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EINSTEIN HALF LIGHTLIKE SUBMANIFOLDS OF CODIMENSION 2

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ABSTRACT. In this paper we study the geometry of Einstein half lightlike submanifolds M of a Lorentz manifold $(\bar{M}(c), \bar{g})$ of constant curvature c, equipped with an integrable screen distribution on M such that the induced connection ∇ is a metric connection and the operator A_u is a screen shape operator.

1. INTRODUCTION

In this paper we develop a theory on differential geometry of Einstein half lightlike submanifolds (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ of constant index q. For this purpose, as the first step, we introduce the induced Ricci curvature tensor *Ric* of M, appear in [6]. In general, the induced Ricci type tensor $R^{(0,2)}$, defined by the method of the geometry of the non-degenerate submanifolds ([1], [10]), is not symmetric ([4], [5], [8]). Therefore $R^{(0,2)}$ has no geometric or physical meaning similar to the Ricci curvature of the non-degenerate submanifolds and it is just a tensor quantity. Hence we need the following definition: *A tensor field $R^{(0,2)}$ of lightlike submanifolds M, given by* (29), *is called its induced Ricci tensor if it is symmetric.* A symmetric $R^{(0,2)}$ tensor will be denoted by *Ric*. In chapter 3, we find the geometric conditions so that the tensor field $R^{(0,2)}$ is Ricci tensor *Ric*.

A next step is to find screen distributions for lightlike submanifolds of codimension 2 ([2], [4]). There are two such classes of submanifolds explained as follows: Let (M, g) be a codimension 2 lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ of constant index q. Then the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ is a vector subbundle of the tangent bundle TM, of rank 1 or 2, where $TM^{\perp}$ is the normal bundle of M, of rank 2. Thus there exists a complementary non-degenerate

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distribution $S(TM)$ of $Rad(TM)$ in TM, called a *screen distribution* on M. Then we have the following orthogonal decomposition

$$TM = Rad(TM) \oplus_{orth} S(TM), \tag{1}$$

where the symbol $\oplus_{orth}$ denotes the orthogonal direct sum. As the geometry of lightlike submanifolds is mainly based on the screen distribution $S(TM)$, we denote such a lightlike submanifold by $(M, g, S(TM))$. The submanifold $(M, g, S(TM))$ is called a *half lightlike* (or *coisotropic*) *submanifold* if rank$(Rad(TM)) = 1$ (or 2).

The purpose of this paper is to study the geometry of Einstein half lightlike submanifolds M of a Lorentz manifold $(\bar{M}(c), \bar{g})$ of constant curvature c, equipped with an integrable screen distribution $S(TM)$ on M such that the induced connection ∇ is a metric connection and the operator A_u is a screen shape operator. This paper contains several new results which are related to the symmetric Ricci tensor or the result: M is a locally product manifold $M = C \times M_\alpha \times M_\beta$, where C is a null curve, and M_α and M_β are leafs of some integrable distributions of M.

2. HALF LIGHTLIKE SUBMANIFOLS

Let $(M, g, S(TM))$ be a half lightlike submanifold of an $(m+3)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ of constant index q. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. We use the same notation for any other vector bundle. Then there exist vector fields $\xi \in \Gamma(Rad(TM))$ and $u \in \Gamma(\mathcal{D})$ such that

$$\bar{g}(\xi, v) = 0, \qquad \bar{g}(u, u) \neq 0, \quad {}^\forall v \in \Gamma(TM^\perp),$$

where $\mathcal{D}$ is a supplementary distribution to $Rad(TM)$ in $TM^\perp$ of rank 1, called a *co-screen distribution* on M. Choose u as a unit vector field with $\bar{g}(u, u) = \epsilon = \pm 1$. Consider the orthogonal complementary distribution $S(TM)^\perp$ to $S(TM)$ in $T\bar{M}$, of rank 3. Certainly ξ and u belong to $\Gamma(S(TM)^\perp)$. Hence we have the following orthogonal decomposition

$$S(TM)^\perp = \mathcal{D} \oplus_{orth} \mathcal{D}^\perp,$$

where $\mathcal{D}^\perp$ is the orthogonal complementary to $\mathcal{D}$ in $S(TM)^\perp$, of rank 2. Then, for any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely defined vector field $N \in \Gamma(ntr(TM)) \subset \Gamma(\mathcal{D}^\perp)$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, u) = 0, \quad {}^\forall X \in \Gamma(S(TM)|_{\mathcal{U}}) \tag{2}$$

if and only if N is given by

$$N = \frac{1}{\bar{g}(\xi, V)} \left\{ V - \frac{\bar{g}(V, V)}{2\bar{g}(\xi, V)} \xi \right\}, \tag{3}$$

where V is a vector field in $\mathcal{D}^{\perp} \subset T\bar{M}$ such that $\bar{g}(\xi, V) \neq 0$. We call $ntr(TM)$, N and $tr(TM) = \mathcal{D} \oplus_{orth} ntr(TM)$ the *null transversal vector bundle, null transversal vector field* and *transversal vector bundle* of M with respect to $S(TM)$ respectively. Then $T\bar{M}$ is decomposed as follows;

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ntr(TM)\} \oplus_{orth} \mathcal{D} \oplus_{orth} S(TM). \end{aligned} \tag{4}$$

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$, P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1) and η a 1-form such that

$$\eta(X) = \bar{g}(X, N), \ {}^{\forall} X \in \Gamma(TM). \tag{5}$$

Then the local Gauss and Weingartan formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)u, \tag{6}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)u, \tag{7}$$

$$\bar{\nabla}_X u = -A_u X + \phi(X)N, \tag{8}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{9}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{10}$$

for any $X, Y \in \Gamma(TM)$, where ∇ and ∇^* are induced linear connections on TM and $S(TM)$ respectively, the bilinear forms B and D on M are called the *local transversal second fundamental form* and *local screen second fundamental form* on TM respectively, C is called the *local radical second fundamental form* on $S(TM)$. A_N, A^*_ξ and A_u are linear operators on $\Gamma(TM)$ and τ, ρ and ϕ are 1-forms on TM.

Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and both B and D are symmetric. From the facts $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, we know that B is independent of the choice of a screen distribution. Also from $D(X, Y) = \epsilon\bar{g}(\bar{\nabla}_X Y, u)$, we have

$$B(X, \xi) = 0, \quad D(X, \xi) = -\epsilon\phi(X), \ {}^{\forall} X \in \Gamma(TM). \tag{11}$$

But we note that B, C, ρ, ϕ and τ depend on the section $\xi \in \Gamma(Rad(TM)|_{\mathcal{U}})$. Because if we take $\bar{\xi} = \alpha\xi$ for some function α, then $\bar{N} = \frac{1}{\alpha}N$ and from (6), (9), (7) and (8) we obtain $\bar{B} = \alpha B$, $\bar{C} = \frac{1}{\alpha}C$, $\bar{\rho} = \frac{1}{\alpha}\rho$, $\bar{\phi} = \alpha\phi$ and $\tau(X) = \bar{\tau}(X) + X(\text{In}\,\alpha)$.

Taking the exterior derivative d on both sides of the last equation, we have $d\tau = d\bar{\tau}$.

The induced connection ∇ on TM is not metric ([2], [4]) and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\,\eta(Z) + B(X, Z)\,\eta(Y), \ ^{\forall} X, Y, Z \in \Gamma(TM). \tag{12}$$

But the connection ∇^* on $S(TM)$ is metric. It is well known that the second fundamental forms and the shape operators of a non-degenerate submanifold are related by means of the metric tensor field. Contrary to this, in the case of lightlike submanifolds, some of the second fundamental forms and the shape operators are interrelated by means of the metric tensor field. More precisely, the above three local second fundamental forms of M and $S(TM)$ are related to their shape operators by

$$B(X, Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{13}$$

$$C(X, PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0, \tag{14}$$

$$\epsilon D(X, PY) = g(A_u X, PY), \qquad \bar{g}(A_u X, N) = \epsilon\rho(X), \tag{15}$$

$$\epsilon D(X, Y) = g(A_u X, Y) - \phi(X)\eta(Y). \tag{16}$$

We show, by (13) and (14), that the operators A^*_ξ and A_N are shape operators related to B and C respectively, called the *radical shape operator* and *transversal shape operator* on TM and $S(TM)$ respectively and both are $\Gamma(S(TM))$-valued. From (13), the transversal shape operator A^*_ξ is real symmetric and satisfies

$$A^*_\xi \xi = 0, \tag{17}$$

that is, ξ is an eigenvector field of A^*_ξ corresponding to the eigenvalue 0. From (9) and (14), we have

$$g(A_N X, Y) - g(X, A_N Y) = C(X, Y) - C(Y, X) = \eta([X, Y]),$$

for any $X, Y \in \Gamma(S(TM))$. Thus the radical shape operator A_N is a self-adjoint on $\Gamma(TM)$ with respect to g (or equivalently, the radical second fundamental form C is symmetric) if and only if the screen distribution $S(TM)$ is integrable.

Theorem 1. *Let $(M, g, S(TM))$ be a codimension 2 half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the following assertions are equivalent:*

(1) *A_u is a self-adjoint on $\Gamma(TM)$ with respect to g.*
(2) *D satisfies $D(X, \xi) = 0$ for all $X \in \Gamma(S(TM))$.*
(3) *$\phi(X) = 0$ for all $X \in \Gamma(S(TM))$.*
(4) *$A_u\xi = \epsilon\rho(\xi)\xi$ i.e., the radical distribution $Rad(TM)$ is invariant vector bundle under A_u.*

(5) $\epsilon D(X, Y) = g(A_u X, Y) - \phi(\xi)\eta(X)\eta(Y)$ *for all* $X, Y \in \Gamma(TM)$.

Proof. From (16) and the fact that D are symmetric, we have

$$g(A_u X, Y) - g(X, A_u Y) = \phi(X)\eta(Y) - \phi(Y)\eta(X), \quad {}^{\forall}X, Y \in \Gamma(TM).$$

Replace Y by ξ in this equation and using the fact that $X = PX + \eta(X)\xi$, we have

$$g(A_u\xi, X) = -\phi(PX).$$

Also, from the first equation in this proof, we have

$$g(A_u PX, PY) - g(PX, A_u PY) = g(A_u\xi, \xi) - g(\xi, A_u\xi) = 0.$$

(1) $\Leftrightarrow$ (3). If $\phi(PX) = 0$ for all $X \in \Gamma(TM)$, then we have $g(\xi, A_u X) - g(A_u\xi, X) = \phi(PX) = 0$. Thus $g(A_u X, Y) = g(X, A_u Y)$ for all $X, Y \in \Gamma(TM)$, i.e., A_u are self-adjoint on $\Gamma(TM)$ with respect to g. Conversely, if A_u are self-adjoint on $\Gamma(TM)$ with respect to g, then we have $\phi(X)\eta(Y) = \phi(Y)\eta(X)$ for all $X, Y \in \Gamma(TM)$. Replace X by PX and Y by ξ in this equation, we have

$$\phi(PX) = 0$$

for all $X \in \Gamma(TM)$.

(2) $\Leftrightarrow$ (3). By the second equation of (11), we have (2) $\Longleftrightarrow$ (3).

(3) $\Leftrightarrow$ (4). If $\phi(PX) = 0$ for all $X \in \Gamma(TM)$, from the second equation in this proof, we have $P(A_u\xi) = 0$. In general, since $A_u X = \epsilon\rho(X)\xi + P(A_u X)$, we get

$$A_u\xi = \epsilon\rho(\xi)\xi.$$

Conversely if $A_u\xi = \epsilon\rho(\xi)\xi$, from the second equation in this proof, we have $\phi(PX) = 0$.

(3) $\Leftrightarrow$ (5). If $\phi(PX) = 0$, then $\phi(X) = \eta(X)\phi(\xi)$. Thus, from (16), we get

$$\epsilon D(X, Y) = g(A_u X, Y) - \phi(\xi)\eta(X)\eta(Y)$$

for all $X, Y \in \Gamma(TM)$. From this equation and (16), the converse is also true. □

Definition 1. A self-adjoint A_u such that $\phi = 0$ on $\Gamma(Rad\,(TM))$ called the *screen shape operators* of M.

Theorem 2. *Let* $(M, g, S(TM))$ *be a codimension* 2 *half lightlike submanifold of a semi-Riemannian manifold* $(\bar{M}, \bar{g})$ *of constant index* q. *Then the operator* A_u, *given by* (8), *is a shape operator of* M *if and only if the* 1*-form* ϕ *vanishes on any* $\mathcal{U}$. *In this case,* ξ *is an eigenvector field of* A_u *corresponding to the eigenvalue* $\epsilon\rho(\xi)$.

Denote by $\bar{R}$ and R the curvature tensors of the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$ and the induced connection ∇ on M respectively. Using the Gauss-Weingarten equations for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M:

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)Z, PW) &= g(R(X, Y)Z, PW) \\ &\quad + B(X, Z)C(Y, PW) - B(Y, Z)C(X, PW) \\ &\quad + \epsilon\{D(X,Z)D(Y,PW) - D(Y,Z)D(X,PW)\},\end{aligned} \tag{18}$$

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)Z, \xi) &= \bar{g}(R(X, Y)Z, \xi) \\ &\quad + D(Y, Z)\phi(X) - D(X, Z)\phi(Y) \\ &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ &\quad + B(Y, Z)\tau(X) - B(X, Z)\tau(Y) \\ &\quad + D(Y, Z)\phi(X) - D(X, Z)\phi(Y),\end{aligned} \tag{19}$$

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)Z, N) &= \bar{g}(R(X, Y)Z, N) \\ &\quad + \epsilon\{D(X, Z)\rho(Y) - D(Y, Z)\rho(X)\},\end{aligned} \tag{20}$$

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)Z, u) &= \epsilon\{(\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) \\ &\quad + B(Y, Z)\rho(X) - B(X, Z)\rho(Y)\},\end{aligned} \tag{21}$$

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)\xi, N) &= \bar{g}(R(X, Y)\xi, N) + \rho(X)\phi(Y) - \rho(Y)\phi(X) \\ &= g(A^*_\xi X, A_N Y) - g(A^*_\xi Y, A_N X) - 2d\tau(X, Y) \\ &\quad + \rho(X)\phi(Y) - \rho(Y)\phi(X),\end{aligned} \tag{22}$$

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)\xi, u) &= g(A^*_\xi X, A_u Y) - g(A_u X, A^*_\xi Y) - 2d\phi(X, Y) \\ &\quad + \phi(X)\tau(Y) - \phi(Y)\tau(X),\end{aligned} \tag{23}$$

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)N, u) &= \epsilon\{D(Y, A_N X) - D(X, A_N Y) + 2d\rho(X, Y) \\ &\quad + \rho(X)\tau(Y) - \rho(Y)\tau(X)\},\end{aligned} \tag{24}$$

$$\begin{aligned}\bar{g}(R(X, Y)PZ, PW) &= g(R^*(X, Y)PZ, PW) \\ &\quad + C(X, PZ)B(Y, PW) - C(Y, PZ)B(X, PW),\end{aligned} \tag{25}$$

for any $X, Y, Z, W \in \Gamma(TM_{|\mathcal{U}})$ where R^* is the curvature tensors of the induced connection ∇^* on $S(TM)$.

The *Ricci curvature tensor*, denoted by $\bar{Ric}$, of $\bar{M}$ is defined by

$$\bar{Ric}(X, Y) = trace\{Z \rightarrow \bar{R}(X, Z)Y\}, \tag{26}$$

for any $X, Y \in \Gamma(T\bar{M})$. Locally, $\bar{Ric}$ is given by

$$\bar{Ric}(X, Y) = \sum_{i=1}^{m+3} \epsilon_i \, \bar{g}(\bar{R}(E_i, X)Y, E_i),$$

where $\{E_1, \ldots, E_{m+3}\}$ is an orthonormal frame field of $T\bar{M}$ and $\epsilon_i \,(= \pm 1)$ denotes the causal character of respective vector field E_i. $\bar{M}$ is called *Ricci flat* if its Ricci tensor vanishes on $\bar{M}$. If $\dim(\bar{M}) > 2$ and

$$\bar{Ric} = \bar{\gamma}\bar{g}, \quad \bar{\gamma} \text{ is a constant},$$

then $\bar{M}$ is an *Einstein manifold.* For $\dim(\bar{M}) = 2$, any $\bar{M}$ is Einstein but $\bar{\gamma}$ is not necessarily constant. The *scalar curvature* $\bar{r}$ is defined by

$$\bar{r} = \sum_{i=1}^{m+3} \epsilon_i \, \bar{Ric}(E_i, E_i). \tag{27}$$

Using the definition of Einstein manifold in (27) implies that $\bar{M}$ is Einstein if and only if $\bar{r}$ is constant and

$$\bar{Ric} = \frac{\bar{r}}{m+3}\,\bar{g}.$$

3. Ricci Curvature Tensors

Consider the induced quasi-orthonormal frame $\{\xi; W_a\}$ on M, where $Rad\,TM = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$ and let $E = \{\xi, W_a; u, N\}$ be the corresponding frame field on $\bar{M}$. Then, by using (26), we obtain

$$\begin{aligned} \bar{Ric}(X, Y) &= \sum_{a=1}^{m} \epsilon_a \, \bar{g}(\bar{R}(W_a, X)Y, W_a) \; + \; \bar{g}(\bar{R}(\xi, X)Y, N) \\ &\quad + \epsilon \, \bar{g}(\bar{R}(u, X)Y, u) \; + \; \bar{g}(\bar{R}(N, X)Y, \xi). \end{aligned} \tag{28}$$

Let $R^{(0,2)}$ denote the induced tensor of type (0, 2) on M given by

$$R^{(0,2)}(X, Y) \; = \; trace\{Z \to R(X, Z)Y\}, \quad \forall\, X, Y \in \Gamma(TM). \tag{29}$$

Using the induced quasi-orthonormal frame $\{\xi; W_a\}$ on M, we obtain

$$R^{(0,2)}(X, Y) \; = \; \sum_{a=1}^{m} \epsilon_a \, g(R(W_a, X)Y, W_a) + \bar{g}(R(\xi, X)Y, N).$$

Substituting the Gauss-Codazzi equations (18) and (20) in (28), then, using the relations (13) and (14), we obtain

$$R^{(0,2)}(X, Y) = \bar{Ric}(X, Y) + B(X, Y)trA_N + D(X, Y)trA_u$$

$$- g(A_N X, A^*_\xi Y) - \epsilon\, g(A_u X, A_u Y) + \rho(X)\phi(Y)$$

$$- \bar{g}(\bar{R}(\xi, Y)X, N) - \epsilon\, \bar{g}(\bar{R}(u, Y)X, u), \tag{30}$$

for any $X, Y \in \Gamma(TM)$. This shows that $R^{(0,2)}$ is not symmetric. A tensor field $R^{(0,2)}$ of M, given by (29), is called its *induced Ricci tensor* if it is symmetric. From now and in the sequal, a symmetric $R^{(0,2)}$ tensor will be denoted by Ric.

Using (30) and the first Bianchi's identity, we obtain

$$R^{(0,2)}(X, Y) - R^{(0,2)}(Y, X) = g(A^*_\xi X, A_N Y) - g(A^*_\xi Y, A_N X)$$
$$+ \rho(X)\phi(Y) - \rho(Y)\phi(Y) - \bar{g}(\bar{R}(X, Y)\xi, N).$$

From this equation and (22), we have

$$R^{(0,2)}(X, Y) - R^{(0,2)}(Y, X) = 2d\tau(X, Y).$$

Theorem 3 ([4]). *Let $(M, g, S(TM))$ be a codimension 2 half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the tensor $R^{(0,2)}$ is a symmetric Ricci tensor Ric, if and only if, each 1-form τ is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$.*

If the ambient manifold $(\bar{M}, \bar{g})$ is a semi-Riemannian manifold $(\bar{M}(c), \bar{g})$ of constant curvature c, then we have $\bar{R}(\xi, Y)X = c\bar{g}(X, Y)\xi$, $\bar{R}(u, X)Y = c\bar{g}(X, Y)u$ and $\bar{Ric}(X, Y) = (m+2)c\,\bar{g}(X, Y)$, Thus we obtain

$$R^{(0,2)}(X, Y) = mcg(X, Y) + B(X, Y)trA_N + D(X, Y)trA_u$$

$$- g(A_N X, A^*_\xi Y) - \epsilon\, g(A_u X, A_u Y) + \rho(X)\phi(Y). \tag{31}$$

Any geodesic of M with respect to an induced connection ∇ is a geodesic of $\bar{M}$ with respect to $\bar{\nabla}$, we say that M is a *totally geodesic*. In this case, we have $B = D = A^*_\xi = \phi = 0$ and $A_u X = \epsilon\rho(X)\xi$ on any $\mathcal{U} \subset M$. Thus we have

Theorem 4. *Any totally geodesic codimension 2 half lightlike submanifold M of a semi-Riemannian manifold $(\bar{M}(c), \bar{g})$ of constant curvature c admits an induced symmetric Ricci tensor. In particular, M is Einstein manifold.*

Now suppose that the induced connection ∇ on M is a metric. It follows from (12) and (13) that both B and A^*_ξ vanish on M. Then, using (19), we obtain

$$D(X, Z)\phi(Y) = D(Y, Z)\phi(X),$$

for any $X, Y \in \Gamma(TM)$. From this equation and (16), we have

$$g\,(\phi(Y)A_u X - \phi(X)A_u Y, Z) = 0, \quad {}^\forall Z \in \Gamma(TM).$$

Thus $\phi(X)A_uY - \phi(Y)A_uX \in \Gamma(Rad\,(TM))$. While, using (15) and (22), we have

$$g\,(\phi(Y)A_uX - \phi(X)A_uY,\, N) = \epsilon\{\rho(X)\phi(Y) - \rho(Y)\phi(X)\} = 2\epsilon d\tau(X,\, Y);$$
$$\phi(Y)A_uX - \phi(X)A_uY = \epsilon\{\rho(X)\phi(Y) - \rho(Y)\phi(X)\}\xi = 2\epsilon d\tau(X,\, Y)\xi.$$

Set $A^*_uX = A_uX - \epsilon\rho(X)\xi$, then A^*_u is $\Gamma(S(TM))$-valued real symmetric and satisfies

$$\phi(Y)A^*_uX = \phi(X)A^*_uY.$$

The *type number* $t(x)$ of M at any point x is the rank of A_u. Let $t(x) > 2$ for any $x \in M$. Suppose there exists a vector field $X_o \in \Gamma(TM)$ such that $\phi(X_o) \neq 0$, then $A^*_uY = fA^*_uX_o$ for any $Y \in \Gamma(TM)$, where f is a smooth function. It follows that the rank of A^*_u is 1. It is a contradiction as rank $A_u > 2$. Thus we have $\phi = 0$ on $\mathcal{U}$.

Theorem 5. *Let $(M,\, g,\, S(TM))$ be a codimension 2 half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}(c),\, \bar{g})$ of constant curvature c such that the induced connection ∇ on M is a metric connction. If the type number satisfies $t(x) > 2$ for any $x \in M$, then each 1-form ϕ from (8) vanishes on any $\mathcal{U} \subset M$. Consequently A_u is a screen shape operator of M.*

From Theorem 5, we know the condition that the operator A_u is a screen shape operator on M is very weak condition. If M is totally geodesic, then $t(x) = 1$ for any $x \in M$. Assume that the induced connection ∇ on M is a metric connection and the operator A_u is a screen shape operator on M. It follow that B, A^*_ξ and ϕ vanish on any $\mathcal{U} \subset M$. Thus, from (31), we have

Theorem 6. *Let $(M,\, g,\, S(TM))$ be a codimension 2 half lightlike submanifold, equipped with a metric connection ∇ and a screen shape operation A_u, of a semi-Riemannian manifold $(\bar{M}(c),\, \bar{g})$ of constant curvature c. Then the tensor field $R^{(0,\,2)}$ of the induced connection ∇ is a symmetric Ricci tensor Ric.*

4. Einstein Submanifolds

Let M be an Einstein half lightlike submanifold of a Lorentz manifold $(\bar{M}(c),\, \bar{g})$ of constant curvature c, equipped with an integrable screen distribution $S(TM)$ on M such that the induced connection ∇ is a metric and A_u is a screen shape operator. Then $S(TM)$ is a Riemannian vector bundle and $\epsilon = 1$. Set $A^*_uX = A_uX - \rho(X)\xi$, then A^*_u is also a shape operator of M related to the local screen second fundamental form D and satisfies

$$D(X, Y) = g(A^*_u X, Y), \quad \bar{g}(A^*_u X, N) = 0, \quad A^*_u \xi = 0. \tag{32}$$

From the third equation of (32), we know that ξ is an eigenvector field of A^*_u corresponding to the eigenvalue 0. Since A^*_u is $\Gamma(S(TM))$-valued real symmetric, A^*_u have m real orthonormal eigenvector fields in $S(TM)$ and is diagonalizable on $S(TM)$.

Consider a frame field of eigenvectors $\{\xi, E_1, \ldots, E_m\}$ of A^*_u such that $\{E_i\}_i$ is an orthonormal frame field of $S(TM)$. Then

$$A^*_u E_i = \lambda_i E_i, \quad 1 \le i \le m.$$

Since ∇ is a metric connection and A_u is a shape operator of M, we have $B = A^*_\xi = \phi = 0$ on M. Also, since $Ric = \gamma g$, the equation (31) reduces to

$$g(A^*_u X, A^*_u Y) - sg(A^*_u X, Y) + (\gamma - mc)g(X, Y) = 0, \tag{33}$$

where $s = trA_u = trA^*_u + \rho(\xi)$. Put $X = Y = E_i$ in (33), λ_i is a solution of

$$x^2 - sx + (\gamma - mc) = 0. \tag{34}$$

The equation (34) has at most two distinct solutions which are real valued functions. Assume that there exists $p \in \{0, 1, \ldots, m\}$ such that $\lambda_1 = \cdots = \lambda_p = \alpha$ and $\lambda_{p+1} = \cdots = \lambda_m = \beta$, by renumbering if necessary. From (34), we have

$$s = \alpha + \beta = p\alpha + (m - p)\beta + \rho(\xi), \qquad \alpha\beta = (\gamma - mc).$$

Now we consider the following four distributions D_α, D_β, D^s_α and D^s_β on M:

$$\Gamma(D_\alpha) = \{X \in \Gamma(TM) \mid A^*_u X = \alpha\, PX\}, \quad D^s_\alpha = D_\alpha \cap S(TM);$$
$$\Gamma(D_\beta) = \{U \in \Gamma(TM) \mid A^*_u U = \beta\, PU\}, \quad D^s_\beta = D_\beta \cap S(TM).$$

Note that $E_1, \ldots, E_p \in \Gamma(D^s_\alpha)$ and $E_{p+1}, \ldots, E_m \in \Gamma(D^s_\beta)$. The equation (34) has only one solution $\iff$ $\alpha = \beta$ $\iff$ $D_\alpha = D_\beta$ $(= TM)$. If $0 < p < m$, then $D_\alpha \ne D_\beta$ and $D_\alpha \cap D_\beta = Rad\,(TM)$. In case $m \ge 2$ and $D_\alpha \ne D_\beta$: If $p = 0$, then α is not an eigenvalue of A^*_u but a root of (34) and $D_\alpha = Rad\,(TM)$; $D_\beta = TM$. If $p = m$, then β is not an eigenvalue of A^*_u but a root of (34) and $D_\alpha = TM$; $D_\beta = Rad\,(TM)$.

Lemma 1. *If $D_\alpha \ne D_\beta$, then $D_\alpha \perp_g D_\beta$ and $D_\alpha \perp_D D_\beta$.*

Proof. If $0 < p < m$, then we have $A^*_u PX = A^*_u X = \alpha PX$ for any $X \in \Gamma(D_\alpha)$ and $A^*_u PU = A^*_u U = \beta PU$ for any $U \in \Gamma(D_\beta)$. Thus the projection morphism P maps $\Gamma(D_\alpha)$ onto $\Gamma(D^s_\alpha)$ and $\Gamma(D_\beta)$ onto $\Gamma(D^s_\beta)$. Since PX and PU are eigenvector fields of the real symmetric operator A^*_u corresponding to the different eigenvalues α and

β respectively. Thus $PX \perp_g PU$ and $g(X, U) = g(PX, PU) = 0$, that is, $D_\alpha \perp_g D_\beta$. Also, since $D(X, U) = g(A^*_u X, U) = \alpha g(PX, PU) = 0$, we get

$$D(D_\alpha, D_\beta) = 0,$$

that is, $D_\alpha \perp_D D_\beta$. On the other hand, if $p = 0$ or $p = m$, then

$$D_\alpha = Rad\,(TM)\,;\;\; D_\beta = TM$$

or

$$D_\alpha = TM\,;\;\; D_\beta = Rad\,(TM)$$

respectively. Thus we have $D_\alpha \perp_g D_\beta$ and $D_\alpha \perp_D D_\beta$. □

Lemma 2. *If $D_\alpha \neq D_\beta$, then $TM = Rad\,(TM) \oplus_{orth} D^s_\alpha \oplus_{orth} D^s_\beta$. If $D_\alpha = D_\beta$, then $TM = Rad\,(TM) \oplus_{orth} D^s_\alpha \oplus_{orth} \{0\}$.*

Proof. If $0 < p < m$, since $\{E_i\}_{1\leq i\leq p}$ and $\{E_a\}_{p+1\leq a\leq m}$ are vector fields of D^s_α and D^s_β respectively and D^s_α and D^s_β are mutually orthogonal vector subbundle of $S(TM)$, we show that D^s_α and D^s_β are non-degenerate distributions of rank p and rank $(m-p)$ respectively and $D^s_\alpha \cap D^s_\beta = \{0\}$. Thus we have

$$S(TM) = D^s_\alpha \oplus_{orth} D^s_\beta.$$

If $D_\alpha \neq D_\beta$ and $p = 0$, then $D^s_\alpha = \{0\}$ and $D^s_\beta = S(TM)$. If $D_\alpha \neq D_\beta$ and $p = m$, then $D^s_\alpha = S(TM)$ and $D^s_\beta = \{0\}$. Also we have

$$S(TM) = D^s_\alpha \oplus_{orth} D^s_\beta.$$

Next, if $D_\alpha = D_\beta$, then $D^s_\alpha = D^s_\beta = S(TM)$. Thus, from (1), we have this lemma. □

Lemma 3. $Im(A^*_u - \alpha P) \subset \Gamma(D^s_\beta)\,;\, Im(A^*_u - \beta P) \subset \Gamma(D^s_\alpha)$.

Proof. From (33), we show that $(A^*_u)^2 - (\alpha+\beta)A^*_u + \alpha\beta P = 0$. If $0 < p < m$. Let $Y \in Im(A^*_u - \alpha P)$, then there exists $X \in \Gamma(TM)$ such that

$$Y = (A^*_u - \alpha P)X.$$

Then $(A^*_u - \beta P)Y = 0$ and $Y \in \Gamma(D_\beta)$. Thus $Im(A^*_u - \alpha P) \subset \Gamma(D_\beta)$. Since the morphism $A^*_u - \alpha P$ maps $\Gamma(TM)$ onto $\Gamma(S(TM))$, we have

$$Im(A^*_u - \alpha P) \subset \Gamma(D^s_\beta).$$

By duality, we also have

$$Im(A^*_u - \beta P) \subset \Gamma(D^s_\alpha).$$

While, if $p=0$, then, since $D^s_\alpha = \{0\}\,;\ D^s_\beta = S(TM)$ and $D_\beta = TM$, we have

$$Im(A^*_u - \alpha P) \subset \Gamma(S(TM))\,;\ A^*_u X = \beta PX$$

for all $X \in \Gamma(TM)$, that is, $Im(A^*_u - \beta P) = \{0\}$ or if $p = m$, then, since $D^s_\alpha = S(TM)\,;\ D^s_\beta = \{0\}$ and $D_\alpha = TM$, we have $A^*_u X = \alpha PX$ for all $\Gamma(TM)$, that is,

$$Im(A^*_u - \alpha P) = \{0\}\,;\ Im(A^*_u - \beta P) \subset \Gamma(S(TM)).$$

□

Lemma 4. *If $D_\alpha \neq D_\beta$, then both D_α and D_β are integrables. In particular, if $S(TM)$ is integrable, then D^s_α and D^s_β are also integrables.*

Proof. If $D_\alpha \neq D_\beta$, for any $X,\,Y \in \Gamma(D_\alpha)$ and any $Z \in \Gamma(TM)$, we have

$$\begin{aligned}(\nabla_X D)(Y,\,Z) &= X\,(\alpha g(PY,\,Z)) - g(A^*_u \nabla_X Y,\,Z) - \alpha g(PY,\,\nabla_X Z)\\ &= (X\alpha) g(PY,\,Z) \;-\; g\,((A^*_u - \alpha P)\nabla_X Y,\,Z)\,.\end{aligned}$$

Using this and the fact that $(\nabla_X D)(Y,\,Z) = (\nabla_Y D)(X,\,Z)$, we have

$$g\,((A^*_u - \alpha P)[X,\,Y],\,Z) = (X\alpha) g(PY,\,Z) - (Y\alpha) g(PX,\,Z). \tag{35}$$

If we take $Z = U \in \Gamma(D_\beta)$, then we have

$$g\,((A^*_u - \alpha P)[X,\,Y],\,U) = 0.$$

Since the distribution D^s_β is non-degenerate and $Im(A^*_u - \alpha P) \subset \Gamma(D^s_\beta)$, we have

$$(A^*_u - \alpha P)[X,\,Y] = 0.$$

Thus $[X,\,Y] \in \Gamma(D_\alpha)$ and D_α is integrable. By duality, D_β is also integrable. On the other hand, if $S(TM)$ is integrable, for any $X,\,Y \in \Gamma(D^s_\alpha)$, we have $[X,\,Y] \in \Gamma(D_\alpha)$ and $[X,\,Y] \in \Gamma(S(TM))$. Thus $[X,\,Y] \in \Gamma(D^s_\alpha)$ and D^s_α is integrable. Also D^s_β is integrable. □

Theorem 7. *Let $(M,\,g,\,S(TM))$ be a codimension 2 Einstein half lightlike submanifold of a Lorentz manifold $(\bar{M}(c),\,\bar{g})$ of constant curvature c, equipped with an integrable screen distribution $S(TM)$ on M such that the induced connection ∇ is a metric connection and the operator A_u is a screen shape operator. Then M is a locally product manifold $C \times M_\alpha \times M_\beta$, where C is a null curve, and M_α and M_β are leafs of some integrable distributions of M.*

Lemma 5. *If $0 < p < m$, then α and β are constants along both D_α and D_β.*

Proof. From (35), for $X, Y \in \Gamma(D_\alpha)$ and any $Z \in \Gamma(TM)$, we get

$$d\alpha(Y)g(PX, Z) = d\alpha(X)g(PY, Z).$$

Since $S(TM)$ is non-degenerate, we have

$$d\alpha(Y)PX = d\alpha(X)PY.$$

Now suppose there exists a vector field $X_o \in \Gamma(D_\alpha)$ such that $d\alpha(X_o) \neq 0$ at each point $x \in M$, then $PY = \delta PX_o$ for any $Y \in \Gamma(D_\alpha)$, where δ is a smooth function. It follows that all vectors from the fiber $(D_\alpha)_x$ are colinear with $(PX_o)_x$. It is a contradiction as $\dim((D_\alpha)_x) = p + 1 > 1$. Thus we have $d\alpha = 0$ on D_α. By duality, we also have $d\beta = 0$ on D_β. These mean that α is a constant along each vector fields in D_α and β is a constant along each vector fields in D_β. While α and β satisfy $\alpha\beta = (\gamma - mc)$ which is a constant. Thus we have this lemma. □

Lemma 6. *If* $0 < p < m$, *for any* $X \in \Gamma(D_\alpha)$ *and any* $U \in \Gamma(D_\beta)$, *we have*

$$\nabla_X U \in \Gamma(D_\beta); \qquad \nabla_U X \in \Gamma(D_\alpha). \tag{36}$$

Proof. From (21), we get

$$(\nabla_X D)(U, Z) = (\nabla_U D)(X, Z), \quad \forall Z \in \Gamma(TM).$$

Using this equation and Lemma 5, we have

$$g\left((A^*_u - \beta P)\nabla_X U, Z\right) = g\left((A^*_u - \alpha P)\nabla_U X, Z\right),$$

for any $Z \in \Gamma(TM)$. Since $S(TM)$ is non-degenerate, we have

$$(A^*_u - \beta P)\nabla_X U = (A^*_u - \alpha P)\nabla_U X.$$

Since the left term of the last equation is in $\Gamma(D^s_\alpha)$ and the right term is in $\Gamma(D^s_\beta)$ and $D^s_\alpha \cap D^s_\beta = \{0\}$, we have

$$(A^*_u - \beta P)\nabla_X U = 0, \qquad (A^*_u - \alpha P)\nabla_U X = 0.$$

This imply that $\nabla_X U \in \Gamma(D_\beta)$ and $\nabla_U X \in \Gamma(D_\alpha)$. □

Lemma 7. *If* $0 < p < m$, *for any* $X, Y \in \Gamma(D_\alpha)$ *and* $U, V \in \Gamma(D_\beta)$, *we have*

$$g(\nabla_Y X, U) = 0; \qquad g(X, \nabla_V U) = 0. \tag{37}$$

Proof. Since $g(X, U) = 0$ and the connection ∇ is a metric one, we have

$$g(\nabla_Y X, U) = -\nabla_Y(g(X, U)) + g(\nabla_Y X, U) + g(X, \nabla_Y U) = 0,$$
$$g(X, \nabla_V U) = -\nabla_V(g(X, U)) + g(\nabla_V X, U) + g(X, \nabla_V U) = 0.$$

□

Lemma 8. *If $0 < p < m$, then we have $c = -\alpha\beta$ and $\gamma = (m-1)c$.*

Proof. Using (36) and (37), for any $X \in \Gamma(D_\alpha)$ and any $U \in \Gamma(D_\beta)$, we have

$$g(R(X, U)U, X) = g(\nabla_X \nabla_U U, X).$$

From the second equation of (37), we know that $\nabla_U U$ has no component of D_α. Since the projection morphism P maps $\Gamma(D_\alpha)$ onto $\Gamma(D^s_\alpha)$ and $\Gamma(D_\beta)$ onto $\Gamma(D^s_\beta)$, and $S(TM) = D^s_\alpha \oplus_{orth} D^s_\beta$, we have

$$\nabla_U U = P(\nabla_U U) + \eta(\nabla_U U)\xi\,; \quad P(\nabla_U U) \in \Gamma(D^s_\beta).$$

It follows that

$$g(R(X, U)U, X) = 0. \tag{38}$$

From (18) and (38), we have

$$(c + \alpha\beta)g(X, X)g(U, U) = 0.$$

Thus $c = -\alpha\beta$. Since $\alpha\beta = (\gamma - mc)$, we show that $\gamma = (m-1)c$. □

Note. Let $D_\alpha \neq D_\beta \neq S(TM)$. Then $c = 0 \iff \alpha = 0$ or $\beta = 0 \iff \gamma = 0$ (unless $m = 1$). In this case, the ambient manifold $\bar{M}$ is a semi-Euclidean manifold and the Ricci curvature Ric of M is flat.

Theorem 8. *Let $(M, g, S(TM))$ be a codimension 2 Einstein half lightlike submanifold of a Lorentz manifold $(\bar{M}(c), \bar{g})$ of constant curvature c, equipped with an integrable screen distribution $S(TM)$ such that the induced connection ∇ is a metric one and the operator A_u is a screen shape one. Then M is a locally product manifold $C \times M_\alpha \times M_\beta$, where C is a null curve, M_α and M_β are leafs of some integrable distributions of M such that*

(1) *If $\gamma \neq mc$: M_α and M_β are p and $(m-p)$-dimensional Riemannian manifolds of constant curvatures; M is totally umbilical and M_α is a point whenever $p = 0$ or M is totally umbilical and M_β is a point whenever $p = m$.*

(2) *If $\gamma = mc$: M_α is a p-dimensional Euclidean manifold or m-dimensional Riemannian manifold of the constant curvature c; M_β is an $(m-p)$-dimensional Riemannian manifold of constant curvature; M is totally umbilical and M_α is a point whenever $p = 0$ or M is totally geodesic and M_β is a point whenever $p = m$.*

Proof. (1) Let $\gamma \neq mc$: In case $(trA_u)^2 \neq 4(\gamma - mc)$. The equation (34) has two non-vanishing distinct solutions α and β. From (18) and (25), we have

$$R^*(X,Y)Z = (c+\alpha^2)\{g(Y,Z)X - g(X,Z)Y\} \text{ on } D^s_\alpha,$$
$$R^*(U,V)W = (c+\beta^2)\{g(V,W)U - g(U,W)V\} \text{ on } D^s_\beta.$$

If $0 < p < m$, then α and β are non-zero distinct constants on both D_α and D_β. Thus the leafs M_α and M_β are Rienannian manifolds of constant curvatures $(c+\alpha^2)$ and $(c+\beta^2)$ respectively and M is a locally product $C \times M_\alpha \times M_\beta$, where C is a null curve and M_α and M_β are Riemannian manifolds of constant curvatures. If $p = 0$, then $D^s_\alpha = \{0\}$; $D^s_\beta = S(TM)$ and $D(X,Y) = \beta g(X,Y)$ for all X, $Y \in \Gamma(TM)$, thus M is totally umbilical, M_α is a point and M_β is an m-dimensional Rienannian manifold of curvature $(c+\beta^2)$. If $p = m$, then $D^s_\alpha = S(TM)$; $D^s_\beta = \{0\}$ and $D(X,Y) = \alpha g(X,Y)$ for all X, $Y \in \Gamma(TM)$, thus M is totally umbilical, M_α is an m-dimensional Rienannian manifold of curvature $(c+\alpha^2)$ and M_β is a point.

In case $(trA_u)^2 = 4(\gamma - mc)$. The equation (34) has only one non-zero constant solution α. From (18) and (25), we have

$$R^*(X,Y)Z = (c+\alpha^2)\{g(Y,Z)X - g(X,Z)Y\} \text{ on } S(TM).$$

Thus M is a locally product $C \times M^* \times \{x\}$, where C is a null curve and M^* is a Riemannian manifolds of constant curvature $(c+\alpha^2)$. Since $D(X,Y) = \alpha g(X,Y)$ for all X, $Y \in \Gamma(TM)$, thus M is totally umbilical.

(2) Let $\gamma = mc$. In case $trA_u \neq 0$. The equation (34) reduces to $x(x-s) = 0$. Let $\alpha = 0$ and $\beta = s$. If $0 < p < m$, then, by Lemma 8, we have $c = \gamma = 0$. Thus the leaf M_α is a Euclidean manifold and the leaf M_β is a Rienannian manifold of constant curvature s^2. If $p = m$, then we have $A^*_u = 0$ or equivalently $D = 0$. Thus M is totally geodesic in $\bar{M}$. Since $D^s_\alpha = S(TM)$; $D^s_\beta = \{0\}$, the leaf M_α of D^s_α is a m-dimensional Riemannian manifold and the leaf M_β of D^s_β is a point. Now consider the frame fields of eigenvectors $\{\xi, E_1, \ldots, E_m\}$ of A^*_u such that $\{E_1, \ldots, E_m\}$ is an orthonormal frame field of $S(TM)$. From (18) and (25), we have

$$\bar{g}(\bar{R}(E_i,E_j)E_j, E_i) = g(R^*(E_i,E_j)E_j, E_i) = c.$$

Thus the sectional curvature K_α of the leaf M_α is given by

$$K_\alpha(E_i,E_j) = \frac{g(R^*(E_i,E_j)E_j, E_i)}{g(E_i,E_i)g(E_j,E_j) - g^2(E_i,E_j)} = c.$$

Thus M is a locally product $C \times M_\alpha \times M_\beta$ where C is a null curve, M_α is a Riemannian manifold of the curvature c and M_β is a point. If $p = 0$. Since $D^s_\alpha = \{0\}$; $D^s_\beta =$

$S(TM)$ and $D(U,V) = \beta g(U,V)$ for all U, $V \in \Gamma(TM)$, M is totally umbilical, M_α is a point and M_β is an m-dimensional Riemannian manifold of constant curvature $(c+\beta^2)$.

In case $trA_u = 0$. The equation (34) has only trivial solution 0. Thus M is a locally product $C \times M^* \times \{x\}$, where C is a null curve and M^* is an m-dimensional Riemannian manifolds of constant curvature c. Since $D(X,Y) = 0$ for all X, $Y \in \Gamma(TM)$, thus M is totally geodesic. □

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