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ORTHOGONAL MULTI-WAVELETS FROM MATRIX FACTORIZATION

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Abstract. Accuracy of the scaling function is very crucial in wavelet theory, or correspondingly, in the study of wavelet filter banks. We are mainly interested in vector-valued filter banks having matrix factorization and indicate how to choose block central symmetric matrices to construct multi-wavelets with suitable accuracy.

1. Introduction

We consider the case of compactly supported multi-wavelets. That is, suppose $\Phi = (\phi_1, \ldots, \phi_r)^T$ are scaling functions, and $\Psi = (\psi_1, \ldots, \psi_r)^T$ are the corresponding wavelet functions, so that the following two-scale equations hold for all $x \in \mathbb{R}$:

$$\Phi(x) = \sum_{n=0}^{M} S_n \, \Phi(2\,x - n), \tag{1.1}$$

$$\Psi(x) = \sum_{n=0}^{M} T_n \, \Phi(2\,x - n). \tag{1.2}$$

Define the corresponding symbol functions as

$$m_0(x) := \frac{1}{2} \sum_{n\in\mathbb{Z}} S_n \, x^n, \qquad m_1(x) := \frac{1}{2} \sum_{n\in\mathbb{Z}} T_n \, x^n.$$

As is well known, the orthonormality of the multi-wavelets implies the following PR condition

$$H(x)\, H^T(1/x) = I_2, \tag{1.3}$$

with

$$H(x) := \begin{pmatrix} m_0(x) & m_0(-x) \\ m_1(x) & m_1(-x) \end{pmatrix}. \tag{1.4}$$

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Up to now, it seems very difficult to find all solutions of the matrix equation (1.3) so people hope to construct some special solutions. A class of PR wavelet filter banks was given in [2] for the general case. In particular, one solution for the multi-wavelets case is as follows:

$$m_k(x) = \frac{1}{2}\left(I_r,\, xI_r\right)\left[\prod_{j=1}^{N} U_j \operatorname{diag}\left(I_r,\, x^2 I_r\right) U_j^T\right] V_k,\ \ k = 0, 1. \tag{1.5}$$

Here N is a fixed positive integer, U_j is any arbitrary $2r \times 2r$ real orthogonal matrix, and

$$V := (V_0,\, V_1) = \begin{pmatrix} I_r & I_r \\ I_r & -I_r \end{pmatrix}. \tag{1.6}$$

The following theorem could be found in [2].

Theorem 1.1. *Suppose filter banks are constructed as in* (1.5)-(1.6)*. Then the PR condition* (1.3) *is satisfied.*

It is well known that the linear phase of filter banks corresponds to symmetry of the related functions. It was also pointed out in [2] that to ensure the uniform linear phase, i.e., to ensure that there exists a natural number s such that $m_k(x) = x^s\, m_k(1/x)$, $k = 0, 1$, we should choose U_j to be r-block central symmetric matrices:

$$U_j = S \begin{pmatrix} P_j & 0 \\ 0 & Q_j \end{pmatrix} S^T,\ \ S = \begin{pmatrix} I_r & -J_r \\ J_r & I_r \end{pmatrix}, \tag{1.7}$$

where P_j, Q_j are r-th real orthogonal matrices and J_r is the r-th reversal matrix.

For later convenience, let $G(x) = \frac{1}{2}\left(I_r,\, xI_r\right)\prod_{j=1}^{N} U_j \operatorname{diag}\left(I_r,\, x^2 I_r\right) U_j^T$.

2. Accuracy conditions of multiple scaling functions

In the following sections, we concentrate on sufficient and necessary conditions so that the scaling functions have accuracy of order p. That is, all polynomials with total degree at most $p-1$ can be reproduced from linear combinations of the multi-integer translates of function Φ.

Gilbert Strang stated in [5] that to ensure accuracy, one must check the value of function m_0 and its derivatives at all aliasing frequencies which seems difficult to compute. By only imposing some conditions on the functions m_0, m_1 at $x = 1$, the author of this paper produced the following theorem, [9].

Theorem 2.1. *If $\Phi = (\phi_1, \phi_2, \ldots, \phi_r)^T$ are scaling functions, and the integer translates of $\phi_1, \ldots, \phi_r$ are linearly independent, moreover, if the corresponding filter bank satisfies the PR condition* (1.3)*, then Φ have accuracy p if and only* *if there are* p *vectors* $\nu_0, \ldots, \nu_{p-1}$, *each* ν_l *being* $r \times 1$ *vector and* $\nu_0 \neq 0$, *such that for all* $j \in \mathbb{Z}_p = \{0, 1, \ldots, p-1\}$:

$$\sum_{l=0}^{j} \binom{j}{l} (2i)^{l-j} \, m_0^{(j-l)}(1) \, \nu_l = 2^j \nu_j, \tag{2.1}$$

$$\sum_{l=0}^{j} \binom{j}{l} (2i)^{l-j} \, m_1^{(j-l)}(1) \, \nu_l = 0. \tag{2.2}$$

By using this theorem, the next proposition is obtained for the filter banks constructed as above.

Proposition 2.2. *If* $\Phi = (\phi_1, \phi_2, \ldots, \phi_r)^T$ *are scaling functions with the integer translates of* $\phi_1, \ldots, \phi_r$ *being linearly independent, and, the corresponding filter banks are constructed as in* (1.5)-(1.6), *then* Φ *have accuracy* p *if and only if there are* p *vectors* $\nu_0, \ldots, \nu_{p-1}$, *each* ν_l *being* $r \times 1$ *vector and* $\nu_0 \neq 0$, *such that for all* $j \in \mathbb{Z}_p$:

$$\sum_{l=0}^{j} \binom{j}{l} (2i)^{l-j} \, G^{(j-l)}(1) \, V_0 \, \nu_l = 2^j \nu_j, \tag{2.3}$$

$$\sum_{l=0}^{j} \binom{j}{l} (2i)^{l-j} \, G^{(j-l)}(1) \, V_1 \, \nu_l = 0. \tag{2.4}$$

By this proposition, we must find p vectors $\nu_0, \ldots, \nu_{p-1}$ to meet the requirements of equations (2.3) and (2.4). The procedure is simplified as follows so that one must only find a nonzero vector ν_0 which is the common eigen-vector corresponding to eigenvalue $\lambda = 0$ of several matrices.

Theorem 2.3. *Under the assumptions of Proposition 2.2, the scaling functions have accuracy* p *if and only if there exist a* $r \times 1$ *vector* $\nu_0 \neq 0$ *such that* ν_0 *are common eigenvector corresponding to eigenvalue* 0 *of matrices* $B_1, \ldots, B_{p-1}$, *that is,*

$$B_n \, \nu_0 = 0, \; n = 1, 2, \ldots, p-1. \tag{2.5}$$

The matrices B_j *are constructed iteratively as*

$$\begin{cases} B_1 = N_1, \\ B_n = N_n + \displaystyle\sum_{j=1}^{n-1} \frac{\binom{n}{j}}{2^j - 1} \, N_{n-j} A_j, \; 2 \leq n \leq p-1 \end{cases} \tag{2.6}$$

and

$$\begin{cases} A_1 = M_1, \\ A_n = M_n + \displaystyle\sum_{j=1}^{n-1} \frac{\binom{n}{j}}{2^j - 1} \, M_{n-j} A_j, \; 2 \leq n \leq p-1 \end{cases} \tag{2.7}$$

with

$$M_j := G^{(j)}(1)\,V_0,\ \ N_j := G^{(j)}(1)\,V_1. \tag{2.8}$$

Furthermore, the solutions of equations (2.3), (2.4) *are given as*

$$\nu_n = \frac{(2i)^{-n}}{2^n-1}\,A_n\,\nu_0,\ \ n=1,2,\ldots,p-1. \tag{2.9}$$

Proof. If the assumptions of Proposition 2.2 are satisfied, one easily checks that

$$G(1)\,V_0 = I_r,\ \ G(1)\,V_1 = 0_r, \tag{2.10}$$

so that for any $r\times 1$ vector $\nu_0 \neq 0$, equations (2.3) and (2.4) for $p=1$. Thus, the corresponding scaling functions have at least accuracy of $p=1$.

From Proposition 2.2, Φ have accuracy $p=2$ if and only if there exists $r\times 1$ vector ν_0,ν_1 with $\nu_0\neq 0$ such that

(2.11)

$$\begin{aligned} &G(1)\,V_0\,\nu_0 = \nu_0, && G(1)\,V_1\,\nu_0 = 0,\\ &(2i)^{-1}G^{(1)}(1)\,V_0\,\nu_0 + G(1)V_0\nu_1 = 2\nu_1, && (2i)^{-1}G^{(1)}(1)\,V_1\,\nu_0 + G(1)V_1\nu_1 = 0. \end{aligned}$$

By using the relations (2.10) and the notations in (2.6)-(2.8), the last equations are equivalent to

$$B_1\,\nu_0 = 0,\ \nu_1 = (2i)^{-1}A_1\,\nu_0,$$

this is just what equations (2.5) and (2.9) states for $p=2$.

Suppose this theorem holds for some $p\geq 2$, next we will prove by induction that it also holds for $p+1$. Proposition 2.2 states that Φ have accuracy $p+1$ if and only if there exists $r\times 1$ vector $\nu_0,\ldots,\nu_{p+1}$ with $\nu_0\neq 0$ such that (2.3) and (2.4) holds for all $j=0,1,\ldots,p$. By induction, this is equivalent to

$$B_n\,\nu_0 = 0,\ \ \nu_n = \frac{(2i)^{-n}}{2^n-1}\,A_n\,\nu_0,\ \ n=1,2,\ldots,p-1; \tag{2.12}$$

$$\sum_{l=0}^{p}\binom{p}{l}(2i)^{l-p}\,G^{(p-l)}(1)\ V_0\,\nu_l = 2^p\nu_p; \tag{2.13}$$

$$\sum_{l=0}^{p}\binom{p}{l}(2i)^{l-p}\,G^{(p-l)}(1)\ V_1\,\nu_l = 0. \tag{2.14}$$

By using (2.10) and (2.12), the left side of equation (2.14) equals to

$$\begin{aligned} &\sum_{l=1}^{p}\binom{p}{l}(2i)^{l-p}\,N_{p-l}\,\nu_l + (2i)^{-p}\,N_p\nu_0\\ &= \sum_{l=1}^{p}\binom{p}{l}(2i)^{l-p}\,N_{p-l}\,\frac{(2i)^{-l}}{2^l-1}\,A_l\,\nu_0 + (2i)^{-p}\,N_p\nu_0\\ &= (2i)^{-p}\left\{\sum_{l=1}^{p}\frac{\binom{p}{l}}{2^l-1}\,N_{p-l}\,A_l + N_p\right\}\nu_0 = (2i)^{-p}\,B_p\,\nu_0. \end{aligned}$$

Similarly, by using (2.10) and (2.12), the left side of equations (2.13) equals to

$$\begin{aligned}&\sum_{l=1}^{p}\binom{p}{l}(2i)^{l-p}\,M_{p-l}\,\nu_l+(2i)^{-p}\,M_p\nu_0+\nu_p\\&=\sum_{l=1}^{p}\binom{p}{l}(2i)^{l-p}\,M_{p-l}\,\frac{(2i)^{-l}}{2^l-1}\,A_l\,\nu_0+(2i)^{-p}\,M_p\nu_0+\nu_p\\&=(2i)^{-p}\left\{\sum_{l=1}^{p}\frac{\binom{p}{l}}{2^l-1}\,M_{p-l}\,A_l+M_p\right\}\nu_0+\nu_p=(2i)^{-p}\,A_p\,\nu_0+\nu_p.\end{aligned}$$

Thus, equations (2.12)-(2.14) are equivalent to

$$B_n\,\nu_0=0,\ \ \nu_n=\frac{(2i)^{-n}}{2^n-1}\,A_n\,\nu_0,\ \ n=1,2,\ldots,p.$$

So we have proved this theorem. □

3. Computation of the derivatives $G^{(j)}(1)$

Theorem 2.3 propose a sufficient and necessary condition for the corresponding scaling functions to have accuracy of degree p. Note that this implies the necessity of computing the derivatives of $G(x)$, that is, the derivatives of the product of several functions. Next, we give the following results concerning the computation of derivatives.

3.1. Derivation of products of functions

The first lemma is classical in mathematical analysis which is called Leibniz's formula.

Lemma 3.1. *Let $f(x)=f_1(x)\,f_2(x)$. Then, for any natural number n, the n-th derivative of function f is*

$$f^{(n)}(x)=\sum_{m=0}^{n}\binom{n}{m}f_1^{(m)}(x)\,f_2^{(n-m)}(x). \tag{3.1}$$

Lemma 3.2. *Let $f(x)=f_1(x)\cdots f_M(x)$. Then, for any natural number n, the n-th derivative of f is*

$$(f_1\cdots f_M)^{(n)}(x)=\sum_{\substack{j_1+\cdots+j_M=n\\ j_i\geq 0}}\frac{n!}{j_1\,!\,j_2\,!\cdots j_M\,!}\,f_1^{(j_1)}(x)\cdots f_M^{(j_M)}(x). \tag{3.2}$$

Proof. We will prove this theorem by induction of M. For $M=1$, this theorem holds naturally. And, Lemma 3.1 states that (3.2) holds for $M=2$.

Suppose this theorem holds for some natural number $M \geq 2$, then, by Lemma 3.1,

$$\begin{aligned}
&(f_1 \cdots f_{M+1})^{(n)}(x) \\
&= \sum_{m=0}^{n} \binom{n}{m} (f_1 \cdots f_M)^{(m)}(x)\, f_{M+1}^{(n-m)}(x) \\
&= \sum_{m=0}^{n} \binom{n}{m} \sum_{\substack{j_1+\cdots+j_M=m\\ j_i\geq 0}} \frac{m!}{j_1\,!\, j_2\,! \cdots j_M\,!} f_1^{(j_1)}(x) \cdots f_M^{(j_M)}(x)\, f_{M+1}^{(n-m)}(x) \\
&= \sum_{\substack{j_1+\cdots+j_{M+1}=n\\ j_i\geq 0}} \frac{(n+1)!}{j_1\,!\, j_2\,! \cdots j_{M+1}\,!} f_1^{(j_1)}(x) \cdots f_{M+1}^{(j_{M+1})}(x).
\end{aligned}$$

Thus, this theorem also holds for $M+1$. □

3.2. Computation of the derivatives $G^{(j)}(1)$

In this section we will concentrate on the filter banks which are constructed in (1.5)-(1.7). Let $f_0 = \frac{1}{2}(I_r,\, xI_r)$, and for $j = 1, \ldots, N$,

$$f_j(x) = U_j \operatorname{diag}\left(I_r,\, x^2 I_r\right) U_j^T.$$

Then their derivatives are

(3.3)

$$f_0^{(k)}(1) = \begin{cases} \frac{1}{2}(I_r, I_r), & k=0,\\ \frac{1}{2}(0_r, I_r), & k=1,\\ 0_{r\times 2r}, & k\geq 2, \end{cases} \quad f_j^{(k)}(1) = \begin{cases} I_{2r}, & k=0,\\ 2\,U_j \operatorname{diag}(0_r, I_r)\, U_j^T, & k=1,2,\\ 0_{2r}, & k\geq 3. \end{cases}$$

Or equivalently,

$$f_0^{(k)}(1) = \frac{1}{2}\left(\delta_k\, I_r,\, (\delta_k + \delta_{k-1}) I_r\right), \tag{3.4}$$

$$f_j^{(k)}(1) = U_j \operatorname{diag}\left(\delta_k\, I_r,\, (\delta_k + 2\delta_{k-1} + 2\delta_{k-2}) I_r\right) U_j^T. \tag{3.5}$$

Thus, by Lemma 3.2, the derivative of G at $x=1$ is given as in the next theorem.

Theorem 3.3. *The derivatives of $G(x)$ is given as*

$$\begin{aligned}
&G^{(k)}(1) \\
&= \frac{1}{2}(I_r,\, I_r) \sum_{\substack{k_1+\cdots+k_N=k,\\ 0\leq k_i\leq 2}} \frac{k!}{k_1\,! \cdots k_M\,!} \prod_{j=1}^{N} U_j \operatorname{diag}\left(\delta_{k_j} 0_r,\, (\delta_{k_j} + 2\delta_{k_j-1} + 2\delta_{k_j-2}) I_r\right) U_j^T \\
&\quad + \frac{1}{2}(0_r,\, I_r) \sum_{\substack{k_1+\cdots+k_N=k-1,\\ 0\leq k_i\leq 2}} \frac{(k-1)!}{k_1\,! \cdots k_M\,!} \prod_{j=1}^{N} U_j \operatorname{diag}\left(\delta_{k_j} 0_r,\, (\delta_{k_j} + 2\delta_{k_j-1} + 2\delta_{k_j-2}) I_r\right) U_j^T.
\end{aligned}$$

Proof. By using Lemma 3.2 and relation (3.3), we have

$$\begin{aligned} G^{(k)}(1) &= \frac{1}{2}\,(I_r,\, I_r) \sum_{\substack{k_1+\cdots+k_N=k \\ 0\le k_i\le 2}} \frac{k!}{k_1\,!\cdots k_M\,!} f_1^{(k_1)}(1)\cdots f_N^{(k_N)}(1) \\ &\quad + \frac{1}{2}\,(0_r,\, I_r) \sum_{\substack{k_1+\cdots+k_N=k-1 \\ 0\le k_i\le 2}} \frac{(k-1)!}{k_1\,!\cdots k_M\,!} f_1^{(k_1)}(1)\cdots f_N^{(k_N)}(1). \end{aligned} \tag{3.6}$$

This combined with relations (3.4)-(3.5) concludes the proof of this theorem. □

Although this theorem gives a close form of the derivative $G^{(j)}(1)$, it is not very convenient for computational purpose. It is implied from relations (3.3) and (3.6) that to compute $G^{(k)}(1)$, we should consider matrix multiplication of the following form $\prod_{q=1}^{M} U_{j_q}\,\mathrm{diag}\,(0_r,\, I_r)\, {U_{j_q}}^T$ where each U_{j_q} is characterized as in (1.7).

Proposition 3.4. *Let U_j be characterized as in* (1.7)*. Then for any $2 \le M \le N$, we have*

$$\prod_{j=1}^{M} U_j \begin{pmatrix} 0_r & \\ & I_r \end{pmatrix} U_j^T = U_1 \begin{pmatrix} 0_r & 0_r \\ 0_r & \displaystyle\prod_{j=1}^{M-1} (Q_j^T\, Q_{j+1} + J_r\, P_j^T\, P_{j+1}\, J_r) \end{pmatrix} U_M^T. \tag{3.7}$$

Proof. We will prove this proposition by induction of M.

(1) For $M = 2$, by using the fact that

$$U_j \begin{pmatrix} 0_r & \\ & I_r \end{pmatrix} U_j^T = S \begin{pmatrix} P_j & \\ & Q_j \end{pmatrix} S^T \begin{pmatrix} 0_r & \\ & I_r \end{pmatrix} S \begin{pmatrix} P_j^T & \\ & Q_j^T \end{pmatrix} S^T, \tag{3.8}$$

we have

$$\begin{aligned} & U_1 \begin{pmatrix} 0_r & \\ & I_r \end{pmatrix} U_1^T\, U_2 \begin{pmatrix} 0_r & \\ & I_r \end{pmatrix} U_2^T \\ = \;& S \begin{pmatrix} P_1 & \\ & Q_1 \end{pmatrix} S^T \begin{pmatrix} 0_r & \\ & I_r \end{pmatrix} S \begin{pmatrix} P_1^T\, P_2 & \\ & Q_1^T\, Q_2 \end{pmatrix} S^T \begin{pmatrix} 0_r & \\ & I_r \end{pmatrix} S \begin{pmatrix} P_2^T & \\ & Q_2^T \end{pmatrix} S^T. \end{aligned}$$

Note that for any $r \times r$ matrices A, B, C, D, the following identities hold:

$$\begin{pmatrix} 0_r & \\ & I_r \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0_r & \\ & I_r \end{pmatrix} = \begin{pmatrix} 0_r & 0_r \\ 0_r & D \end{pmatrix}, \tag{3.9}$$

$$S \begin{pmatrix} P_1^T\, P_2 & \\ & Q_1^T\, Q_2 \end{pmatrix} S^T = \begin{pmatrix} * & * \\ * & Q_1^T\, Q_2 + J_r\, P_1^T P_2 J_r \end{pmatrix}. \tag{3.10}$$

Thus, we have

$$U_1 \begin{pmatrix} 0_r & \\ & I_r \end{pmatrix} U_1^T\, U_2 \begin{pmatrix} 0_r & \\ & I_r \end{pmatrix} U_2^T = U_1 \begin{pmatrix} 0_r & 0_r \\ 0_r & Q_1^T\, Q_2 + J_r\, P_1^T\, P_2\, J_r \end{pmatrix} U_2^T.$$

That is, we have proved the relation (3.7) for $M = 2$.

(2) Suppose relation (3.7) holds for some $M \geq 2$, then, by induction,

$$\prod_{j=1}^{M+1} U_j \begin{pmatrix} 0_r & \\ & I_r \end{pmatrix} U_j^T \tag{3.11}$$
$$= U_1 \begin{pmatrix} 0_r & 0_r \\ 0_r & \prod_{j=1}^{M-1} (Q_j^T Q_{j+1} + J_r P_j^T P_{j+1} J_r) \end{pmatrix} U_M^T U_{M+1} \begin{pmatrix} 0_r & \\ & I_r \end{pmatrix} U_{M+1}^T.$$

Note that for any $r \times r$ matrices A, B, C, D, F the following identities hold:

$$\begin{pmatrix} 0_r & 0_r \\ 0_r & F \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0_r & \\ & I_r \end{pmatrix} = \begin{pmatrix} 0_r & 0_r \\ 0_r & F D \end{pmatrix},$$

$$U_M^T U_{M+1} = S \begin{pmatrix} P_M^T P_{M+1} & \\ & Q_M^T Q_{M+1} \end{pmatrix} S^T$$
$$= \begin{pmatrix} * & * \\ * & Q_M^T Q_{M+1} + J_r P_M^T P_{M+1} J_r \end{pmatrix}.$$

Consequently,

$$\prod_{j=1}^{M+1} U_j \begin{pmatrix} 0_r & \\ & I_r \end{pmatrix} U_j^T = U_1 \begin{pmatrix} 0_r & 0_r \\ 0_r & \displaystyle\prod_{j=1}^{M} (Q_j^T Q_{j+1} + J_r P_j^T P_{j+1} J_r) \end{pmatrix} U_{M+1}^T.$$

That is, we have proved that this proposition holds for $M + 1$. □

Another question concerning the formula (3.6) is as follows: how to characterize the set $\{(j_1, \ldots, j_N) : j_1 + \cdots + j_N = k,\ j_i \in \{0, 1, 2\}\}$. In fact we only have to consider the following simpler sets

$$S_{N,k} := \{(j_1, \ldots, j_N) : j_1 \geq j_2 \geq \cdots \geq j_N, j_1 + \cdots + j_N = k,\ j_i \in \{0, 1, 2\}\}. \tag{3.12}$$

Proposition 3.5. *Given a natural number N, suppose $S_{N,k}$ are defined in* (3.12) *for all nonnegative integers k. Then, we have*

$$S_{N,k} = \begin{cases} \{(0, 0, \ldots, 0)\}, & k = 0, \\ \{(1, 0, \ldots, 0)\}, & k = 1, \\ \{(\overbrace{1, \ldots, 1}^{k}, 0, \ldots, 0)\} \cup \tilde{S}_{N,k-2}, & 2 \leq k \leq N, \\ \{(2 - j_N, \ldots, 2 - j_1) : (j_1, \ldots, j_N) \in S_{N,2N-k}\}, & N + 1 \leq k \leq 2N, \\ \varnothing, & k \geq 2N + 1, \end{cases} \tag{3.13}$$

with $\tilde{S}_{N,k} := \{(2, j_1, \ldots, j_{N-1}) : (j_1, \ldots, j_N) \in S_{N,k}\}$. Moreover, the cardinalities are

$$\#(S_{N,k}) = \begin{cases} [\frac{k}{2}] + 1 & 0 \leq k \leq N, \\ N - [\frac{k-1}{2}] & N + 1 \leq k \leq 2N, \\ 0 & k \geq 2N + 1. \end{cases} \tag{3.14}$$

Proof. To verify the equalities (3.13), we only have to prove that the third equality holds for all $2 \le k \le N$, that is, $S_{N,k} = \{(\overbrace{1,\dots,1}^{k},0,\dots,0)\} \cup \tilde{S}_{N,k-2}$. On the one hand, for any $(j_1,\dots,j_N) \in S_{N,k}$, we have either $j_1 = 2$ or $j_1 = 1$. If $j_1 = 2$, then it is easy to verify that $(j_2,\dots,j_N,0) \in S_{N,k-2}$, thus, $(j_1,\dots,j_N) \in \tilde{S}_{N,k-2}$; in the case that $j_1 = 1$, we have $(j_1,\dots,j_N) = (\overbrace{1,\dots,1}^{k},0,\dots,0)$. So we have prove that

$$S_{N,k} \subseteq \left\{(\overbrace{1,\dots,1}^{k},0,\dots,0)\right\} \bigcup \tilde{S}_{N,k-2}.$$

On the other hand, firstly we know that $(\overbrace{1,\dots,1}^{k},0,\dots,0) \in S_{N,k}$. For any $(2,j_1,\dots,j_{N-1}) \in \tilde{S}_{N,k-2}$ where $(j_1,\dots,j_N) \in S_{N,k-2}$, one can check easily that $j_N = 0$. Otherwise, if $j_N \ge 1$, then it is implied that $j_1 + \cdots + j_N \ge N > k-2$. Thus, we have $(2,j_1,\dots,j_{N-1}) \in S_{N,k}$. Consequently, we have proved that $S_{N,k} \supseteq \{(\overbrace{1,\dots,1}^{k},0,\dots,0)\} \cup \tilde{S}_{N,k-2}$. Combining the two result, the proof of (3.13) is finished. The equality (3.14) can be easily checked by using the results of (3.13). □

Proposition 3.5 produces a recursive method to construct the sets $S_{N,k}$. In what follows we will show exactly what $S_{N,k}$ are.

Proposition 3.6. *Let the sets $S_{N,k}$ be defined as in* (3.12)*. Then for any $0 \le k \le N$, we have*

$$S_{N,k} = \bigcup_{j=0}^{[\frac{k}{2}]} \left\{(\overbrace{2,\dots,2}^{j},\overbrace{1,\dots,1}^{k-2j},\overbrace{0,\dots,0}^{N-k+j})\right\}. \tag{3.15}$$

And, for those $N+1 \le k \le 2N$, we have

$$S_{N,k} = \bigcup_{j=0}^{[N-\frac{k}{2}]} \left\{(\overbrace{2,\dots,2}^{k+j-N},\overbrace{1,\dots,1}^{2N-k-2j},\overbrace{0,\dots,0}^{j})\right\}. \tag{3.16}$$

Proof. One can easily checks that equality (3.16) is implied by (3.13) and (3.15). We will prove by induction of k that equality (3.15) holds for all $0 \le k \le N$. Firstly, it is easy to verify that this holds for $k = 0, 1$. Furthermore, the fact that $S_{N,2} = \{(2,0,0,\dots,0),(1,1,0,\dots,0)\}$ implies that this equality also holds for $k = 2$.

Suppose there are some $2 \le k \le N-1$, such that equality (3.15) holds for all $0 \le n \le k$. By (3.13), we have

$$S_{N,k+1} = \{(\overbrace{1,\dots,1}^{k+1},0,\dots,0)\} \cup \tilde{S}_{N,k-1}, \tag{3.17}$$
$$\text{with} \quad \tilde{S}_{N,k-1} = \{(2,j_1,\dots,j_{N-1}) : (j_1,\dots,j_N) \in S_{N,k-1}\}.$$

As was pointed in the proof of the last proposition, for any $(j_1,\dots,j_N)\in S_{N,k-1}$, we have $j_N=0$. Thus, by induction,

$$\tilde{S}_{N,k-1}=\bigcup_{j=0}^{[\frac{k-1}{2}]}\left\{(\overbrace{2,\dots,2}^{j+1},\overbrace{1,\dots,1}^{k-1-2j},\overbrace{0,\dots,0}^{N-k+j})\right\},$$

so that

$$S_{N,k+1} = \left\{(\overbrace{1,\dots,1}^{k+1},0,\dots,0)\right\}\cup\bigcup_{j=0}^{[\frac{k-1}{2}]}\left\{(\overbrace{2,\dots,2}^{j+1},\overbrace{1,\dots,1}^{k-1-2j},\overbrace{0,\dots,0}^{N-k+j})\right\},$$

$$= \bigcup_{j=0}^{[\frac{k+1}{2}]}\left\{(\overbrace{2,\dots,2}^{j},\overbrace{1,\dots,1}^{k+1-2j},\overbrace{0,\dots,0}^{N-k-1+j})\right\}. \tag{3.18}$$

Note that to prove the equality (3.18), we have used the equality $[\frac{k+1}{2}]-1=[\frac{k-1}{2}]$. □

Combining the results in relation (3.6), and Propositions 3.4, 3.6, we propose the following theorem which seems more convenient to compute $G^{(j)}(1)$.

Theorem 3.7. *For $0\le k\le N$, we have*

$$\sum_{\substack{k_1+\cdots+k_N=k\\0\le k_i\le 2}}\frac{k!}{k_1!\cdots k_M!}f_1^{(k_1)}(1)\cdots f_N^{(k_N)}(1)$$

$$=\sum_{j=0}^{[\frac{k}{2}]}\binom{k-j}{j}\sum_{(1\le p_1<\cdots<p_{k-j}\le N)}2^{k-2j}\,k!\,U_{p_1}\begin{pmatrix}0_r & 0_r\\ 0_r & \prod_{l=1}^{k-j-1}(Q_{p_l}^T Q_{p_{l+1}}+J P_{p_l}^T P_{p_{l+1}} J)\end{pmatrix}U_{p_{k-j}}^T.$$

And, for $N+1\le k\le 2N$, we have

$$\sum_{\substack{k_1+\cdots+k_N=k\\0\le k_i\le 2}}\frac{k!}{k_1!\cdots k_M!}f_1^{(k_1)}(1)\cdots f_N^{(k_N)}(1)$$

$$=\sum_{j=0}^{[N-\frac{k}{2}]}\binom{N-j}{k+j-N}\sum_{(1\le p_1<\cdots<p_{N-j}\le N)}\frac{k!}{2^{k+j-N}}U_{p_1}\begin{pmatrix}0_r & 0_r\\ 0_r & \prod_{l=1}^{N-j-1}(Q_{p_l}^T Q_{p_l+1}+J P_{p_l}^T P_{p_l+1} J)\end{pmatrix}U_{p_{N-j}}^T.$$

Proof. For any n-tuples $(a_1,\dots,a_n)$, denotes by $P(a_1,\dots,a_n)$ the set of permutations of $a_1,\dots,a_n$. First consider the case $0\le k\le N$, we have

$$\sum_{\substack{k_1+\cdots+k_N=k\\0\le k_i\le 2}}\frac{k!}{k_1!\cdots k_M!}f_1^{(k_1)}(1)\cdots f_N^{(k_N)}(1)$$

$$=\sum_{(j_1,\dots,j_N)\in S_{N,k}}\ \sum_{(k_1,\dots,k_N)\in P(j_1,\dots,j_N)}\frac{k!}{k_1!\cdots k_M!}f_1^{(k_1)}(1)\cdots f_N^{(k_N)}(1)$$

$$= \sum_{(j_1,\ldots,j_N)\in S_{N,k}} \sum_{(k_1,\ldots,k_N)\in P(j_1,\ldots,j_N)} \frac{k!}{j_1!\cdots j_M!} f_1^{(k_1)}(1)\cdots f_N^{(k_N)}(1)$$

$$= \sum_{j=0}^{[\frac{k}{2}]} [\sum_{(k_1,\ldots,k_N)\in P(e_{j,k})} \frac{k!}{2^j} f_1^{(k_1)}(1)\cdots f_N^{(k_N)}(1)]$$

where the N-tuples $e_{j,k}$ are defined as $e_{j,k} := (\overbrace{2,\ldots,2}^{j},\overbrace{1,\ldots,1}^{k-2j},\overbrace{0,\ldots,0})$, note that the last equality holds due to Proposition 3.6. Next, we will show clearly what the summation in the bracket is. For any given $j = 0, 1, \ldots, [\frac{k}{2}]$, let

$$C_1 := \sum_{(k_1,\ldots,k_N)\in P(e_{j,k})} \frac{k!}{2^j} f_1^{(k_1)}(1)\cdots f_N^{(k_N)}(1),$$

$$C_2 := \sum_{(q_1,\ldots,q_{k-j})\in p(\tilde{e}_{j,k})} \sum_{1\le p_1<\cdots<p_{k-j}\le N} \frac{k!}{2^j} f_{p_1}^{(q_1)}(1)\cdots f_{p_{k-j}}^{(q_{k-j})}(1),$$

where $\tilde{e}_{j,k} := (\overbrace{2,\ldots,2}^{j},\overbrace{1,\ldots,1}^{k-2j})$, we claim that $C_1 = C_2$. On the one hand, it is straightforward to prove that the terms of summations are equal since $\binom{N}{j}\binom{N-j}{k-2j} = \binom{k-j}{j}\binom{N}{k-j}$. On the other hand, we will verify that any summation term of C_1 emerges also in C_2. For fixed permutation of $e_{j,k}$, it is implied from relation (3.3) that only $k-j$ terms in the product $\frac{k!}{2^j} f_1^{(k_1)}(1)\cdots f_N^{(k_N)}(1)$ counts. Thus, there exists $1 \le p_1 < \cdots < p_{k-j} \le N$, and $q_1, \ldots, q_{k-j}$ being permutation of $\tilde{e}_{j,k}$ so that $\frac{k!}{2^j} f_1^{(k_1)}(1)\cdots f_N^{(k_N)}(1) = \frac{k!}{2^j} f_{p_1}^{(q_1)}(1)\cdots f_{p_{k-j}}^{(q_{k-j})}(1)$. This concludes the proof of identity $C_1 = C_2$. So we have

$$\sum_{\substack{k_1+\cdots+k_N=k \\ 0\le k_i\le 2}} \frac{k!}{k_1!\cdots k_M!} f_1^{(k_1)}(1)\cdots f_N^{(k_N)}(1)$$

$$= \sum_{j=0}^{[\frac{k}{2}]} [\sum_{(q_1,\ldots,q_{k-j})\in P(\tilde{e}_{j,k})} \sum_{1\le p_1<\cdots<p_{k-j}\le N} \frac{k!}{2^j} f_{p_1}^{(q_1)}(1)\cdots f_{p_{k-j}}^{(q_{k-j})}(1)]$$

$$= \sum_{j=0}^{[\frac{k}{2}]} \binom{k-j}{j} \sum_{(1\le p_1<p_2<\cdots<p_{k-j}\le N)} \frac{k!}{2^j} f_{p_1}^{(q_1)}(1)\cdots f_{p_{k-j}}^{(q_{k-j})}(1)$$

$$= \sum_{j=0}^{[\frac{k}{2}]} \binom{k-j}{j} \sum_{(1\le p_1<p_2<\cdots<p_{k-j}\le N)} 2^{k-2j}\, k!\, U_{p_1} \begin{pmatrix} 0_r & 0_r \\ 0_r & \prod_{l=1}^{k-j-1} (Q_{p_l}^T Q_{1+p_l} + J P_{p_l}^T P_{1+p_l} J) \end{pmatrix} U_{p_{k-j}}^T.$$

It should be noted that we have used Proposition 3.4 and the following facts:

(1) for any $j = 1, \ldots, N$, $f_j^{(1)}(1) = f_j^{(2)}(1) = 2U_j \begin{pmatrix} 0_r & \\ & I_r \end{pmatrix} U_j^T$;

(2) the number of permutations of $(\overbrace{2, \ldots, 2}^{m}, \overbrace{1, \ldots, 1}^{n})$ is $\binom{m+n}{m}$.

By the same trick we can prove the theorem for the case $N+1 \leq k \leq 2N$. $\square$

4. Numerical examples

Consider the case $r = 2$. Assume that filter banks are constructed as in (1.5)-(1.7) where the 2×2 real orthogonal matrices P_j, Q_j are

$$P_j = \begin{pmatrix} \cos \alpha_j & \sin \alpha_j \\ -\sin \alpha_j & \cos \alpha_j \end{pmatrix}, \; Q_j = \begin{pmatrix} \cos \beta_j & \sin \beta_j \\ -\sin \beta_j & \cos \beta_j \end{pmatrix}. \tag{4.19}$$

Let $\gamma_j = \alpha_j + \beta_j$, then the matrices B_1, B_2 defined in Theorem 2.3 are

$$B_1 = -\lambda_3 I_2, \, B_2 = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{pmatrix},$$

with parameters

$$\begin{aligned}
\lambda_1 &= \frac{1}{2} + \sum_{j=1}^{N} \cos \gamma_j, \\
\lambda_2 &= -N + \sum_{j=1}^{N} \sin \gamma_j - (N+1) \sum_{j=1}^{N} \cos \gamma_j + \sum_{1 \leq k < j \leq N} \sin(\gamma_j - \gamma_k), \\
\lambda_3 &= -N - \sum_{j=1}^{N} \sin \gamma_j - (N+1) \sum_{j=1}^{N} \cos \gamma_j - \sum_{1 \leq k < j \leq N} \sin(\gamma_j - \gamma_k).
\end{aligned}$$

Thus, by Theorem 2.3, we have the following sufficient and necessary conditions for the corresponding scaling functions to have accuracy $p = 2, 3$.

Theorem 4.1. *When $r = 2$, and wavelet filter banks are constructed as in* (1.5)-(1.7) *and* (4.19), *then the corresponding scaling functions have at least accuracy of order $p = 2$ if and only if*

$$\sum_{j=1}^{N} \cos \gamma_j = -\frac{1}{2}. \tag{4.20}$$

Moreover, the corresponding scaling functions have at least accuracy of order $p = 3$ if and only if in addition to (4.20), *the following equality holds*:

$$\sum_{j=1}^{N} \sin \gamma_j + \sum_{1 \leq k < j \leq N} \sin(\gamma_j - \gamma_k) = \pm \frac{1}{2}. \tag{4.21}$$

Proof. By Theorem 2.3, the scaling functions have at least second accuracy if and only if there exists nonzero 2×1 vector ν_0 such that $B_1 \nu_0 = -\lambda_1 \nu_0 = 0$, this reduces to $\lambda_1 = 0$.

Similarly, the scaling functions have at least third accuracy if and only if there exists nonzero 2×1 vector ν_0 such that $B_1 \nu_0 = -\lambda_1 \nu_0 = 0, B_2 \nu_0 = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{pmatrix} \nu_0 = 0$, this reduces to either of the following two equalities:

$$\begin{cases} \lambda_1 = 0, \\ \lambda_2 = 0, \end{cases} \qquad \begin{cases} \lambda_1 = 0, \\ \lambda_3 = 0. \end{cases}$$

□

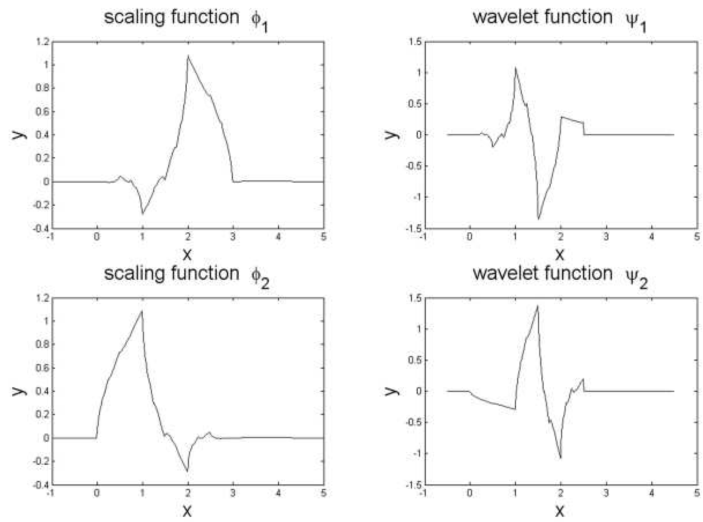

FIGURE 1. Wavelet and scaling function with second accuracy

By equation (4.20), to obtain second accuracy and the minimal length of the filters, we should choose $N = 1$, that is, $\cos \gamma_1 = -\frac{1}{2}$. In fact, in this case, ψ_2, ϕ_2 are the Daubechies's wavelet function db2 and the corresponding scaling function. The graph of the above functions are plotted in Figure 1 .

On the other hand, to obtain third accuracy and minimal length, we have to choose $N = 2$. Here the equations (4.20) and (4.21) have four solutions:

$$\begin{aligned} \gamma_1 &= \pi + \arcsin 0.5374, & \gamma_2 &= -\arcsin 0.9392; \\ \gamma_1 &= \pi + \arcsin 0.9756, & \gamma_2 &= \pi - \arcsin 0.9600; \\ \gamma_1 &= \pi - \arcsin 0.5374, & \gamma_2 &= -\arcsin 0.9392; \\ \gamma_1 &= \pi - \arcsin 0.9756, & \gamma_2 &= \pi + \arcsin 0.9600. \end{aligned}$$

Take the first solution, and we present the graph of the above functions in Figure 2.

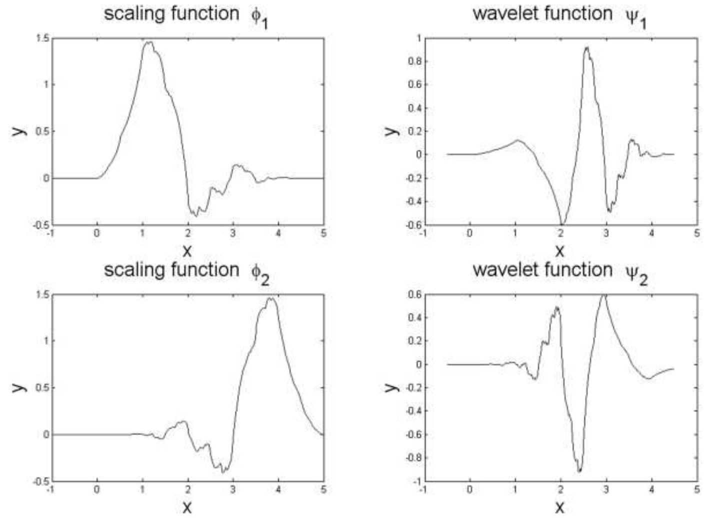

FIGURE 2. Wavelet and scaling function with third accuracy

References

[1] C. Cabrelli, C. Heil, and U. Molter, *Accuracy of lattice translates of several multidimensional refinable functions*, J. Approx. Theory **95** (1998), no. 1, 5–52.

[2] Q. H. Chen, C. A. Micchelli, S. Peng, and Y. Xu, *Multivariate filter banks having matrix factorizations*, SIAM J. Matrix Anal. Appl. **25** (2003), no. 2, 517–531.

[3] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, 61. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[4] C. deBoor, R. A. Devore, and A. Ron, *Approximation orders of FSI spaces in* $L_2(\mathbf{R}^d)$, Constr. Approx. **14** (1998), no. 4, 631–652.

[5] C. Heil, G. Strang, and V. Strela, *Approximation by translates of refinable functions*, Numer. Math. **73** (1996), no. 1, 75–94.

[6] R. Q. Jia, *Approximation properties of multivariate wavelets*, Math. Comp. **67** (1998), no. 222, 647–665.

[7] Q. Lian, H. Xiao, and Q. Chen, *Some properties on multivariate filter banks with a matrix factorization*, Progr. Natur. Sci. (English Ed.) **15** (2005), no. 2, 115–125.

[8] G. Plonka, *Approximation order provided by refinable function vectors*, Constr. Approx. **13** (1997), no. 2, 221–244.

[9] H. Xiao, *Lattice structure for paraunitary linear-phase filter banks with accuracy*, Acta Math. Sin. (Engl. Ser.) **22** (2006), no. 3, 679–688.

[10] ———, *Piecewise Linear Spectral Sequences and Wavelet Filter Banks,* Doctoral Thesis, Graducate School of CAS, May 2005.

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