that $P - \{0\}$ consists exactly of elements of R that are not weakly prime to I. If $r \in R$ is not weakly prime to I, then $r \in w(I) \subseteq P$. Next, assume that $s \in P - \{0\}$. Since every non-zero element of I is not weakly prime to I, we can assume that $s \notin I$. Therefore, $s \in P - I \subseteq \Gamma_I(R)$ implies that $sb \in I$ for some $b \in R - I$. Since $(P - I) \cap Z_I(R) = \emptyset$, we must have $sb \neq 0$; hence s is not weakly prime to I. Thus I is a P-weakly primal ideal of R. □

Set $R = \mathbb{Z}/24\mathbb{Z}$, $I = 8\mathbb{Z}/24\mathbb{Z}$ and $P = 2\mathbb{Z}/24\mathbb{Z}$. Then I is a P-primal ideal of R. Hence, by Proposition 2.2, $\Gamma_I(R) = P-I = \{\overline{2}, \overline{4}, \overline{6}, \overline{10}, \overline{12}, \overline{14}, \overline{18}, \overline{20}, \overline{22}\}$. It is easy to check that $\Gamma_I(R) = Z_I(R)$. While I is not a weakly prime ideal of R because $\overline{0} \neq \overline{2}.\overline{4} \in I$ with $\overline{2}, \overline{4} \notin I$. This example shows that if $\Gamma_I(R) = Z_I(R)$, I need not necessarily be weakly prime. But the converse holds:

Theorem 4.4. *Let I be a weakly prime ideal of a commutative ring R. Then $\Gamma_I(R) = Z_I(R)$. In particular, $\Gamma_I(R)$ is a subgraph of $\Gamma(R)$.*

Proof. Suppose that I is a weakly prime ideal of R. Then I is I-weakly primal by [7, Theorem 3], so Lemma 4.2 gives $\Gamma_I(R) = Z_I(R)$. □

The proof of the following corollary can be found in [7, Proposition 18], but our proof here will be different.

Corollary 4.5. *Let I be an ideal of an integral domain R. Then I is primal if and only if it is weakly primal.*

Proof. Clearly, $Z_I(R) = \emptyset$. Now the assertion follows from Proposition 2.2 and Theorem 4.3. □

Proposition 4.6. *Assume that S is a multiplicatively closed subset of a ring R and let I be a P-weakly primal ideal of R with $P \cap S = \emptyset$. Then $S^{-1}(\Gamma_I(R)) = \Gamma_{S^{-1}I}(S^{-1}R)$.*

Proof. By [7, Proposition 9], $S^{-1}I$ is a $S^{-1}P$-weakly primal ideal of $S^{-1}R$. Then Lemma 4.2 gives

$$\Gamma_{S^{-1}I}(S^{-1}R) = (S^{-1}P - S^{-1}I) \cup (Z_{S^{-1}I}(S^{-1}R)).$$

It is enough to show that

$$S^{-1}[(P - I) \cup Z_I(R)] = (S^{-1}P - S^{-1}I) \cup (Z_{S^{-1}I}(S^{-1}R)).$$

First, assume that $a/s \in (S^{-1}P - S^{-1}I) \cup (Z_{S^{-1}I}(S^{-1}(R)$. If

$$a/s \in Z_{S^{-1}I}(S^{-1}R),$$

then $(a/s)(b/t) = ab/st = 0 \in S^{-1}I$ for some $b/t \in S^{-1}(R) - S^{-1}I$, so $b \notin I$ and $ab = 0 \in I$; hence $a \in (P - I) \cup Z_I(R)$. Therefore, $a/s \in S^{-1}[(P - I) \cup Z_I(R)]$. If $a/s \in (S^{-1}P - S^{-1}I)$, then $a/s \notin S^{-1}I$ (so $a \notin I$) and $(a/s)(b/t) = ab/st \in S^{-1}I$ for some $b/t \notin S^{-1}I$ (so $b \notin I$). If $ab/st = 0$, then $ab = 0 \in I$. If $ab/st \neq 0$, then $ab \in I$ by [7, Lemma 8]. It follows that $a \in (P - I) \cup Z_I(R)$. Thus $(S^{-1}P - S^{-1}I) \cup (Z_{S^{-1}I}(S^{-1}R) \subseteq S^{-1}[(P - I) \cup Z_I(R)]$. Next, suppose that $a/s \in S^{-1}[(P - I) \cup Z_I(R)]$. Then $ab \in I$ for some $b \notin I$. By [7, Lemma 8], $b/1 \notin S^{-1}I$. Clearly, $(a/s)(b/1) = ab/s \in S^{-1}I$. If $(a/s)(b/1) = 0$, then $a/s \in Z_{S^{-1}I}(S^{-1}R))$. If $(a/s)(b/1) \notin 0$, then a/s is not weakly prime to $S^{-1}I$, so $a/s \in S^{-1}P - S^{-1}I$; hence $S^{-1}[(P - I) \cup Z_I(R)] \subseteq (S^{-1}P - S^{-1}I) \cup (Z_{S^{-1}I}(S^{-1}R))$, so the proof is complete. □

Theorem 4.7. *Assume that S is a multiplicatively closed subset of a ring R which consists of regular elements of R and let I be a P-weakly primal ideal of R with $P \cap S = \emptyset$. Then* $\mathrm{diam}(\Gamma_I(R)) = \mathrm{diam}(\Gamma_{S^{-1}I}(S^{-1}R))$.

Proof. Suppose that $\mathrm{diam}(\Gamma_I(R)) = 1$. For every distinct vertices $a/s, b/t$ of $\Gamma_{S^{-1}I}(S^{-1}R))$, Proposition 4.6 gives a and b are distinct elements of $\Gamma_I(R)$, so $ab \in I$; hence $(a/s)(b/t) \in S^{-1}I$. Thus $\mathrm{diam}(\Gamma_{S^{-1}I}(S^{-1}R)) = 1$. If $\mathrm{diam}(\Gamma_{S^{-1}I}(S^{-1}R)) = 1$, then for every distinct vertices a, b of $\Gamma_I(R) = (P - I) \cup Z_I(R)$, we must have $a/1, b/1 \in S^{-1}((P - I) \cup Z_I(R)) = \Gamma_{S^{-1}I}(S^{-1}R)$

by Proposition 4.6, so $(a/1)(b/1) \in S^{-1}I$. If $ab = 0$, then $ab \in I$. If $ab \neq 0$, then [7, Lemma 8] gives $ab \in I$. Thus $\operatorname{diam}(\Gamma_I(R)) = 1$.

The proof of the cases when diameters are 2 and 3 is similar to that in case $\operatorname{diam}(\Gamma_I(R)) = 1$ and Theorem 2.16, and we omit it. □

Theorem 4.8. *Assume that S is a multiplicatively closed subset of a ring R which consists of regular elements of R and let I be a P-weakly primal ideal of R with $P \cap S = \emptyset$. Then* $\operatorname{gr}(\Gamma_I(R)) = \operatorname{gr}(\Gamma_{S^{-1}I}(S^{-1}R))$.

Proof. By [13, Theorem 5.5], the girth of every ideal-based zero-divisor graph of a commutative ring, when finite, is either 3 or 4. By using Proposition 4.6 and [7, Lemma 8] the proof is similar to that in the Theorem 2.18 and we omit it. □

Theorem 4.9. *Assume that S is a multiplicatively closed subset of a ring R which consists of regular elements of R and let P be a weakly prime ideal of R with $P \cap S = \emptyset$. Then the following hold*:

(i) $\operatorname{diam}(\Gamma_P(R)) = \operatorname{diam}(\Gamma_{S^{-1}P}(S^{-1}R))$.

(ii) $\operatorname{gr}(\Gamma_P(R)) = \operatorname{gr}(\Gamma_{S^{-1}P}(S^{-1}R))$.

Proof. By [7, Theorem 3], every weakly prime ideal of R is weakly primal. Also, $S^{-1}P$ is weakly prime ideal of $S^{-1}R$ by [3, Proposition 13]. Now the assertion follows from Theorem 4.7 and Theorem 4.8. □

Let $Q(R)$ be the total quotient ring of R. It is proved in [1, Theorem 2.2] (see also [12, Theorem 1.1]) that the graphs $\Gamma(R)$ and $\Gamma(Q(R))$ are isomorphic. So these two graphs have the same diameters and same girths. This theorem is basic and several important results follows from it. Now consider the following remark:

Remark 4.10. Suppose that $Q(R) = T^{-1}R$ is the total quotient ring of R. We know that 0 always is a weakly prime ideal of R (by definition). Therefore, by Theorem 4.9, $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma_0(R)) = \operatorname{diam}(\Gamma(Q(R))$. Thus Theorem 4.9 is a generalization of [12, Theorem 1.1].

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