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Optimal step stress accelerated life tests for the exponential distribution under periodic inspection and type I censoring

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Abstract

In this paper, the inferences of data obtained from periodic inspection and type I censoring for the step-stress accelerated life test are studied. The exponential distribution with a failure rate function that a log-linear function of stress and the tampered failure rate model are considered. The maximum likelihood estimators of the model parameters are estimated and also the optimal stress change time which minimize the asymptotic variance of maximum likelihood estimators of parameters is determined. A numerical example will be given to illustrate the proposed inferential procedures and the sensitivity of the asymptotic variance of the estimated mean by the guessed parameters is investigated.

Keywords: Asymptotic variance, exponential distribution, Fisher information, optimum plan, periodic inspection, tampered failure rate model.

1. Introduction

The accelerated life testings (ALTs) are used to reduce the long testing time under environment conditions. Testing units are on greater stress than use stress and then accelerated life testing quickly yields information on test unit. The lifetimes of test units can be continuously or intermittently inspected in the step-stress ALTs. There are three types of models that have been commonly used on analysis of step-stress ALTs. They are the tampered random variable (TRV) model by DeGroot and Goel (1979), the cumulative exposure (CE) model by Nelson (1980) and the tampered failure rate (TFR) model by Bhattacharyya and Soejoeti (1989).

The periodic inspection of test is frequently used to be possible further reduction in time and cost, but earlier studies assumed continuous inspection for ALTs. The data obtained from periodic inspection consists of only the number of failures in the inspection intervals.

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There are few works which are considered these problems when only inspection and censored data are available.

Yum and Choi (1989) first studied asymptotic optimal ALTs plans for periodic inspection and type I censoring. Nelson (1990), Miller and Nelson (1983) studied the design to determine the optimal stress change time for two-step stress ALTs and their results were extended to the case of Type I censoring under periodic observation by Bai *et al.* (1989). Seo and Yum (1993) proposed several approximate maximum likelihood estimators (MLEs) for the mean and compared them by a Monte Carlo simulation when the lifetime distribution is exponential. Islam and Ahmad (1994) studied the optimal ALTs plans for the case of the lifetime at a stress level follow Weibull distribution under the assumptions of periodic inspection and type I censoring under the constant stress ALTs. Xiong and Ji (2004) studied the statistical inference of model parameters and optimum test plans analysis using only grouped and Type I censored data obtained from a step-stress ALTs. Ahmad *et al.* (2006) generalized the previous works on the design of ALTs for periodic inspection and Type I censoring under the constant stress ALTs. Moon and Kim (2006) studied parameter estimation of the two-parameter exponential distribution under three step-stress ALTs. Moon (2008) considered the estimation of model parameters and optimum plans based on grouped and Type I censored data obtained from three step-stress ALTs, assuming that the lifetime of test units follows an exponential distribution under the TFR model.

In this paper, we consider the estimation of model parameters and optimal plans to search the stress changing time based on periodic inspection with Type I censoring under two step-stress ALTs, assuming that the lifetime of test units follows an exponential distribution under the TFR model. In Section 2, we describe the model and some necessary assumptions. In Section 3, MLEs of the parameters are obtained and the optimal two step-stress plan that minimizes the asymptotic variance of the MLE of logarithm of the mean lifetime at use stress is presented. The proposed inferential procedures are illustrated in Section 4.

2. Model and assumptions

For step-stress ALTs, all test units are simultaneously put on stress x_1 and inspections are conducted at specified times $t_{11,}t_{12,}\cdots,t_{1K(1)}$ until a preassigned time τ_1, but if all units do not fail before time τ_1, the surviving units are subjected to a stronger stress x_2 and observed at specified times $t_{21,}t_{22,}\cdots,t_{2K(2)}$ until censoring time τ_2.

At these stress levels, we assume that the lifetimes (T) of test units identically and independently follows an exponential distribution.

At stress x_i, $i=1,2$, the number of failures n_{ij} corresponding p_{ij}, probability of failures in the interval $(t_{ij-1},t_{ij}]$, $i=1,2, j=1,2cdots,K(i)$ where $K(i)$ is the number of inspections, are recorded. Also $t_{10}=0$, $t_{1K(1)}=\tau_1$ and $t_{20}=\tau_1$, $t_{2K(2)}=\tau_2$.

Some useful notations are introduced as follows.

1. n_{ij} is the number of failed test units during the inspection time interval $(t_{i,j-1},t_{ij}]$ at the stress x_i, $i=1,2$, $j=1,2,\cdots,K(i)$ and n_c is the censored test units at a fixed censoring time τ_2.
2. $n_1=\sum_{j=1}^{K(1)} n_{1j}, n_2=\sum_{j=1}^{K(2)} n_{2j}$ and $n_c=n-(n_1+n_2)$

3. $p_{ij} = P(t_{ij-1} < T \le t_{ij}), i = 1, 2, j = 1, 2, \cdots, K(i)$ and $p_c = P(\tau_2 < T < \infty)$, where $t_{10} = 0$, $t_{20} = \tau_1$.

Suppose that stress response relationship of each test unit has the log-linear function with the stress variable x_i, which is given by

$$\log \theta_i = \beta_0 + \beta_1 x_i, \quad i = 1, 2, \tag{2.1}$$

where β_0 and β_1 are unknown model parameters.

The numbers of failures n_{ij}, $i = 1, 2, j = 1, 2, \cdots, K(i)$ are used to estimate the model parameters β_0 and β_1 and then the model is extrapolated to make inferences on the mean lifetime, percentiles, the reliability, failure rates, etc. under the use condition.

The probability distribution function $f(t)$ for a test unit lifetime T at stress x_1 is given by

$$f(t) = \begin{cases} \dfrac{1}{\theta_1} \exp\left(-\dfrac{t}{\theta_1}\right), & t \le \tau_1 \\ \dfrac{1}{\theta_2} \exp\left(-\dfrac{\tau_1}{\theta_1} - \dfrac{t - \tau_1}{\theta_2}\right), & t > \tau_1 \end{cases}. \tag{2.2}$$

We use the following notations to simplify equations

$$u_{ij-1}^{(k)}(\beta_0, \beta_1) = u_{ij-1}^{(k)} = (t_{ij-1} - \tau_{i-1})\, x_i^k \exp(-\beta_0 - \beta_1 x_i) + \tau_{i-1} x_{i-1}^k \exp\left(-\beta_0 - \beta_1 x_{i-1}\right),$$
$$u_{ij}^{(k)}(\beta_0, \beta_1) = u_{ij}^{(k)} = (t_{ij} - \tau_{i-1})\, x_i^k \exp(-\beta_0 - \beta_1 x_i) + \tau_{i-1} x_{i-1}^k \exp\left(-\beta_0 - \beta_1 x_{i-1}\right)$$

for $i = 1, 2, j = 1, 2, \cdots, K(i)$ and $k = 0, 1$ where $x_0 = 0$ and $\tau_0 = 0$.

3. Maximum likelihood estimators and optimum plan

Now, we obtain MLEs of the model parameters β_0 and β_1 and study the optimum test plan for searching the stress change times minimizing the asymptotic variance of the MLE of logarithm of mean lifetime at use stress. The likelihood function is given by

$$L \propto \prod_{i=1}^{2} \prod_{j=1}^{K(i)} p_{ij}^{n_{ij}} \cdot p_c^{n_c}$$

where for $i = 1, 2, j = 1, 2, \cdots, K(i)$,

$$p_{1j} = P\left(t_{1j-1} < T \le t_{1j}\right) = \exp\left(-u_{1j-1}^{(0)}\right) - \exp\left(-u_{1j}^{(0)}\right),$$
$$p_{2j} = P\left(t_{2j-1} < T \le t_{2j}\right) = \exp\left(-u_{2j-1}^{(0)}\right) - \exp\left(-u_{2j}^{(0)}\right),$$
$$p_c = P\left(T > \tau_2\right) = \exp\left(-u_{2K(2)}^{(0)}\right).$$

Thus, the log likelihood function is a function of unknown parameters β_0 and β_1 given by as follows:

$$\begin{aligned}\log L(\beta_0,\beta_1) &\propto \sum_{i=1}^{2}\sum_{j=1}^{K(i)} n_{ij}\log p_{ij} + n_c \log p_c \\ &\propto \sum_{i=1}^{2}\sum_{j=1}^{K(i)} n_{ij}\log\left[\exp(-u_{ij-1}^{(0)}) - \exp(-u_{ij}^{(0)})\right] + n_c\log\left[\exp(-u_{2K(2)}^{(0)})\right].\end{aligned}$$

The MLEs for the model parameters β_0 and β_1 can be obtained by solving the following equation in (3.1) using the Newton-Raphson method.

$$\frac{\partial}{\partial\beta_k}\log L(\beta_0,\beta_1) = \sum_{i=1}^{2}\sum_{j=1}^{K(i)} n_{ij}\cdot\frac{1}{p_{ij}}\left(\frac{\partial p_{ij}}{\partial\beta_k}\right) + n_c\cdot\frac{1}{p_c}\left(\frac{\partial p_c}{\partial\beta_k}\right) = 0 \tag{3.1}$$

for $k = 0, 1$ where

$$\begin{aligned}\frac{\partial p_{1j}}{\partial\beta_k} &= \frac{x_1^k}{\theta_1}\left(t_{1j-1}\exp(-u_{1j-1}^{(0)}) - t_{1j}\exp(-u_{1j}^{(0)})\right), \\ \frac{\partial p_{2j}}{\partial\beta_k} &= \frac{x_2^k}{\theta_2}\left(t_{2j-1}\exp(-u_{2j-1}^{(0)}) - t_{2j}\exp(-u_{2j}^{(0)})\right) - \tau_1\left(\frac{x_2^k}{\theta_2} - \frac{x_1^k}{\theta_1}\right)p_{2j}, \\ \frac{\partial p_c}{\partial\beta_k} &= u_{2K(2)}^{(k)}\exp\left(-u_{2K(2)}^{(0)}\right).\end{aligned}$$

The Fisher information matrix F is defined as $F = (f_{st})$, $s, t = 1, 2$ and obtained by taking the expected value of the second partial and mixed partial derivatives of $\log L(\beta_0, \beta_1)$ with respect to β_0 and β_1 given by

$$\begin{aligned}\frac{\partial^2\log L(\beta_0,\beta_1)}{\partial\beta_k^2} &= \sum_{i=1}^{2}\sum_{j=1}^{K(i)}\frac{n_{ij}}{p_{ij}}\left[\frac{\partial^2 p_{ij}}{\partial\beta_k^2} - \frac{1}{p_{ij}}\left(\frac{\partial p_{ij}}{\partial\beta_k}\right)^2\right] + \frac{n_c}{p_c}\left[\frac{\partial^2 p_c}{\partial\beta_k^2} - \frac{1}{p_c}\left(\frac{\partial p_c}{\partial\beta_k}\right)^2\right], \\ \frac{\partial^2\log L(\beta_0,\beta_1)}{\partial\beta_k\partial\beta_l} &= \sum_{i=1}^{2}\sum_{j=1}^{K(i)}\frac{n_{ij}}{p_{ij}}\left[\frac{\partial^2 p_{ij}}{\partial\beta_k\partial\beta_l} - \frac{1}{p_{ij}}\left(\frac{\partial p_{ij}}{\partial\beta_k}\right)\left(\frac{\partial p_{ij}}{\partial\beta_l}\right)\right] \\ &\quad + \frac{n_c}{p_c}\left[\frac{\partial^2 p_c}{\partial\beta_k\partial\beta_l} - \frac{1}{p_c}\left(\frac{\partial p_c}{\partial\beta_k}\right)\left(\frac{\partial p_c}{\partial\beta_l}\right)\right],\end{aligned}$$

where $k \neq l = 0, 1$.

The expected value of the second partial and mixed partial derivatives of $\log L(\beta_0, \beta_1)$

with respect to β_0 and β_1 are

$$
\begin{aligned}
f_{ss} = -E\left(\frac{\partial^2 \log L}{\partial \beta_k^2}\right) =& n\left\{\sum_{i=1}^{2}\sum_{j=1}^{K(i)}\left[\frac{1}{p_{ij}}\left(\frac{\partial p_{ij}}{\partial \beta_k}\right)^2 - \left(\frac{\partial^2 p_{ij}}{\partial \beta_k^2}\right)\right] + \frac{1}{p_c}\left(\frac{\partial p_c}{\partial \beta_k}\right)^2 - \frac{\partial^2 p_c}{\partial \beta_k^2}\right\} \\
=& n\left\{\sum_{i=1}^{2} w_{ik} + w_{ck}\right\}, \\
f_{st} = -E\left(\frac{\partial^2 \log L}{\partial \beta_k \partial \beta_l}\right) =& n\sum_{i=1}^{2}\sum_{j=1}^{K(i)}\left[\frac{1}{p_{ij}}\left(\frac{\partial p_{ij}}{\partial \beta_k}\right)\left(\frac{\partial p_{ij}}{\partial \beta_l}\right) - \left(\frac{\partial^2 p_{ij}}{\partial \beta_k \partial \beta_l}\right)\right] \\
& + n\left[\frac{1}{p_c}\left(\frac{\partial p_c}{\partial \beta_k}\right)\left(\frac{\partial p_c}{\partial \beta_l}\right) - \left(\frac{\partial^2 p_c}{\partial \beta_k \partial \beta_l}\right)\right] \\
=& n\left\{\sum_{i=1}^{2} Q_i + Q_c\right\},
\end{aligned}
$$

for $k \neq l = 0, 1,\ s \neq t = 1, 2$, where for $i = 1, 2$,

$$
\begin{aligned}
w_{ik} &= \sum_{j=1}^{K(i)}\left[\frac{1}{p_{ij}}\left(\frac{\partial p_{ij}}{\partial \beta_k}\right)^2 - \frac{\partial^2 p_{ij}}{\partial \beta_k^2}\right], \quad w_{ck} = \left[\frac{1}{p_c}\left(\frac{\partial p_c}{\partial \beta_k}\right)^2 - \frac{\partial^2 p_c}{\partial \beta_k^2}\right], \\
Q_i &= \sum_{j=1}^{K(i)}\left[\frac{1}{p_{ij}}\left(\frac{\partial p_{ij}}{\partial \beta_k}\right)\left(\frac{\partial p_{ij}}{\partial \beta_l}\right) - \frac{\partial^2 p_{ij}}{\partial \beta_k \partial \beta_l}\right], \quad Q_c = \left[\frac{1}{p_c}\left(\frac{\partial p_c}{\partial \beta_k}\right)\left(\frac{\partial p_c}{\partial \beta_l}\right) - \frac{\partial^2 p_c}{\partial \beta_k \partial \beta_l}\right] \quad (3.2)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 p_{1j}}{\partial \beta_k^2} &= \left((u_{1j-1}^{(k)})^2 - u_{1j-1}^{(2k)}\right)\exp(-u_{1j-1}^{(0)}) - \left((u_{1j}^{(k)})^2 - u_{1j}^{(2k)}\right)\exp(-u_{1j}^{(0)}), \\
\frac{\partial^2 p_{1j}}{\partial \beta_k \beta_l} &= u_{1j-1}^{(l)}\left(u_{1j-1}^{(k)} - 1\right)\exp(-u_{1j-1}^{(0)}) - u_{1j}^{(l)}\left(u_{1j}^{(k)} - 1\right)\exp(-u_{1j}^{(0)}), \\
\frac{\partial^2 p_{2j}}{\partial \beta_k^2} &= \left((u_{2j-1}^{(k)})^2 - u_{2j-1}^{(2k)}\right)\exp(-u_{2j-1}^{(0)}) - \left((u_{2j}^{(k)})^2 - u_{2}^{(2k)}\right)\exp(-u_{2j}^{(0)}), \\
\frac{\partial^2 p_{2j}}{\partial \beta_k \beta_l} &= u_{2j-1}^{(l)}\left(u_{2j-1}^{(k)} - 1\right)\exp(-u_{2j-1}^{(0)}) - u_{2j}^{(l)}\left(u_{2j}^{(k)} - 1\right)\exp(-u_{2j}^{(0)}), \\
\frac{\partial^2 p_c}{\partial \beta_k^2} &= \left((u_{2K(2)}^{(k)})^2 - u_{2K(2)}^{(2k)}\right)\exp\left(-u_{2K(2)}^{(0)}\right), \\
\frac{\partial^2 p_c}{\partial \beta_k \beta_l} &= u_{2K(2)}^{(l)}\left(u_{2K(2)}^{(k)} - 1\right)\exp\left(-u_{2K(2)}^{(0)}\right).
\end{aligned}
$$

We consider the optimum plan for determining τ_1 for two-step stress ALTs, which minimizes the asymptotic variance of MLE of logarithm of mean lifetime.

The asymptotic covariance matrix V of $\widehat{\beta}_0$ and $\widehat{\beta}_1$ is given by

$$
V = F^{-1} = (f_{st})^{-1}.
$$

Then the asymptotic variance of $\log \widehat{\theta}_0$ is given by

$$nAvar(\log \widehat{\theta}_0) = n(1, x_0)V(1, x_0)'$$
$$= \frac{w_{11} + w_{21} + w_{c1}}{(w_{10} + w_{20} + w_{c0})(w_{11} + w_{21} + w_{c1}) - (Q_1 + Q_2 + Q_c)^2}. \tag{3.3}$$

The optimal change time τ_1^* for two-step stress ALTs minimizing the asymptotic variance, $nAvar(\log \widehat{\theta}_0)$, in (3.3) is unique solution of the equations given by

$$\begin{aligned} f(\tau_1) = &\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right)[2(Q_1 + Q_2 + Q_c)(w_{11} + w_{21} + w_{c1})(Q_2 + Q_c) \\ &- (w_{21} + w_{c1})(Q_1 + Q_2 + Q_c)^2 - (w_{20} + w_{c0})(w_{11} + w_{21} + w_{c1})^2] \\ &+ (p_c + \sum_{j=1}^{K(2)} p_{2j})[\left(\frac{x_2^2}{\theta_2} - \frac{x_1^2}{\theta_1}\right)(Q_1 + Q_2 + Q_c)^2 \\ &- 2\left(\frac{x_2}{\theta_2} - \frac{x_1}{\theta_1}\right)(Q_1 + Q_2 + Q_c)(w_{11} + w_{21} + w_{c1}) \\ &+ \left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right)(w_{11} + w_{21} + w_{c1})^2], \end{aligned} \tag{3.4}$$

where $w_{ik}, w_{ck}, Q_i, Q_c, i = 1, 2$ are given in (3.2).

4. Examples

The data from periodic inspections in ALTs consists of only the number of failures in each inspection interval $(t_{ij-1}, t_{ij}]$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, K(i)$ where m is stress level and $K(i)$ is the number of inspection in each stress level.

The data are simulated from (2.2) to obtain the MLEs for model parameters β_0, β_1 and optimal stress change time τ_1, based on $\beta_0 = 3.0$, $\beta_1 = -2.0$, $x_1 = 0.6$, $x_2 = 1.0$, $\tau_1 = 4.1933$ and $\tau_2 = 8.5682$.

Also the number of inspections on each stress is $K(i) = 3, i = 1, 2$ and the probabilities of failure, p_{ij} in inspection intervals $(t_{ij-1}, t_{ij}]$, $i = 1, 2$, $j = 1, 2, 3$ were assumed to be $p_{11} = 0.2$, $p_{12} = 0.15$, $p_{13} = 0.15$ on stress x_1 and $p_{21} = 0.2$, $p_{22} = 0.1$, $p_{23} = 0.1$ on stress x_2.

The optimal stress change time τ_1^* minimizing the asymptotic variance of $nAvar(\log \widehat{\theta}_0)$ at use stress x_0 in (3.3) was obtained as $\tau_1^* = 4.75664$ by solving (3.4).

Now, we obtain MLEs of β_0 and β_1 using the optimal stress change time τ_1^*. The 40 test units are simultaneously put on stress $x_1 = 0.6$ and inspections are conducted three times at specified times $t_{11} = 1.34994$, $t_{12} = 2.60608$ and stress change time $t_{13} = \tau_1^* = 4.75664$, but if all units do not fail before time τ_1^*, the surviving units are subjected to a stronger stress $x_2 = 1.0$ and also observed at specified times $t_{21} = 5.58186, t_{22} = 6.68403$ and censoring time $\tau_2 = 8.5682$.

The number of failed test units at each interval $(t_{ij-1}, t_{ij}]$, $i = 1, 2, j = 1, 2, 3$, where $t_{10} = 0$ and $t_{20} = \tau_1^*$ were $n_{11} = 6, n_{12} = 5, n_{13} = 7$ on stress x_1 and $n_{21} = 4, n_{22} = 5, n_{23} = 7$

on stress x_2 and the number of censoring units was $n_c = 6$. By Newton-Raphson method, we found the MLEs $\widehat{\beta}_0 = 3.03314$ and $\widehat{\beta}_1 = -1.96122$ of model parameters β_0 and β_1.

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