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SOME PROPERTIES OF TENSOR CENTRE OF GROUPS

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Abstract. Let $G \otimes G$ be the tensor square of a group G. The set of all elements a in G such that $a \otimes g = 1_{\otimes}$, for all g in G, is called the *tensor centre* of G and denoted by $Z^{\otimes}(G)$. In this paper some properties of the tensor centre of G are obtained and the capability of the pair of groups (G, G') is determined. Finally, the structure of $J_2(G)$ will be described, where $J_2(G)$ is the kernel of the map $\kappa : G \otimes G \to G'$.

1. Introduction

Let G and H be two groups which act on themselves by conjugation ${}^{g}g' = gg'g^{-1}$ and on each other in such a way that the following compatibility conditions are satisfied:

$$^{(^{g}h)}g' = {}^{g}(^{h}(^{g^{-1}}g')),\ ^{(^{h}g)}h' = {}^{h}(^{g}(^{h^{-1}}h'))$$

for all g, g' in G and h, h' in H.

The non-abelian tensor product $G \otimes H$ was introduced by R. Brown and J.-L. Loday in [2] as the group generated by all the symbols $g \otimes h$ subject to the following relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h),\ g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$$

for all g, g' in G, h, h' in H.

If $G = H$, then $G \otimes G$ is considered to be the non-abelian tensor square, so that it will be focused in this paper. There exists an action of G on $G \otimes G$ satisfying ${}^{g}(g' \otimes h) = {}^{g}g' \otimes {}^{g}h$ for all g, g', h in G. The following relations hold for all g, g', h, h' in G (see [1] for more details).

(i) ${}^{g}(g^{-1} \otimes h) = (g \otimes h)^{-1} = {}^{h}(g \otimes h^{-1})$;
(ii) ${}^{(g\otimes h)}(g' \otimes h') = {}^{[g,h]}(g' \otimes h')$;
(iii) $[g, h] \otimes h' = (g \otimes h)^{h'}(g \otimes h)^{-1}$;
(iv) $g' \otimes [g, h] = {}^{g'}(g \otimes h)(g \otimes h)^{-1}$;
(v) $[g \otimes h, g' \otimes h'] = [g, h] \otimes [g', h']$.

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Clearly there is an epimorphism κ from $G\otimes G$ onto G' given by $g\otimes h \mapsto [g,h]$, so that the kernel of κ is denoted by $J_2(G)$. The subgroup $J_2(G)$ of $G \otimes G$ is central and its elements are fixed under the action of G by [1, Proposition 4].

Let G be a group and N a normal subgroup G. A *relative central extension* of the pair (G, N) consists of a group homomorphism $\sigma : M \longrightarrow G$ together with an action of G on M such that:

(i) $\sigma(M) = N$;

(ii) $\sigma({}^g m) = g(\sigma m)g^{-1}$ for all $g \in G$, $m \in M$;

(iii) ${}^{\sigma(m)}m' = mm'm^{-1}$ for all m, m' in M;

(iv) ker $(\sigma) \subseteq Z_G(M)$, where $Z_G(M)$ is the set of all elements m in M, which are fixed under the action of G.

The pair (G, N) is *capable* if it admits a relative central extension such that ker σ is equal to $Z_G(M)$. Let $\nabla(G)$ be the subgroup of $J_2(G)$, generated by all the elements $g \otimes g$ for all g in G. Define $G \wedge G = G \otimes G/\nabla(G)$, which is called the *exterior square* of G. The set of all elements $g \in G$ such that $g \wedge g' = 1$ for all $g' \in G$, is said to be the *exterior centre* of G and denoted by $Z^\wedge(G)$. In [4], it is shown that G is capable if and only if $Z^\wedge(G) = 1$. In Section 2 using this fact, we give some properties of this important subgroup.

Finally, in Section 2 we give an exact sequence related to central subgroup $Z_G(G \otimes G)$ and the tensor centre $Z^{\otimes}(G)$. In Section 3, we show that for any two groups A and B, $\nabla(A \times B) \cong \nabla(A * B)$ and then give the structure of the factor group of $J_2(A \times B)$. Note that $A * B$ means the free product of the groups A and B.

2. Tensor centre of a group

In this section, the concept of tensor centre $Z^{\otimes}(G)$ of a group G is discussed and some of its properties are obtained. In fact, an exact sequence related to the tensor centre is given and it is proved that $Z^{\otimes}(A \times B) = Z^{\otimes}(A) \times Z^{\otimes}(B)$, when $(|A|, |B|) = 1$.

Lemma 2.1. *Let N be a normal subgroup of a finite group G. Then the sequence*

$$0 \longrightarrow N \cap Z^{\otimes}(G) \longrightarrow Z^{\otimes}(G) \longrightarrow Z^{\otimes}(G/N)$$

is exact.

Proof. The result is obtained by using the epimorphism $G \otimes G \longrightarrow G/N \otimes G/N$. □

Note that the above lemma with $N = G^{'}$ implies the exactness of the following sequence

$$0 \longrightarrow G^{'} \cap Z^{\otimes}(G) \longrightarrow Z^{\otimes}(G) \longrightarrow Z^{\otimes}(G/G^{'}),$$

in which $Z^{\otimes}(G/G^{'}) = 0$ by [3, Proposition 18], and hence $Z^{\otimes}(G) \leq G^{'}$.

Now, we determine the centre of G as a factor group. To do this, consider the epimorphism $\kappa : G \otimes G \overset{[\,,\,]}{\longrightarrow} G^{'}$ with the kernel $J_2(G)$. The restriction of κ to $Z_G(G \otimes G)$ and the fact that $Z^{\otimes}(G) \leq G^{'}$, imply the following:

Proposition 2.2. *If G is a finite group, then the following sequence*

$$0 \longrightarrow J_2(G) \longrightarrow Z_G(G \otimes G) \longrightarrow Z^{\otimes}(G) \longrightarrow 0$$

is exact.

Corollary 2.3. *If G is a finite group and $Z^{\otimes}(G)$ is trivial, then the pair $(G, G^{'})$ is capable.*

Proof. It is easy to check that the map κ defined as above is a relative central extension and so the result holds. □

Extending the construction of $Z_G(G \otimes G)$, one can easily show that $Z_{G'}(G \otimes G)$ is equal to the centre of $G \otimes G$. Put $K = \langle t \in G' \mid t \otimes s = 1_{G \otimes G}, \forall\ s \in G' \rangle$ and as in Proposition 2.2, the following sequence is exact:

$$0 \longrightarrow J_2(G) \longrightarrow Z_{G'}(G \otimes G) \longrightarrow K \longrightarrow 0.$$

It is easily seen that $\mathrm{Inn}(G \otimes G) \cong G^{'}/K$.

Proposition 2.4. *Under the above assumption, if G is a finite group and K is trivial, then $\mathrm{Inn}(G \otimes G)$ is isomorphic with $G^{'}$. In particular, $G^{'}$ is a capable group.*

Suppose A and B are two groups. Consider the tensor product $(A \times B) \otimes (A \times B)$ and put

$$M^{\otimes} = \langle (a, 1) \otimes (1, b)) \mid a \in A,\ b \in B) \rangle.$$

Using this notation, the following lemma is obtained immediately.

Lemma 2.5. *Under the above assumption, $e(M^{\otimes}) \mid \gcd(e(A), e(B))$, where $e(X)$ is the exponent of the group X.*

Now if $(a, b) \in Z^{\otimes}(A \times B)$, then $a \in Z^{\otimes}(A)$ and $b \in Z^{\otimes}(B)$, which implies that the homomorphism

$$\theta : Z^{\otimes}(A \times B) \longrightarrow Z^{\otimes}(A) \times Z^{\otimes}(A)$$

is injection. Clearly, θ is also surjective when $\gcd(e(A), e(B)) = 1$. For, let $a \in Z^{\otimes}(A)$ and $b \in Z^{\otimes}(B)$, then for all (c, d) in $A \times B$ and using Lemma 2.5, we obtain

$$\begin{aligned}(a, b) \otimes (c, d) &= ((a, 1)(1, b) \otimes (c, 1)(1, d)) \\ &= [(1, b) \otimes (c, 1)][(1, b) \otimes (1, d)][(a, 1) \otimes (c, 1)][(a, 1) \otimes (1, d)] = 1.\end{aligned}$$

Hence, the pair (a, b) is in $Z^{\otimes}(A \times B)$ which shows that θ is an isomorphism and so

$$Z^{\otimes}(A \times B) \cong Z^{\otimes}(A) \times Z^{\otimes}(A).$$

By the similar method one can show that

$$Z^{\wedge}(A \times B) \cong Z^{\wedge}(A) \times Z^{\wedge}(A),$$

if $\gcd(e(A), e(B)) = 1$. In particular, if $G = G_{p_1} \times \cdots \times G_{p_k}$ is a nilpotent group with the Sylow p-subgroups G_{p_i}, then

$$Z^{\wedge}(G) = Z^{\wedge}(G_{p_1}) \times \cdots \times Z^{\wedge}(G_{p_k}).$$

From the above discussion, the following result is obtained, which was already shown in [8, Corollary 3.5].

Proposition 2.6. *If* $\gcd(e(A), e(B)) = 1$*, then the direct product* $A \times B$ *is capable if and only if* A*,* B *are capable.*

3. Some properties of $J_2(G)$

In this section, we focus on $J_2(G)$ in order to establish a similar exact sequence to Ganea with familiar terms. More precisely, the abelian groups in Ganea sequence, given in [5], are the homomorphic images of the abelian groups in our exact sequence. In addition, using a fact of [6] we also establish a result on $J_2(A \times B)$.

Let us recall the following well-known commutative diagram with exact rows and central columns given in [1]:

$$
(1) \qquad \begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
H_3(G) & \longrightarrow & \Gamma(G^{ab}) & \longrightarrow & J_2(G) & \longrightarrow & H_2(G) & \longrightarrow & 0 \\
= \downarrow & & = \downarrow & & \downarrow & & \downarrow & & \\
H_3(G) & \longrightarrow & \Gamma(G^{ab}) & \longrightarrow & G \otimes G & \longrightarrow & G \wedge G & \longrightarrow & 1, \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

where Γ is the Whitehead's quadratic functor and G^{ab} is the abelianized of G (see [9]).

The following result of [1] is needed for our further investigation.

Proposition 3.1. *If* N *is a normal subgroup of a given group* G *with the central extension*

$$1 \longrightarrow N \longrightarrow G \stackrel{\pi}{\longrightarrow} G/N \longrightarrow 1,$$

then the following sequence is exact

$$(N \otimes G) \times (G \otimes N) \stackrel{l}{\longrightarrow} G \otimes G \stackrel{\pi \otimes \pi}{\longrightarrow} G/N \otimes G/N \longrightarrow 1,$$

in which Im $l \leq J_2(G)$.

Now, by the above discussion we are able to prove the following

Theorem 3.2. *Under the above assumptions and notations, the sequence*

$$(*) \qquad 0 \longrightarrow \mathrm{Im}\ l \xrightarrow{inc} J_2(G) \xrightarrow{\pi_1} J_2(G/N) \xrightarrow{\kappa_1} G^{'} \cap N \longrightarrow 0$$

is exact, where $\kappa_1(\bar{g} \otimes \bar{h}) = [g,h]$ *for all* $\bar{g}$, $\bar{h}$ *in* G/N *and* π_1 *is the restriction of* $\pi \otimes \pi$ *to* $J_2(G)$ *as in Proposition 3.1.*

Proof. The kernel of the homomorphism π_1 is equal to $\ker(\pi \otimes \pi) \cap J_2(G)$ but $\ker\pi_1 = \mathrm{Im}\ l \leq J_2(G)$, which gives the exactness of the left side of $(*)$. On the other hand, κ_1 is well-defined, since N is a central subgroup of G. It is easily seen that $\mathrm{Im}\pi_1 = \ker\kappa_1$, as required. □

Corollary 3.3. *Under the above assumptions,*
(i) $Z_G(G \otimes G) \cong J_2(G/Z^{\otimes}(G))$;
(ii) *if* $N \leq Z^{\otimes}(G)$, *then* $e(J_2(G/N)) \mid e(J_2(G))\ e(N)$.

Proof. Part (i) follows from the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_2(G) & \longrightarrow & Z_G(G \otimes G) & \longrightarrow & Z^{\otimes}(G) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J_2(G) & \longrightarrow & J_2(G/Z^{\otimes}(G)) & \longrightarrow & Z^{\otimes}(G) & \longrightarrow & 0 \end{array}$$

and the proof of (ii) follows easily. □

Remark 3.4. Note that the sequence

$$0 \longrightarrow Z_G(G \otimes G) \longrightarrow G \otimes G \longrightarrow G'/Z^{\otimes}(G) \longrightarrow 1$$

is exact and when G is a finite group, it is equivalent to the following exact sequence.

$$0 \longrightarrow J_2(G/Z^{\otimes}(G)) \longrightarrow G/Z^{\otimes}(G) \otimes G/Z^{\otimes}(G) \longrightarrow G'/Z^{\otimes}(G) \longrightarrow 1,$$

as $G \otimes G \cong G/Z^{\otimes}(G) \otimes G/Z^{\otimes}(G)$.

Consider the Ganea exact sequence as in [5]

$$G \otimes Z \xrightarrow{\gamma} M(G) \longrightarrow M(G/Z) \longrightarrow G^{'} \cap Z \longrightarrow 0,$$

where Z is a central subgroup of G and $M(G)$ denotes the Schur multiplier of G.

Now, we exhibit a close relation between the above sequence and the exact sequence $(*)$.

Clearly the following diagram of exact sequences are commutative.

$$\begin{array}{ccccccccc}
(G\otimes Z)\times(Z\otimes G) & \xrightarrow{l} & J_2(G) & \longrightarrow & J_2(G/Z) & \longrightarrow & G'\cap Z & \longrightarrow & 0\\
\downarrow\alpha & & \downarrow & & \downarrow & & \downarrow= & & \\
G\otimes Z & \xrightarrow{\gamma} & M(G) & \longrightarrow & M(G/Z) & \longrightarrow & G'\cap Z & \longrightarrow & 0,\\
\downarrow & & \downarrow & & \downarrow & & & & \\
0 & & 0 & & 0 & & & &
\end{array}$$

where $\alpha((g\otimes z),(z'\otimes g'))=(g\otimes z)(g'\otimes z')^{-1}$ for all g,g' in G and z,z' in Z. In particular, the following diagram is commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_2(G) & \longrightarrow & J_2(G/Z^{\otimes}(G)) & \longrightarrow & Z^{\otimes}(G) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M(G) & \longrightarrow & M(G/Z^{\otimes}(G)) & \longrightarrow & Z^{\otimes}(G) & \longrightarrow & 0.
\end{array}$$

Clearly, if we replace the left hand side of the above diagram with $\mathrm{Im}l$ and $\mathrm{Im}\gamma$, respectively. Then the diagram still remains commutative.

Let A and B be two arbitrary groups then in [6], N. D. Gilbert has shown that

$$J_2(A*B)\cong J_2(A)\times J_2(B)\times(A^{ab}\otimes B^{ab}).$$

Finally, in the remaining part of the paper we present a similar isomorphism for $J_2(A\times B)$. Clearly, by the diagram (1) the following sequences are exact.

$$\Gamma((A*B)^{ab})\xrightarrow{\lambda}J_2(A*B)\longrightarrow H_2(A*B)\longrightarrow 0,$$

$$\Gamma((A\times B)^{ab})\xrightarrow{\lambda'}J_2(A\times B)\longrightarrow H_2(A\times B)\longrightarrow 0.$$

Using the above discussion we have the following:

Proposition 3.5. *There is an isomorphism between* $\mathrm{Im}\lambda$ *and* $\mathrm{Im}\lambda'$, *i.e.,*

$$\nabla(A*B)\cong\nabla(A\otimes B).$$

Proof. We know that $\mathrm{Im}\lambda$ is generated by the elements $x\otimes x$ for all x in $A*B$ and $\mathrm{Im}\lambda'$ is generated by the elements $(a,b)\otimes(a,b)$ for all (a,b) in $A\times B$. The epimorphism $A*B\longrightarrow A\times B$ induces an epimorphism

$$\alpha:(A*B)\otimes(A*B)\longrightarrow(A\times B)\otimes(A\times B).$$

Note that, the restriction of α to $\mathrm{Im}\lambda$ is again an epimorphism onto $\mathrm{Im}\lambda'$. So, it is enough to find a left inverse for $\alpha|_{\mathrm{Im}\lambda}$. Clearly there is a homomorphism

$$(A\times B)\otimes(A\times B)\xrightarrow{\zeta}(A\otimes B)\times(A\otimes B)$$

such that $\zeta((a,b)\otimes(c,d))=((a\otimes c),(b\otimes d))$ for all a,c in A and b,d in B. Now, observe that $\mathrm{Im}\lambda'$ is mapped into $J_2(A)\times J_2(B)$ by the homomorphism ζ. In [6, Proposition 3.1], it is shown that $\eta:J_2(A)\times J_2(B)\longrightarrow J_2(A*B)$ is also a homomorphism. Therefore

$$\eta(\zeta|_{\mathrm{Im}\lambda'}((a,b)\otimes(a,b))=(a\otimes a)(b\otimes b)\in J_2(A*B)$$

for all (a,b) in $A\times B$. On the other hand, the abelianization of $A\times B$ induces a homomorphism

$$\sigma:(A\times B)\otimes(A\times B)\longrightarrow(A^{ab}\otimes A^{ab})\oplus(A^{ab}\otimes B^{ab})\oplus(B^{ab}\otimes A^{ab})\oplus(B^{ab}\otimes B^{ab}).$$

Assume π_2 is the projection to the second summand. Then for all (a,b) in $A\times B$,

$$\alpha(\pi_2(\sigma|_{Im\lambda'}((a,b)\otimes(a,b)))) = (a\otimes b)(b\otimes a)$$

would be in $J_2(A*B)$, where α is the well-defined homomorphism from $A^{ab}\otimes B^{ab}$ into $J_2(A*B)$, as given in [6]. Now, let the map $\theta : \mathrm{Im}\lambda' \longrightarrow \mathrm{Im}\lambda \leq J_2(A*B)$ be the product of $\eta(\zeta|_{\mathrm{Im}\lambda'})$ by $\alpha(\pi_2(\sigma|_{\mathrm{Im}\lambda'}))$. Then θ is a homomorphism and $\theta(\alpha|_{\mathrm{Im}\lambda})$ is the identity, since $J_2(A*B)$ is abelian and

$$\theta((a,b)\otimes(a,b)) = (a\otimes a)(b\otimes b)(a\otimes b)(b\otimes a) = ab\otimes ab.$$

This completes the assertion. □

Now, one observes that the map $\mu : A*B \longrightarrow A\times B$ induces a homomorphism

$$\bar{\mu} : J_2(A*B) \longrightarrow J_2(A\times B),$$

which is injective and so $J_2(A*B) \trianglelefteq J_2(A\times B)$.

Finally, we are able to prove the following:

Theorem 3.6. *Let A and B be two groups. Then*

$$J_2(A\times B)/J_2(A*B) \cong A^{ab}\otimes B^{ab}.$$

Proof. Clearly, by [7]

$$H_2(A\times B) \cong H_2(A)\times H_2(B)\times A^{ab}\otimes B^{ab}$$

and

$$H_2(A*B) \cong H_2(A)\times H_2(B),$$

where $H_2(X)$ is the second homology of the group X. Thus $H_2(A\times B)/H_2(A*B) \cong A^{ab}\otimes B^{ab}$ and hence there exists an epimorphism

$$J_2(A\times B) \longrightarrow H_2(A\times B) \longrightarrow H_2(A\times B)/H_2(A*B)$$

such that $J_2(A*B)$ is contained in the kernel of the composition maps. So we have the following surjective

$$J_2(A\times B)/J_2(A*B) \longrightarrow H_2(A\times B)/H_2(A*B),$$

which gives the result. □

In particular, if A and B are two groups such that $A\otimes B = 1$, then according to the previous theorem $J_2(A\times B) \cong J_2(A*B) \cong J_2(A)\times J_2(B)$.

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