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2×2 INVERTIBLE MATRICES OVER WEAKLY STABLE RINGS

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Abstract. A ring R is a weakly stable ring provided that $aR + bR = R$ implies that there exists $y \in R$ such that $a + by \in R$ is right or left invertible. In this article, we characterize weakly stable rings by virtue of 2×2 invertible matrices over them. It is shown that a ring R is a weakly stable ring if and only if for any $A \in GL_2(R)$, there exist two invertible lower triangular L and K and an invertible upper triangular U such that $A = LUK$, where two of L, U and K have diagonal entries 1. Related results are also given. These extend the work of Nagarajan et al.

1. Introduction

A ring R is said to have stable range one provided that $aR + bR = R$ with $a, b \in R$ implies that there exists $y \in R$ such that $a + by \in GL_1(R)$. This condition plays an important role in non-stable K-theory (cf. [8]). It is instructive to represent an invertible matrix as a product of triangular matrices (cf. [11]). The group of all 2×2 invertible lower triangular matrices over a ring R is denoted by $\mathfrak{L}$, i.e., $\mathfrak{L} = \{(a_{ij}) \in GL_2(R) \mid a_{ij} = 0$ whenever $i < j$, $1 \leq i,\ j \leq 2\}$. The group of all 2×2 invertible upper triangular matrices over a ring R is denoted by $\mathfrak{U}$, i.e., $\mathfrak{U} = \{(a_{ij}) \in GL_2(R) \mid a_{ij} = 0$ whenever $i > j$, $1 \leq i,\ j \leq 2\}$. Obviously, $\mathfrak{L}$ and $\mathfrak{U}$ are both subgroups of the 2-dimensional general linear group $GL_2(R)$ of R. In [14, Theorem 1], Vaserstein and Wheland proved that if R has stable range one, then every invertible $n \times n$ matrix over R can be written as $A = LUK$, where $L, K \in \mathfrak{L}$ and $U \in \mathfrak{U}$. In [12, Theorem 3.1], Nagarajan et al. proved that for commutative rings the converse is true. Furthermore, the author and Chen extended these results to any associative ring. We proved that a ring R has stable range one if and only if for any $A \in GL_2(R)$, there exist $L, K \in \mathfrak{L}, U \in \mathfrak{U}$ such that $A = LUK$, where U and K have diagonal entries 1 (cf. [4, Theorem 2.2]).

A ring R is said to be a weakly stable ring provided that $aR + bR = R$ implies that there exists $y \in R$ such that $a + by \in R$ is right or left invertible. This concept is left-right symmetric. It is a natural generalization of that of

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stable range one (cf. [9] and [15-16]). The class of weakly stable rings is very large. It includes rings having stable range one, one-sided unit-regular rings (cf. [2]), exchange rings satisfying comparability axiom (cf. [7] and [13]), simple separative exchange rings ([13]), etc. The weakly stable property is Morita invariant; hence, every regular square matrix over weakly stable rings is the product of an idempotent matrix and a right or left invertible matrix. The author showed that every regular square matrix over weakly stable exchange rings is the sum of an idempotent matrix and a right or left invertible matrix (cf. [3, Theorem 2]). Let A be a right R-module having the finite exchange property. If A is expressible as a direct sum of isomorphic indecomposable submodules, then $\mathrm{End}_R(A)$ is a weakly stable ring (cf. [6]).

The main purpose of this article is to study triangular factorizations of 2×2 invertible matrices over weakly stable rings. We prove that a ring R is a weakly stable ring if and only if for any $A \in GL_2(R)$, there exist $L \in \mathfrak{L}, U \in \mathfrak{U}, K \in \mathfrak{L}$ such that $A = LUK$, where two of L, U and K have diagonal entries 1. Related result are also obtained.

2. Triangular factorizations

We now extend [4, Theorem 2.2] to weakly stable rings.

Theorem 2.1. *Let R be a ring. Then the following are equivalent*:

(1) *R is a weakly stable ring.*

(2) *For any $A \in GL_2(R)$, there exist $L \in \mathfrak{L}$, $U \in \mathfrak{U}$, $K \in \mathfrak{L}$ such that $A = LUK$, where two of L, U and K have diagonal entries* 1.

Proof. (1) $\Rightarrow$ (2) Let $A = (a_{ij}) \in GL_2(R)$. Then $a_{11}R + a_{12}R = R$. Since R is a weakly stable ring, there exists $y \in R$ such that $a_{11} + a_{12}y = u \in R$ is right or left invertible. Assume that $uv = 1$ for a $v \in R$. Then

$$\begin{aligned}\begin{pmatrix} u & a_{12} \\ a_{21}+a_{22}y & a_{22}\end{pmatrix} &= \begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & va_{12} \\ a_{21}+a_{22}y & a_{22}\end{pmatrix} \\ &= \begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & *\end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1\end{pmatrix}.\end{aligned}$$

Thus, we see that $A = \left(\begin{smallmatrix} * & 0 \\ * & *\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & * \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & * \\ -y & 1\end{smallmatrix}\right) \in \mathfrak{L}\mathfrak{U}\mathfrak{L}$.

Assume that $vu = 1$ for a $v \in R$. Then

$$\begin{aligned}\begin{pmatrix} u & a_{12} \\ a_{21}+a_{22}y & a_{22}\end{pmatrix} &= \begin{pmatrix} 1 & a_{12} \\ (a_{21}+a_{22}y)v & a_{22}\end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ * & 1\end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & *\end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}.\end{aligned}$$

Hence, $A = \left(\begin{smallmatrix} 1 & 0 \\ * & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} u & * \\ 0 & *\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -y & 1\end{smallmatrix}\right) \in \mathfrak{L}\mathfrak{U}\mathfrak{L}$, as required.

(2) $\Rightarrow$ (1) Given $ax + b = 1$ in R, then $\left(\begin{smallmatrix} a & b \\ -1 & x\end{smallmatrix}\right) = \left(\begin{smallmatrix} x & xa-1 \\ 1 & a\end{smallmatrix}\right)^{-1} \in GL_2(R)$. By assumption, we have a factorization $\left(\begin{smallmatrix} a & b \\ -1 & x\end{smallmatrix}\right) = LUK \in \mathfrak{L}\mathfrak{U}\mathfrak{L}$, where two of L, U

and K have diagonal entries 1. If every diagonal entry of L and U is 1, then

$$\begin{pmatrix} a & b \\ -1 & x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}.$$

Thus, we have $z, u \in R$ such that

$$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} a & b \\ -1 & x \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} * & 0 \\ * & u \end{pmatrix}.$$

This implies that $x + zb = u$. Clearly, $u \in R$ is left invertible. In view of [2, Proposition 1], there exists $y \in R$ such that $a + by \in R$ is right invertible.

If every diagonal entry of U and K is 1, then

$$\begin{pmatrix} a & b \\ -1 & x \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}.$$

Thus, we have $y, u \in R$ such that

$$\begin{pmatrix} a & b \\ -1 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} u & * \\ 0 & * \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

This implies that $a + by = u$. Clearly, $u \in R$ is right invertible. Hence, $a + by \in R$ is right invertible.

If every diagonal entry of L and K is 1, then

$$\begin{pmatrix} a & b \\ -1 & x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}.$$

Thus, we have $y, u \in R$ such that

$$\begin{pmatrix} a & b \\ -1 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ * & * \end{pmatrix}.$$

This implies that $a + by = u$. In this case, $u \in R$ is right invertible. Therefore we conclude that R is a weakly stable ring. □

Corollary 2.2. *Let R be a weakly stable ring. Then every 2×2 invertible matrix is similar to the product of two invertible triangular matrices.*

Proof. Let $A \in GL_2(R)$. In view of Theorem 2.1, there exist $L \in \mathfrak{L}$, $U \in \mathfrak{U}$, $K \in \mathfrak{L}$ such that $A = LUK$, where two of L, U and K have diagonal entries 1. Clearly, either L or K has diagonal entries 1. If L has diagonal entries 1, then $L^{-1}AL = U(KL)$ is the product of an upper triangular matrix U and a lower triangular matrix KL. If K has diagonal entries 1, then $KAK^{-1} = (KL)U$ is the product of a lower triangular matrix KL and an upper triangular matrix U, and therefore we complete the proof. □

Let R be the collection of $\aleph_0 \times \aleph_0$ matrices with entries from a field F, the form

$$\left\{ \begin{pmatrix} [A] & & & \\ & q & & \\ & & q & \\ & & & \ddots \end{pmatrix}_{\aleph_0 \times \aleph_0} \mid A \in M_n(F) \text{ for some } n, q \in F \right\}.$$

Then R is a regular ring satisfying the comparability axiom. Hence, it is a weakly stable ring. In view of Corollary 2.2, every 2×2 invertible matrix over R is similar to the product of two invertible triangular matrices.

An element $e \in R$ is infinite if there exist orthogonal idempotents $f, g \in R$ such that $e = f + g$ while $eR \cong fR$ and $g \neq 0$. A simple ring is said to be purely infinite if every nonzero right ideal of R contains an infinite idempotent. It is well known that a ring R is a purely infinite, simple ring if and only if it is not a division ring and for any nonzero $a \in R$ there exist $s, t \in R$ such that $sat = 1$ (cf. [1]). The class of purely infinite simple rings is rather large. All purely infinite simple C^*-algebras and all directly infinite regular rings over which every finitely generated projective right module is free are purely infinite simple ring. We note that every purely infinite, simple ring is weakly stable. According to Corollary 2.2, every 2×2 invertible matrix over purely infinite, simple rings is similar to the product of two invertible triangular matrices.

Recall that a ring R is a Hermite ring if for any 1×2 matrix (a, b), there exists a $Q \in GL_2(R)$ such that $(a, b)Q = (*, 0)$. As is well known, R is a Hermite ring if and only if for every 2×2 matrix A over R, there is an invertible Q such that AQ is a lower triangular matrix.

Corollary 2.3. *Let R be a ring. Then the following are equivalent*:

(1) For any $A \in M_2(R)$, there exist a lower triangular L and $U \in \mathfrak{U}$, $K \in \mathfrak{L}$ such that $A = LUK$, and that two of L, U and K have diagonal entries 1 if $A \in GL_2(R)$.
(2) R is a weakly stable, Hermite ring.

Proof. (1) $\Rightarrow$ (2) In view of Theorem 2.1, R is a weakly stable ring. Let $A \in M_2(R)$. Then there exist a lower triangular L and $U \in \mathfrak{U}$, $K \in \mathfrak{L}$ such that $A = LUK$. Hence, $A\big(UK\big)^{-1} = L \in M_2(R)$ is lower triangular; hence, R is a Hermite ring.

(2) $\Rightarrow$ (1) Let $A \in M_2(R)$. By [12, Theorem 2.5], we have an invertible Q such that $AQ \in M_2(R)$ is lower triangular. In view of Theorem 2.1, $Q^{-1} \in \mathfrak{LUL}$, and therefore we obtain the result. $\square$

Let $\mathbb{F}$ be a field, and let $A = \left(\begin{smallmatrix} 0 & 0 \\ 1_{\mathbb{F}} & 0 \end{smallmatrix}\right)$. In view of Corollary 2.3, A is the product of three triangular matrices. By an explicit computation, A cannot be expressed as LUK where L, K are lower triangular and U is upper triangular, diagonal entries of L and K are equal to $1_{\mathbb{F}}$.

Example 2.4. Let V be an infinite-dimensional vector space over a division ring D, and let $R = \mathrm{End}_D(V)$. Then R is a weakly stable ring. Let $\{x_1, x_2, \dots, x_n, \dots\}$ be a basis of V. Define $\tau : V \to V$ given by $\tau(x_i) = x_{i+1}$ $(i = 1, 2, \dots)$ and $\sigma : V \to V$ given by $\sigma(x_1) = 0$ and $\sigma(x_i) = x_{i-1}$ $(i = 2, 3, \dots)$. Then $\sigma\tau = 1_V$, while $\tau\sigma \neq 1_V$. Let $A = \begin{pmatrix} \tau & 1_V - \tau\sigma \\ -2\cdot 1_V & \sigma \end{pmatrix}$. Then $A = \begin{pmatrix} 1_V & 0 \\ -\sigma & 1_V \end{pmatrix} \begin{pmatrix} \tau & 1_V - \tau\sigma \\ -1_V & \sigma \end{pmatrix} \in GL_2(R)$. By virtue of Theorem 2.1, A can be expressed as the product of three invertible triangular matrices. In fact, we have

$$A = \begin{pmatrix} 1_V & 0 \\ -\sigma & 1_V \end{pmatrix} \begin{pmatrix} \tau & 1_V - \tau\sigma \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} 1_V & 0 \\ -\tau & 1_V \end{pmatrix}.$$

In this case, A cannot be expressed as LUK, where $L \in \mathfrak{L}$, $U \in \mathfrak{U}$, $K \in \mathfrak{L}$ and in L and U or in U and K all the diagonal entries are equal to 1_V.

Theorem 2.5. *Let R be a ring. Then the following are equivalent*:

(1) *R is a weakly stable ring.*

(2) *For any $A \in GL_2(R)$, there exist $U \in \mathfrak{U}$, $L \in \mathfrak{L}$, $W \in \mathfrak{U}$ such that $A = ULW$, where two of U, L and W have diagonal entries* 1.

Proof. (1) $\Rightarrow$ (2) For any $A \in GL_2(R)$, we have $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(R)$. In view of Theorem 2.1, we can find $L \in \mathfrak{L}, U \in \mathfrak{U}, K \in \mathfrak{L}$ such that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = LUK$, where two of L, U and K have diagonal entries 1. This implies that $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} LUK \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let

$$U' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} L \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad L' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$W' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} K \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $A = U'L'W'$, where $U' \in \mathfrak{U}, L' \in \mathfrak{L}, W' \in \mathfrak{U}$ and two of L', U' and K' have diagonal entries 1.

(2) $\Rightarrow$ (1) For any $A \in GL_2(R)$, we have $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(R)$. By assumption, there exist $U \in \mathfrak{U}, L \in \mathfrak{L}, W \in \mathfrak{U}$ such that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = ULW$, where two of U, L and W have diagonal entries 1. This implies that $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ULW \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let

$$L' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} L \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$K' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $A = L'U'K'$, where $L' \in \mathfrak{L}, U' \in \mathfrak{U}, K' \in \mathfrak{L}$ and two of U', L' and W' have diagonal entries 1. Therefore R is a weakly stable ring by Theorem 2.1. □

We note that Theorem 2.5 implies that every 2×2 invertible matrix over a weakly stable ring admits a triangular matrix by two elementary transformations.

Corollary 2.6. *Let R be a ring. Then the following are equivalent*:

(1) *R has stable range one.*
(2) *R is directly finite and for any $A \in GL_2(R)$, there exist $L \in \mathfrak{L}$, $U \in \mathfrak{U}$, $K \in \mathfrak{L}$ such that $A = LUK$, where two of L, U and K have diagonal entries* 1*.*
(3) *R is directly finite and for any $A \in GL_2(R)$, there exist $U \in \mathfrak{U}$, $L \in \mathfrak{L}$, $W \in \mathfrak{U}$ such that $A = ULW$, where two of U, L and W have diagonal entries* 1*.*

Proof. (1) $\Rightarrow$ (2) It is obvious from [4, Lemma 2.1].

(2) $\Rightarrow$ (1) In view of Theorem 2.1, R is a weakly stable ring. As R is directly finite, all right or left invertible elements in R is invertible; hence, R has stable range one.

(1) $\Leftrightarrow$ (3) is similar by Theorem 2.5. □

Example 2.7. Let V be an infinite-dimensional vector space over a division ring D, and let $R = \mathrm{End}_D(V)$. Construct τ and σ as in Example 2.4. Let $A = \left(\begin{smallmatrix} \sigma & -2\cdot 1_V \\ 1_V - \tau\sigma & \tau \end{smallmatrix}\right)$. Then

$$A = \begin{pmatrix} 0 & 1_V \\ 1_V & 0 \end{pmatrix} \begin{pmatrix} \tau & 1_V - \tau\sigma \\ -2\cdot 1_V & \sigma \end{pmatrix} \begin{pmatrix} 0 & 1_V \\ 1_V & 0 \end{pmatrix} \in GL_2(R).$$

By virtue of Theorem 2.5, A can be expressed as the product of three invertible triangular invertible matrices. In fact, we have

$$A = \begin{pmatrix} 1_V & -\sigma \\ 0 & 1_V \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 1_V - \tau\sigma & \tau \end{pmatrix} \begin{pmatrix} 1_V & -\tau \\ 0 & 1_V \end{pmatrix}.$$

In this case, A cannot be expressed as ULW, where $U \in \mathfrak{U}, L \in \mathfrak{L}, W \in \mathfrak{U}$ and in U and L or in L and W all the diagonal entries are equal to 1_V.

Example 2.8. Let V be an infinite-dimensional vector space over a division ring D, and let $R = \mathrm{End}_D(V)$. Then $\left(\begin{smallmatrix} 0 & 1_V \\ 1_V & 0 \end{smallmatrix}\right) \notin \mathfrak{L}\mathfrak{U}, \mathfrak{U}\mathfrak{L}$, where all the diagonal entries of one of $\mathfrak{L}$ and $\mathfrak{U}$ are 1_V. Thus, we do need three matrices in Theorem 2.1 and Theorem 2.5.

3. Extensions

Theorem 3.1. *Let R be a ring. Then the following are equivalent*:

(1) *R is a weakly stable ring.*
(2) *For any $A \in GL_2(R)$, there exist $L, K \in \mathfrak{L}, U \in \mathfrak{U}$ such that $LAKU = I_2$, where two of L, K and U have diagonal entries* 1*.*

Proof. (1) $\Rightarrow$ (2) Let $A = (a_{ij}) \in GL_2(R)$. Then $a_{11}R + a_{12}R = R$. Since R is a weakly stable ring, we have $y \in R$ such that $a_{11} + a_{12}y = u \in R$ is right or left invertible. Assume that $uv = 1$ for a $v \in R$. Then

$$A \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 1 - vu & v \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & * \\ * & * \end{pmatrix}.$$

It is easy to verify that $\left(\begin{smallmatrix} u & 0 \\ 1-vu & v \end{smallmatrix}\right) = \left(\begin{smallmatrix} v & 1-vu \\ 0 & u \end{smallmatrix}\right)^{-1} \in GL_2(R)$. Hence we can find $w \in U(R)$ such that $\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right) A \left(\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} u & 0 \\ 1-vu & v \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & w \end{smallmatrix}\right)$.
Let

$$L = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix},$$
$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u & 0 \\ 1-vu & v \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & w^{-1} \end{pmatrix}.$$

Then $L \in \mathfrak{L}, K \in \mathfrak{L}, U \in \mathfrak{U}$ and $LAKU = I_2$ with every entry of L and K is 1.

Assume that $vu = 1$ for a $v \in R$. Then

$$\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} v & 0 \\ 1-uv & u \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = I_2.$$

Let

$$L = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} v & 0 \\ 1-uv & u \end{pmatrix}, \ K = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, \ U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

Then $L \in \mathfrak{L}$, $K \in \mathfrak{L}$, $U \in \mathfrak{U}$ and $LAKU = I_2$, where every entry of K and U is 1.

(2) $\Rightarrow$ (1) Given $ax + b = 1$ in R, then we have $\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) \in GL_2(R)$. By assumption, we can find $L \in \mathfrak{L}, K \in \mathfrak{L}, U \in \mathfrak{U}$ such that $L\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) KU = I_2$, where two of L, K and U have diagonal entries 1. If every diagonal entry of L and K is 1, then we have $w, y, z \in R$ such that

$$\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}\begin{pmatrix} a & b \\ -1 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} z & * \\ 0 & * \end{pmatrix} = I_2.$$

This implies that

$$\begin{pmatrix} a+by & b \\ -1+xy & x \end{pmatrix}\begin{pmatrix} z & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix}.$$

Hence, $(a + by)z = 1$, and so $a + by \in R$ is right invertible.

If every diagonal entry of K and U is 1, then we can find $w, y, z \in R$ such that

$$\begin{pmatrix} z & 0 \\ * & * \end{pmatrix}\begin{pmatrix} a & b \\ -1 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} = I_2.$$

This implies that

$$\begin{pmatrix} z & 0 \\ * & * \end{pmatrix}\begin{pmatrix} a+by & b \\ -1+xy & x \end{pmatrix} = \begin{pmatrix} 1 & -w \\ 0 & 1 \end{pmatrix}.$$

Hence, $z(a + by) = 1$, and so $a + by \in R$ is left invertible.

If every diagonal entry of L and U is 1, then we can find $w, y, z \in R$ such that

$$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} a & b \\ -1 & x \end{pmatrix}\begin{pmatrix} * & 0 \\ * & y \end{pmatrix}\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} = I_2.$$

This implies that

$$\begin{pmatrix} a & b \\ za-1 & x+zb \end{pmatrix}\begin{pmatrix} * & 0 \\ * & y \end{pmatrix} = \begin{pmatrix} 1 & -w \\ 0 & 1 \end{pmatrix}.$$

Hence, $(x+zb)y = 1$, and so $x + zb \in R$ is right invertible. In view of [2, Proposition 1], there exists $t \in R$ such that $a + bt \in R$ is left invertible. Therefore R is a weakly stable ring. □

Recall that a ring R is regular provided that for every $a \in R$ there exists $x \in R$ such that $a = axa$. A regular ring R is said to satisfy the comparability axiom provided that, for any idempotents $e, f \in R$, there exist elements $s \in eRf$ and $t \in fRe$ such that $st = e$ or $ts = f$. (Cf. [7].)

Corollary 3.2. *Let R be a regular ring satisfying the comparability axiom. Then for any $A \in GL_2(R)$, there exist $L, K \in \mathfrak{L}, U \in \mathfrak{U}$ such that $LAKU = I_2$, where two of L, K and U have diagonal entries* 1.

Proof. According to [2, Theorem 8], R is a weakly stable ring. Therefore we complete the proof by Theorem 3.1. □

Theorem 3.3. *Let R be a ring. Then the following are equivalent*:

1. *R is a weakly stable ring.*
2. *For any $A \in GL_2(R)$, there exist $U, W \in \mathfrak{U}, K \in \mathfrak{L}$ such that $UAWK = I_2$, where two of U, W and K have diagonal entries* 1.

Proof. (1) ⇒ (2) For any $A \in GL_2(R)$, we have $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) A \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in GL_2(R)$. In view of Theorem 3.1, we can find $L, K \in \mathfrak{L}, U \in \mathfrak{U}$ such that $LAKU = I_2$, where two of L, K and U have diagonal entries 1. Thus,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} L \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} KU \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = I_2.$$

Let

$$U' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} L \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad W' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} K \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$K' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $U'AW'K' = I_2$ and $U', W' \in \mathfrak{U}$, $K' \in \mathfrak{L}$, where two of U', W' and K' have diagonal entries 1.

(2) ⇒ (1) For any $A \in GL_2(R)$, we have $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) A \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in GL_2(R)$. By assumption, there exist $U, W \in \mathfrak{U}, K \in \mathfrak{L}$ such that $U \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) A \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) WK = I_2$, where two of U, W and K have diagonal entries 1. Let

$$L' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$U' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} K \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $L'AK'U' = I_2$ and $L', K' \in \mathfrak{L}, U' \in \mathfrak{U}$, where two of L', K' and U' have diagonal entries 1. According to Theorem 3.1, R is a weakly stable ring. $\square$

Corollary 3.4. *Let R be a regular ring satisfying the comparability axiom. Then for any $A \in GL_2(R)$, there exist $U, W \in \mathfrak{U}, K \in \mathfrak{L}$ such that $UAWK = I_2$, where two of U, W and K have diagonal entries* 1.

Proof. As in the proof of Corollary 3.2, R is a weakly stable ring, and therefore we complete the proof by Theorem 3.3. $\square$

Recall that a ring R is one-sided unit-regular in case for each $a \in R$, there exists a right or left invertible $u \in R$ such that $a = aua$. Now we add some new characterizations of such regular rings.

Corollary 3.5. *Let R be a regular ring. Then the following are equivalent*:

(1) *R is one-sided unit-regular.*
(2) *For any $A \in GL_2(R)$, there exist $L \in \mathfrak{L}, U \in \mathfrak{U}, K \in \mathfrak{L}$ such that $A = LUK$, where two of L, U and K have diagonal entries* 1.
(3) *For any $A \in GL_2(R)$, there exist $U \in \mathfrak{U}, L \in \mathfrak{L}, W \in \mathfrak{U}$ such that $A = ULW$, where two of U, L and W have diagonal entries* 1.
(4) *For any $A \in GL_2(R)$, there exist $L, K \in \mathfrak{L}, U \in \mathfrak{U}$ such that $LAKU = I_2$, where two of L, U and K have diagonal entries* 1.
(5) *For any $A \in GL_2(R)$, there exist $U, W \in \mathfrak{U}, K \in \mathfrak{L}$ such that $UAWK = I_2$, where two of U, W and K have diagonal entries* 1.

Proof. By virtue of [2, Theorem 8], a regular ring R is one-sided unit-regular if and only if it is a weakly stable ring. Therefore the result follows from Theorem 2.1, Theorem 2.4, Theorem 3.1, and Theorem 3.3. $\square$

4. Regular ideals

An ideal I of a ring R is said to be regular in case for any $a \in I$ there exists $x \in I$ such that $a = axa$. We say that I is weakly stable if $aR + bR = R$ with $a \in 1 + I, b \in R$ implies that there exists $y \in R$ such that $a + by \in R$ is right or left invertible. In what follows, we use $GL_2(I)$ to denote $GL_2(R) \bigcap \big(I_2 + M_2(I)\big)$.

Lemma 4.1. *Let I be a regular ideal of a ring R. Then the following are equivalent*:

(1) *I is a weakly stable ideal.*
(2) *eRe is a weakly stable ring for any idempotents $e \in I$.*

Proof. (1)$\Rightarrow$(2) Let $e = e^2 \in I$. Given $ax + b = e$ with $a, x, b \in eRe$, we have $(a + 1 - e)(x + 1 - e) + b = 1$ with $a + 1 - e \in 1 + I$. Thus, there exists $y \in R$ such that $a + 1 - e + by \in R$ is right or left invertible. Assume that $(a + 1 - e + by)u = 1$ for a $u \in R$. Then $(1 - e)u = 1 - e$, hence $eue = ue$. So $(a + b(eye))(eue) = e$. Assume that $u(a + 1 - e + by) = 1$ for a $u \in R$. Then $(eue)(a + b(eye)) = e$, and so eRe is weakly stable.

(2)⇒(1) Given $aR+bR=R$ with $a\in 1+I$ and $b\in R$, since I is a regular ideal, we have $e=e^2\in I$ such that $1-a=(1-a)e$. Suppose $ar+bs=(1-a)e$ for some $r,s\in R$. Then $eae(e+ere)+ebse=e$. As eRe is a weakly stable ring, we have $z\in eRe$ such that $eae+ebsez=u\in eRe$ is right or left invertible. Let $w=(1-e)ae+(1-e)bsez$. Assume that $uv=e$ for some $v\in eRe$. Then $(ae+bsez)(v-wv+1-e)=wv+e$ and $a(1-e)(v-wv+1-e)=-wv+1-e$. Thus, $(a+bsez)(v-wv+1-e)=1$. Assume that $vu=e$ for some $v\in eRe$. Then $(v-wv+1-e)(ae+bsez)=e$ and $(v-wv+1-e)a(1-e)=1-e$. Thus, $(v-wv+1-e)(a+bsez)=1$. Therefore $a+b(sez)\in R$ is right or left invertible, as required. □

Theorem 4.2. *Let I be a regular ideal of a ring R. Then the following are equivalent*:

(1) *I is a weakly stable ideal.*
(2) *For any $A\in GL_2(I)$, there exist $L\in\mathfrak{L},U\in\mathfrak{U},K\in\mathfrak{L}$ such that $A=LUK$, where two of L,U and K have diagonal entries 1.*
(3) *For any $A\in GL_2(I)$, there exist $U\in\mathfrak{U},L\in\mathfrak{L},W\in\mathfrak{U}$ such that $A=ULW$, where two of U,L and W have diagonal entries 1.*

Proof. (1) ⇒ (2) For any $A\in GL_2(I)$, we have $I_2-A\in M_2(I)$. In view of [5, Lemma 3.9], there exists an idempotents $e\in I$ such that $I_2-A\in M_2(eRe)$. Let $B=I_2-A$. Then $I_2-B\in GL_2(R)$. This implies that $\left(\begin{smallmatrix} e & 0\\ 0 & e\end{smallmatrix}\right)-B\in GL_2(eRe)$. By virtue of Lemma 4.1, eRe is a weakly stable ring. Using Theorem 2.1, we can find $L\in\mathfrak{L},U\in\mathfrak{U},K\in\mathfrak{L}$ such that $\left(\begin{smallmatrix} e & 0\\ 0 & e\end{smallmatrix}\right)-B=LUK$, where two of L,U and K have diagonal entries e. Let

$$L'=\begin{pmatrix} 1-e & 0\\ 0 & 1-e\end{pmatrix}+L,\quad U'=\begin{pmatrix} 1-e & 0\\ 0 & 1-e\end{pmatrix}+U,$$
$$K'=\begin{pmatrix} 1-e & 0\\ 0 & 1-e\end{pmatrix}+K.$$

Then $A=\left(\begin{smallmatrix} 1-e & 0\\ 0 & 1-e\end{smallmatrix}\right)+LUK=L'U'K'$, where two of L',U' and K' have diagonal entries 1, as required.

(2) ⇒ (1) Given $ax+b=e$ with $a,x,b\in eRe$, then $(a+1-e)(x+1-e)+b=1$. It is easy to verify that

$$\begin{aligned}&\begin{pmatrix} 1 & 0\\ e-1 & 1\end{pmatrix}\begin{pmatrix} a+1-e & b\\ -e & x+1-e\end{pmatrix}\begin{pmatrix} 1 & x+1-e\\ 0 & 1\end{pmatrix}\\ =&\begin{pmatrix} 1 & 0\\ e-1 & 1\end{pmatrix}\begin{pmatrix} a+1-e & 1\\ -e & 1-e\end{pmatrix}\\ =&\begin{pmatrix} a+1-e & 1\\ -1 & 0\end{pmatrix}.\end{aligned}$$

Hence, $\left(\begin{smallmatrix} a+1-e & b\\ -e & x+1-e\end{smallmatrix}\right)\in GL_2(I)$. By assumption, there exist $L\in\mathfrak{L}$, $U\in\mathfrak{U}$, $K\in\mathfrak{L}$ such that $\left(\begin{smallmatrix} a+1-e & b\\ -e & x+1-e\end{smallmatrix}\right)=LUK$, where two of L,U and K have diagonal entries 1. As in the proof of Theorem 2.1, we can find $y\in R$ such that

$a+1-e+by \in R$ is right or left invertible. Assume that $(a+1-e+by)u = 1$ for a $u \in R$. Then $(1-e)ue = 0$, and so $ue = eue$. Hence, $\big(a + b(eye)\big)(eue) = e$. Assume that $u(a+1-e+by) = 1$ for a $u \in R$. Then $(eue)\big(a+b(eye)\big) = e$. As a result, $a + b(eye) \in eRe$ is right or left invertible, i.e., eRe is a weakly stable ring. Therefore I is a weakly stable ideal by Lemma 4.1.

(2) ⇒ (3) For any $A \in GL_2(I)$, we see that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) A \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in GL_2(I)$. By assumption, there exist $L \in \mathfrak{L}, U \in \mathfrak{U}, K \in \mathfrak{L}$ such that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) A \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = LUK$, where two of L, U and K have diagonal entries 1. Let

$$U' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} L \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad L' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$W' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} K \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $A = U'L'W'$, where $U' \in \mathfrak{U}, L' \in \mathfrak{L}, W' \in \mathfrak{U}$ and two of U', L' and W' have diagonal entries 1.

(3) ⇔ (2) is proved in the same manner. □

Corollary 4.3. *Let R be a regular ring, and let $A \in GL_2(R)$. If $M_2(R)(I_2 - A)M_2(R)$ is weakly stable, then the following hold*:

(2) *There exist $L \in \mathfrak{L}, U \in \mathfrak{U}, K \in \mathfrak{L}$ such that $A = LUK$, where two of L, U and K have diagonal entries* 1.

(3) *There exist $U \in \mathfrak{U}, L \in \mathfrak{L}, W \in \mathfrak{U}$ such that $A = ULW$, where two of U, L and W have diagonal entries* 1.

Proof. Let $M_2(R)(I_2 - A)M_2(R)$ be a weakly stable ideal of $M_2(R)$. Then we can find an ideal I of R such that $M_2(R)(I_2 - A)M_2(R) = M_2(I)$. By hypothesis, $M_2(I)$ is a weakly stable ideal of $M_2(R)$. Choose $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(R)$. Then $I \cong eM_2(I)e$ and $R \cong eM_2(R)e$. Clearly, I is a regular ideal of R. For any idempotent $efe \in eM_2(I)e$ with $f \in M_2(I)$, we see that $(efe)eM_2(R)e(efe) = (efe)M_2(R)(efe)$. Obviously, $efe \in M_2(I)$ is an idempotent. In view of Lemma 4.1, $(efe)M_2(R)(efe)$ is a weakly stable ring, then so is $(efe)eM_2(R)e(efe)$. By using Lemma 4.1 again, $eM_2(I)e$ is a weakly stable ideal of $eM_2(R)e$. Hence I is a weakly stable ideal of R, and therefore we complete the proof by Theorem 4.2. □

Theorem 4.4. *Let I be a regular ideal of a ring R. Then the following are equivalent*:

(1) *I is a weakly stable ideal.*

(2) *For any $A \in GL_2(I)$, there exist $L, K \in \mathfrak{L}, U \in \mathfrak{U}$ such that $LAKU = I_2$, where two of L, K and U have diagonal entries* 1.

(3) *For any $A \in GL_2(I)$, there exist $U, W \in \mathfrak{U}, K \in \mathfrak{L}$ such that $UAWK = I_2$, where two of U, W and K have diagonal entries* 1.

Proof. (1) ⇒ (2) Let $A \in GL_2(I)$ and $B = I_2 - A$. As in the proof of Theorem 4.2, we can find an idempotent $e \in I$ such that $\left(\begin{smallmatrix} e & 0 \\ 0 & e \end{smallmatrix}\right) - B \in GL_2(eRe)$ and eRe is a weakly stable ring. By virtue of Theorem 3.3, we have $L \in \mathfrak{L}, U \in$

$\mathfrak{U}, K \in \mathfrak{L}$ such that $L\big(\left(\begin{smallmatrix} e & 0 \\ 0 & e \end{smallmatrix}\right) - B\big)UK = \left(\begin{smallmatrix} e & 0 \\ 0 & e \end{smallmatrix}\right)$, where two of L, U and K have diagonal entries e. Let

$$L' = \begin{pmatrix} 1-e & 0 \\ 0 & 1-e \end{pmatrix} + L, \quad U' = \begin{pmatrix} 1-e & 0 \\ 0 & 1-e \end{pmatrix} + U,$$
$$K' = \begin{pmatrix} 1-e & 0 \\ 0 & 1-e \end{pmatrix} + K.$$

Then $L'AK'U' = L'(I_2 - B)K'U' = I_2$, where two of L', K' and U' have diagonal entries 1.

(2) $\Rightarrow$ (1) Given $ax+b=e$ with $a, x, b \in eRe$, then $(a+1-e)(x+1-e)+b=1$. As in the proof of Theorem 4.2, $\left(\begin{smallmatrix} a+1-e & b \\ -e & x+1-e \end{smallmatrix}\right) \in GL_2(I)$; hence, there exist $L, K \in \mathfrak{L}, U \in \mathfrak{U}$ such that $L\left(\begin{smallmatrix} a+1-e & b \\ -e & x+1-e \end{smallmatrix}\right)KU = I_2$, where two of L, K and U have diagonal entries 1. As in the proof of Theorem 3.1, we can find $y \in R$ such that $a+1-e+by \in R$ is right or left invertible. Analogously to the discussion in Theorem 4.2, we show that $a+b(eye) \in eRe$ is right or left invertible, i.e., eRe is a weakly stable ring. According Lemma 4.1, I is a weakly stable ideal.

(2) $\Leftrightarrow$ (3) is proved in the same manner. $\square$

Following Lu et al., an ideal I of a regular ring R is said to satisfy 1-comparability if for any $x, y \in I$, either xR is isomorphic to a submodule of yR or yR is isomorphic to a submodule of xR (see [10]). Let e be a primitive idempotent in a regular ring R. By virtue of [10, Example 1.2], ReR is an ideal satisfying 1-comparability.

Corollary 4.5. *Let I be an ideal of a regular ring R. If I satisfies 1-comparability, then the following hold*:

1. (1) *For any $A \in GL_2(I)$, there exist $L, K \in \mathfrak{L}, U \in \mathfrak{U}$ such that $LAKU = I_2$, where two of L, K and U have diagonal entries 1.*
2. (2) *For any $A \in GL_2(I)$, there exist $U, W \in \mathfrak{U}, K \in \mathfrak{L}$ such that $UAWK = I_2$, where two of U, W and K have diagonal entries 1.*

Proof. It is easy to verify that I is a weakly stable ideal of R. Therefore we complete the proof by Theorem 4.2 and Theorem 4.4. $\square$

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