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# EIGENVALUES OF COUNTABLY CONDENSING MAPS

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Abstract. Using an index theory for countably condensing maps, we show the existence of eigenvalues for countably $k$-set contractive maps and countably condensing maps in an infinite dimensional Banach space $X$, under certain condition that depends on the quantitative characteristic, that is, the infimum of all $k \geq 1$ for which there is a countably $k$-set-contractive retraction of the closed unit ball of $X$ onto its boundary.

## 1. Introduction

A starting point of our investigation is a result of Guo [5] which states as follows:

*Let $\Omega$ be a bounded open subset of an infinite dimensional Banach space $X$. If $F : \overline{\Omega} \to X$ is a compact map such that $\inf_{x\in\partial\Omega} \|Fx\| > 0$ and $Fx \neq \lambda x$ for all $x \in \partial\Omega$ and all $\lambda \in (0, 1]$, then* $\operatorname{ind}(F, \Omega) = 0$.

It has been attempted to extend the result to strict-set contractions; see [2, 9, 12]. In a recent work [4], some generalizations to these contractions and condensing maps are obtained, under suitable condition that depends on the quantitative characteristic $R_\gamma$, where $\gamma$ is a measure of noncompactness on $X$. This means the infimum of all $k \geq 1$ for which there is a $k$-$\gamma$-contractive retraction of the closed unit ball of $X$ onto its boundary. See [3, 4, 6, 11].

It is known in [5] that the above result is closely related to the problem of finding solutions for nonlinear equations. It is natural to consider this problem for a large class of countably condensing maps, roughly speaking, condensing on countable subsets of the space. The use of such countable sets to solve nonlinear equations can be found in [7].

Motivated by the work [4], our goal in the present paper is to study nonlinear eigenvalue problem for countably $k$-set contractive maps and countably condensing maps. To this end, we introduce an index theory for countably condensing maps due to Väth [10] and the corresponding characteristic $R^c_\gamma$ given in [6]. Using a result of [6], we give an extension of Guo's result to countably $k$-set contractive maps and obtain an eigenvalue result for these maps which includes the well known Birkhoff-Kellogg theorem as a special case. Moreover,

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we present the existence of eigenvalues for countably condensing maps. As its application, we discuss nonlinear eigenvalue problem in a more concrete situation, as in [4, 5].

## 2. An index theory

In this section, we introduce a fixed point index theory for countably condensing maps due to Väth [10].

Given a nonempty subset $\Omega$ of a Banach space $X$, the closure, the boundary, the convex hull, and the closed convex hull of $\Omega$ in $X$ are denoted by $\overline{\Omega}$, $\partial\Omega$, $\operatorname{co}\Omega$, and $\overline{\operatorname{co}}\,\Omega$, respectively.

A function $\gamma : \{M \subset X : M \text{ is bounded}\} \to [0, \infty)$ is said to be a *measure of noncompactness* on $X$ if it satisfies the following properties:

(1) $\gamma(\overline{\operatorname{co}}\, M) = \gamma(M)$;
(2) $\gamma(M) = 0$ if and only if $M$ is relatively compact;
(3) $\gamma(M \cup N) = \max\{\gamma(M), \gamma(N)\}$;
(4) $\gamma(M + N) \le \gamma(M) + \gamma(N)$; and
(5) $\gamma(\alpha M) = |\alpha|\gamma(M)$ for all $\alpha \in \mathbb{R}$.

Note that the Kuratowski or the Hausdorff measure of noncompactness has the above properties; see [1].

Let $\Omega$ be a nonempty subset of $X$ and $\gamma$ a measure of noncompactness on $X$, and $k \ge 0$. A continuous map $F : \Omega \to X$ is said to be *countably $k$-$\gamma$-contractive* if $\gamma(F(C)) \le k\gamma(C)$ for each countable bounded set $C \subset \Omega$. A continuous map $F : \Omega \to X$ is said to be *countably $\gamma$-condensing* if $\gamma(F(C)) < \gamma(C)$ for each countable bounded set $C \subset \Omega$ with $\gamma(C) > 0$. Moreover, if $\gamma(F(B)) \le k\gamma(B)$ for all bounded sets $B \subset \Omega$, then $F$ is called *$k$-$\gamma$-contractive.* If $\gamma(F(B)) < \gamma(B)$ for all bounded sets $B \subset \Omega$ with $\gamma(B) > 0$, then $F$ is called *$\gamma$-condensing.*

A continuous homotopy $H : [0, 1] \times \Omega \to X$ is said to be *countably $\gamma$-condensing* if $\gamma(H([0, 1] \times C)) < \gamma(C)$ for each countable bounded set $C \subset \Omega$ with $\gamma(C) > 0$.

To define an index for countably condensing maps, we need the following result which is a particular form of Corollary 2.1 in [10].

**Proposition 2.1.** *Let $\Omega$ be an open bounded nonempty subset of a Banach space $X$. If $F : \overline{\Omega} \to X$ is a countably $\gamma$-condensing map, then there exists a compact convex nonempty subset $S$ of $X$ such that*

(1) *$F(\overline{\Omega} \cap S)$ is a subset of $S$; and*
(2) *the relation $x \in \overline{\operatorname{co}}\,[\{Fx\} \cup S]$ implies $x \in S$.*

*Such a closed convex nonempty set $S$ is called a fundamental set for $F$.*

Let $\Omega$ be an open bounded nonempty set in a Banach space $X$. Suppose that $F : \overline{\Omega} \to X$ is a countably $\gamma$-condensing map such that $Fx \neq x$ for all

$x \in \partial\Omega$. In view of Proposition 2.1, there exists a compact fundamental set $S$ for $F$ such that

$$F(\overline{\Omega} \cap S) \subset S.$$

Let $R : X \to S$ be any retraction onto $S$. Since $I - F \circ R \neq 0$ on the boundary of $R^{-1}(\Omega)$, the Leray-Schauder degree for the compact map $F \circ R$, $d_{LS}(I - F \circ R, R^{-1}(\Omega), 0)$, is defined. Now we define a fixed point index for the countably $\gamma$-condensing map $F$ as

$$\operatorname{ind}(F, \Omega) := d_{LS}(I - F \circ R,\ R^{-1}(\Omega), 0).$$

The fixed point index for $\gamma$-condensing maps, where $\gamma$ is the Kuratowski measure of noncompactness, has been developed by Nussbaum [8]. In case when $F$ is a compact map, the above indices agree with the Leray-Schauder degree.

We give some properties of the above index which will be used later; see [10, Theorem 1.3].

**Proposition 2.2.** *Let $\Omega$ be an open bounded nonempty set in a Banach space $X$. Let $F : \overline{\Omega} \to X$ be a countably $\gamma$-condensing map such that $Fx \neq x$ for all $x \in \partial\Omega$. Then it has the following properties*:

(1) *If* $\operatorname{ind}(F, \Omega) \neq 0$, *then $F$ has a fixed point in $\Omega$.*
(2) (*Normalization*) *If $F \equiv 0$ and $0 \in \Omega$, then* $\operatorname{ind}(F, \Omega) = 1$.
(3) (*Homotopy invariance*) *If $H : [0, 1] \times \overline{\Omega} \to X$ is a countably $\gamma$-condensing homotopy such that $H(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial\Omega$, then*

$$\operatorname{ind}(H(0, \cdot), \Omega) = \operatorname{ind}(H(1, \cdot), \Omega).$$

(4) *If $G : \overline{\Omega} \to X$ is a countably $\gamma$-condensing map such that $Gx = Fx$ for all $x \in \partial\Omega$, then* $\operatorname{ind}(G, \Omega) = \operatorname{ind}(F, \Omega)$.

Let $X$ be an infinite dimensional Banach space. We denote the closed ball of radius $r > 0$ by $B_r(X) = \{x \in X : \|x\| \leq r\}$ and its boundary by $S_r(X)$, respectively.

Consider the following two quantitative characteristics:

$$R_\gamma(X) = \inf\{k \geq 1 : \text{there is a } k\text{-}\gamma\text{-contractive retraction } R : B_1(X) \to S_1(X)\}$$

$$R^c_\gamma(X) = \inf\{k \geq 1 : \text{there is a countably } k\text{-}\gamma\text{-contractive retraction } R : B_1(X) \to S_1(X)\}$$

and $R^c_\gamma(X) = \infty$ if no such a contractive retraction exists. The fact that $k \geq 1$ is always possible, by Mönch's fixed point theorem [7]; see also [10]. It is obvious that $R_\gamma(X) \geq R^c_\gamma(X) \geq 1$. For example, if $\gamma$ is the Hausdorff measure of noncompactness, then $R^c_\gamma(C([0, 1])) = R_\gamma(C([0, 1])) = 1$ and $R^c_\gamma(L_p([0, 1])) = R_\gamma(L_p([0, 1])) = 1$ $(1 \leq p < \infty)$; see [3, 11].

In what follows, $\Omega$ will always be a bounded open subset of an infinite dimensional Banach space $X$ which contains the origin 0. For simplicity, we write $R^c_\gamma$ for $R^c_\gamma(X)$.

## 3. The countably $k$-set contractive case

In this section, we apply index theory to show the existence of an eigenvalue for countably $k$-set contractive maps, where the condition depends on the characteristic $R^c_\gamma$.

The following lemma stated in [6, Theorem 5] is a key tool in proving our results. For completeness we give the proof here; see also [4, Lemma 3.1].

**Lemma 3.1.** *Let $F : \overline{\Omega} \to X$ be a countably $k$-$\gamma$-contractive map for some $k > 0$ such that $kR^c_\gamma < 1$. Suppose that*

$$\inf_{x\in\partial\Omega} \|Fx\| > \|y\| \qquad \text{for all } y \in \Omega$$

*and $F$ has no fixed points on $\partial\Omega$. Then* $\operatorname{ind}(F, \Omega) = 0$.

*Proof.* Choose a real number $k_1$ such that $k_1 \geq R^c_\gamma$, $kk_1 < 1$, and there is a countably $k_1$-$\gamma$-contractive retraction $R : B_1(X) \to S_1(X)$. Set $\delta := \inf_{x\in\partial\Omega} \|Fx\|$ and define $R_\delta : B_\delta(X) \to S_\delta(X)$ by

$$R_\delta(x) := \delta R\left(\frac{x}{\delta}\right) \qquad \text{for } x \in B_\delta(X).$$

It is easily checked that $R_\delta$ is also a countably $k_1$-$\gamma$-contractive retraction. Consider a map $G : \overline{\Omega} \to X$ defined by

$$Gx := L(Fx) \qquad \text{for } x \in \overline{\Omega},$$

where

$$Lx = \begin{cases} R_\delta(x) & \text{if } x \in B_\delta(X), \\ x & \text{if } x \in X \setminus B_\delta(X). \end{cases}$$

Then $G$ is countably $kk_1$-$\gamma$-contractive and hence countably $\gamma$-condensing and we have

$$\inf_{x\in\overline{\Omega}} \|Gx\| \geq \delta > \|y\| \qquad \text{for all } y \in \Omega.$$

Since $G = F$ on $\partial\Omega$ and $G$ has no fixed points in $\Omega$, it follows from Proposition 2.2 that

$$\operatorname{ind}(F, \Omega) = \operatorname{ind}(G, \Omega) = 0. \qquad \square$$

We first give an extension of Guo's result to countably $k$-$\gamma$-contractive maps which is analogous to Theorem 3.2 of [4] when $F$ is a $k$-$\gamma$-contraction.

**Theorem 3.2.** *Suppose that $F : \overline{\Omega} \to X$ is a countably $k$-$\gamma$-contractive map for some $k > 0$ such that $kR^c_\gamma < 1$ and it satisfies*

$$\inf_{x\in\partial\Omega} \|Fx\| > kR^c_\gamma \sup_{y\in\Omega} \|y\|.$$

*If $Fx \neq \lambda x$ for all $x \in \partial\Omega$ and all $\lambda \in (kR^c_\gamma, 1]$, then* $\operatorname{ind}(F, \Omega) = 0$.

*Proof.* We can find a real number $\varepsilon > 0$ such that $kR^c_\gamma + \varepsilon < 1$ and

$$\inf_{x\in\partial\Omega} \|Fx\| > (kR^c_\gamma + \varepsilon) \sup_{y\in\Omega} \|y\|.$$

Set $k_1 := k/(kR^c_\gamma + \varepsilon)$ and define a map $G : \overline{\Omega} \to X$ by

$$Gx := \frac{1}{kR^c_\gamma + \varepsilon} Fx \qquad \text{for } x \in \overline{\Omega}.$$

Then $G$ is countably $k_1$-$\gamma$-contractive, $k_1 R^c_\gamma < 1$, and $G$ has no fixed points on $\partial\Omega$, by taking $\lambda = kR^c_\gamma + \varepsilon$ in the hypothesis. Lemma 3.1 implies that $\operatorname{ind}(G, \Omega) = 0$. Consider a homotopy $H : [0,1] \times \overline{\Omega} \to X$ defined by

$$H(t,x) := tFx + (1-t)Gx \qquad \text{for } (t,x) \in [0,1] \times \overline{\Omega}.$$

Then $H$ is countably $\gamma$-condensing. In fact, for each countable set $C \subset \overline{\Omega}$ with $\gamma(C) > 0$ we have

$$\begin{aligned}\gamma(H([0,1] \times C)) &\le \gamma(\operatorname{co}[F(C) \cup G(C)]) \\ &\le \max\{\gamma(F(C)), \gamma(G(C))\} \\ &< \gamma(C)\end{aligned}$$

because $F$ and $G$ are countably $\gamma$-condensing on $\overline{\Omega}$. As in the proof of Theorem 3.2 in [4], we have by hypothesis

$$Fx \ne (t + \frac{1-t}{kR^c_\gamma + \varepsilon})^{-1} x, \quad \text{that is,} \quad H(t,x) \ne x$$

for all $(t,x) \in [0,1] \times \partial\Omega$. We conclude by Proposition 2.2 that

$$\operatorname{ind}(F, \Omega) = \operatorname{ind}(G, \Omega) = 0. \qquad \square$$

The following statement is Lemma 1 of [5] given in the introduction.

**Corollary 3.3.** *Suppose that $F : \overline{\Omega} \to X$ is a compact map that satisfies*

$$\inf_{x\in\partial\Omega} \|Fx\| > 0.$$

*If $Fx \ne \lambda x$ for all $x \in \partial\Omega$ and all $\lambda \in (0,1]$, then* $\operatorname{ind}(F, \Omega) = 0$.

*Proof.* Choose a real number $k > 0$ such that $kR^c_\gamma < 1$ and

$$\inf_{x\in\partial\Omega} \|Fx\| > kR^c_\gamma \sup_{y\in\Omega} \|y\|.$$

Theorem 3.2 is applicable because $F$ is countably $k$-$\gamma$-contractive. $\square$

Now we show the existence of an eigenvalue for countably $k$-$\gamma$-contractive maps; see [4, Corollary 3.5] for $k$-$\gamma$-contractions.

**Theorem 3.4.** *Let $F : \overline{\Omega} \to X$ be a countably $k$-$\gamma$-contractive map for any $k > 0$. If*

$$\inf_{x\in\partial\Omega} \|Fx\| > kR^c_\gamma \sup_{y\in\Omega} \|y\|,$$

*then there exist a real number $\lambda > kR^c_\gamma$ and a vector $x \in \partial\Omega$ such that $Fx = \lambda x$.*

*Proof.* Assume on the contrary that

$$Fx \neq \lambda x \qquad \text{for all } x \in \partial\Omega \text{ and all } \lambda > kR_\gamma^c.$$

Choose a real number $\varepsilon > 0$ such that

$$\inf_{x\in\partial\Omega} \|Fx\| > (kR_\gamma^c + \varepsilon) \sup_{y\in\Omega} \|y\|.$$

Let $G : \overline{\Omega} \to X$ be a countably $\gamma$-condensing map defined by

$$Gx := \frac{1}{kR_\gamma^c + \varepsilon} Fx \qquad \text{for } x \in \overline{\Omega}.$$

As in the proof of Theorem 3.2, a similar argument shows that $\operatorname{ind}(G, \Omega) = 0$. Now consider a countably $\gamma$-condensing homotopy $H : [0,1] \times \overline{\Omega} \to X$ defined by

$$H(t,x) := (1-t)Gx \qquad \text{for } (t,x) \in [0,1] \times \overline{\Omega}.$$

Our assumption implies that $H(t,x) \neq x$ for all $x \in \partial\Omega$ and $t \in [0,1]$. From $H(1,x) = 0$ for all $x \in \overline{\Omega}$ and $0 \in \Omega$, it follows by Proposition 2.2 that

$$\operatorname{ind}(G,\Omega) = \operatorname{ind}(H(1,\cdot),\Omega) = 1,$$

which is impossible. We conclude that there exist a real number $\lambda > kR_\gamma^c$ and a point $x \in \partial\Omega$ such that $Fx = \lambda x$. □

Our result reduces to the well known Birkhoff-Kellogg theorem.

**Corollary 3.5.** *If $F : \overline{\Omega} \to X$ is a compact map such that*

$$\inf_{x\in\partial\Omega} \|Fx\| > 0,$$

*then $F$ has at least an eigenvalue $\lambda > 0$ whose corresponding vector $x$ lies in $\partial\Omega$.*

*Proof.* Choose a real number $k > 0$ such that

$$\inf_{x\in\partial\Omega} \|Fx\| > kR_\gamma^c \sup_{y\in\Omega} \|y\|.$$

Apply Theorem 3.4. □

## 4. The countably condensing case

In this section, we present an eigenvalue result for countably condensing maps. As its application, we study nonlinear eigenvalue problem in a more concrete situation.

We give a simple proof of the following countably condensing version; compare with Theorem 4.1 of [4] in the condensing case.

**Theorem 4.1.** *Suppose that $F : \overline{\Omega} \to X$ is a countably $\gamma$-condensing map that satisfies*

$$\inf_{x\in\partial\Omega} \|Fx\| > R_\gamma^c \sup_{y\in\Omega} \|y\|.$$

*Then* $\operatorname{ind}(F,\Omega) = 0$.

*Proof.* Choose $\varepsilon > 0$ such that

$$\inf_{x\in\partial\Omega} \|Fx\| > (R_\gamma^c + \varepsilon) \sup_{y\in\Omega} \|y\|.$$

Set $k := 1/(R_\gamma^c + \varepsilon)$ and define $G : \overline{\Omega} \to X$ by

$$Gx := kFx \qquad \text{for } x \in \overline{\Omega}.$$

Then $G$ is countably $k$-$\gamma$-contractive. It is easily verified that

$$\inf_{x\in\partial\Omega} \|Gx\| > \sup_{x\in\partial\Omega} \|x\|.$$

Since $kR_\gamma^c < 1$ and $G$ has no fixed points on $\partial\Omega$, Lemma 3.1 implies that $\operatorname{ind}(G, \Omega) = 0$. Consider a homotopy $H : [0,1] \times \overline{\Omega} \to X$ defined by

$$H(t,x) := tFx + (1-t)Gx \qquad \text{for } (t,x) \in [0,1] \times \overline{\Omega}.$$

As before, $H$ is obviously countably $\gamma$-condensing. For all $(t,x) \in [0,1] \times \partial\Omega$, we have

$$\|H(t,x)\| = \|(t + k(1-t))Fx\| \geq k\|Fx\| = \|Gx\| > \|x\|$$

and thus $H(t,x) \neq x$. Consequently, Proposition 2.2 implies that

$$\operatorname{ind}(F, \Omega) = \operatorname{ind}(G, \Omega) = 0.$$

□

Now we have an existence result on eigenvalues of countably condensing maps; see [4, Corollary 4.2] for condensing maps.

**Theorem 4.2.** *Let $F : \overline{\Omega} \to X$ be a countably $\gamma$-condensing map such that*

$$\inf_{x\in\partial\Omega} \|Fx\| > R_\gamma^c \sup_{y\in\Omega} \|y\|.$$

*Then there exist a real number $\lambda > R_\gamma^c$ and a vector $x \in \partial\Omega$ such that $Fx = \lambda x$.*

*Proof.* Notice that each countably $\gamma$-condensing map is countably 1-$\gamma$-contractive. Applying Theorem 3.4, the conclusion follows. □

As an application of Theorem 4.2, we observe nonlinear eigenvalue problem in a concrete situation; see [4,5].

**Theorem 4.3.** *Let $X = L_p([0,1])$ $(p > 1)$ and let $\Omega$ be a bounded open subset of $X$ with $0 \in \Omega$. Assume that*

(1) *$k : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous nonnegative function such that*

$$\alpha = \min \left\{ \int_0^1 k(t,s)\, dt : s \in [0,1] \right\} > 0.$$

(2) *$L : \overline{\Omega} \to X$ is an integral operator defined by*

$$Lx(t) := \int_0^1 k(t,s)|x(s)|^p \, ds \qquad \text{for } t \in [0,1].$$

(3) *$G : \overline{\Omega} \to X$ is a countably $\gamma$-condensing map such that $G(\partial\Omega)$ is bounded, where $\gamma$ is the Hausdorff measure of noncompactness.*

*Consider nonlinear operator $F : \overline{\Omega} \to X$ given by*

$$Fx = Lx + Gx.$$

*Set*

$$\beta = \inf_{x\in\partial\Omega} \|x\|, \qquad d = \sup_{y\in\Omega} \|y\|, \qquad \textit{and} \quad \delta = \sup_{x\in\partial\Omega} \|Gx\|.$$

*If $\alpha\beta^p > \delta + d$, then $F$ has a positive eigenvalue whose corresponding vector lies in $\partial\Omega$.*

*Proof.* For each $x \in \overline{\Omega}$, we obtain from Jensen's inequality that

$$\begin{aligned} \alpha\|x\|^p = \int_0^1 \alpha|x(s)|^p\, ds &\le \int_0^1 \left(\int_0^1 k(t,s)\, dt\right) |x(s)|^p\, ds \\ &= \int_0^1 \int_0^1 k(t,s)|x(s)|^p\, ds\, dt \\ &\le \left(\int_0^1 \left|\int_0^1 k(t,s)|x(s)|^p\, ds\right|^p dt\right)^{\frac{1}{p}} = \|Lx\|. \end{aligned}$$

For all $x \in \partial\Omega$, we have

$$\|Fx\| \ge \|Lx\| - \|Gx\| \ge \alpha\beta^p - \delta > d$$

and hence

$$\inf_{x\in\partial\Omega} \|Fx\| > \sup_{y\in\Omega} \|y\|.$$

Notice that $R^c_\gamma(L_p([0,1])) = R_\gamma(L_p([0,1])) = 1$ by Theorem 4.5 of [3]. Since $L$ is compact and hence $F$ is countably $\gamma$-condensing, Theorem 4.2 implies that $F$ has an eigenvalue $\lambda > 0$ with corresponding vector $x \in \partial\Omega$. □

In addition, a similar argument shows that Theorem 4.3 holds provided that $G$ is a countably $k$-$\gamma$-contractive map and $\alpha\beta^p > \delta + kd$, where Theorem 3.4 is used in place of Theorem 4.2.

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