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ON THE CONVERGENCE OF NEWTON'S METHOD AND LOCALLY HÖLDERIAN INVERSES OF OPERATORS

IOANNIS K. ARGYROS

ABSTRACT. A semilocal convergence analysis is provided for Newton's method in a Banach space. The inverses of the operators involved are only locally Hölderian. We make use of a point-based approximation and center-Hölderian hypotheses for the inverses of the operators involved. Such an approach can be used to approximate solutions of equations involving nonsmooth operators.

1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1}$$

where F is a continuous operator defined on a closed subset D of a Banach space X with values in a Banach space Y.

Here we continue our work initiated in [3], where Newton's method was used to approximate x^*. In [3] we assumed that F^{-1} is locally p-Hölderian for $p \in [0, 1)$. Here we assume F^{-1} is locally p-"Hölderian" for $p > 1$. The case $p = 1$ has been considered in [6].

The benefits of our approach and the advantages over earlier works (see [5], [6] and the references there) have already been explained in [3].

2. PRELIMINARIES

We need the following definition of point-based approximation:

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Definition 1. Let f be an operator from a closed subset D of a metric space (X, d) into a normed linear space Y, let $x_0 \in D$, and $p \geq 1$. We say f has a point-based approximation (PBA) on D at $x_0 \in D$ if there exist an operator $A\colon D \times D \to Y$ and scalars ℓ, ℓ_0 such that for each u, and v in D,

$$\|f(v) - A(u,v)\| \leq \ell d(u,v)^p \tag{2}$$

$$\|[A(u,x) - A(v,x)] - [A(u,y) - A(v,y)]\| \leq 2\ell d(u,v)^p \tag{3}$$

and

$$\|[A(u,v) - A(x_0,x)] - [A(u,y) - A(x_0,y)]\| \leq 2\ell_0 d(u,v)^p \tag{4}$$

for all $x, y \in D$.

Justifications/choices of operator A have already been given in [3].

To avoid repetitions we assume familiarity of the reader with Definition 2, Lemmas 1 and 2 in [3] (which hold for $p \geq 1$). Note that according to Definition 2 in [3], F^{-1} (if it exists) is $\frac{1}{p}$-Hölderian with modulus $\delta^{-1/p}$.

From now on we also assume $p > 1$.

3. Semilocal Convergence

We need the following result on fixed points:

Theorem 1. *Let $Q\colon D \subset X \to X$ be an operator, p, q scalars with $p > 1$, $q \geq 0$, and x_0 a point in D such that*

$$\|Q(x) - Q(y)\| \leq q\|x - y\|^p \quad \text{for all } x, y \in D; \tag{5}$$

equation

$$2^{p-1}qr^p - r + \|x_0 - Q(x_0)\| = 0 \tag{6}$$

has a unique positive solution r in $I = \left[\|x_0 - Q(x_0)\|, \frac{1}{2}q^{\frac{1}{1-p}}\right)$;

and

$$U(x_0, r) = \{x \in X \mid \|x - x_0\|\} \leq r \subseteq D.$$

Then sequence $\{x_n\}$ $(n \geq 0)$ generated by successive substitutions

$$x_{n+1} = Q(x_n) \quad (n \geq 0) \tag{7}$$

converges to a unique fixed point $x^* \in U(x_0, r)$ of operator Q, so that for all $n \geq 1$

$$\|x_{n+1} - x_n\| \leq d\|x_n - x_{n-1}\| \leq d^n\|x_0 - Q(x_0)\| \tag{8}$$

and

$$\|x_n - x^*\| \le \frac{d^n}{1-d}\|x_0 - Q(x_0)\| \tag{9}$$

where,

$$d = (2r)^{p-1}q. \tag{10}$$

Proof. By the definition of r we have $x_1 \in U(x_0, r)$. Assume $x_k \in U(x_0, r)$ for $k = 0, 1, \dots, n$. Then x_{n+1} is defined by (7). By (5) and (7) we can have in turn:

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|Q(x_n) - Q(x_{n-1})\| \le q\|x_n - x_{n-1}\|^p \\
&\le q\|x_n - x_{n-1}\|^{p-1}\|x_n - x_{n-1}\| \\
&\le q(\|x_n - x_0\| + \|x_0 - x_{n-1}\|)^{p-1}\|x_n - x_{n-1}\| \\
&\le q(2r)^{p-1}\|x_n - x_{n-1}\| = d\|x_n - x_{n-1}\| \le d^n\|x_1 - x_0\|, \qquad (11)
\end{aligned}$$

which shows (8).

Moreover for all $m = 0, 1, 2, \dots$ we have:

$$\begin{aligned}
\|x_{n+m} - x_n\| &\le \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\
&\le (d^{n+m-1} + \cdots + d^n)\|x_0 - Q(x_0)\| \\
&\le \frac{1-d^m}{1-d} d^n \|x_0 - Q(x_0)\|. \qquad (12)
\end{aligned}$$

It follows from (10) and (12) that sequence $\{x_n\}$ is Cauchy in a Banach space X, and as such it converges to some $x^* \in U(x_0, r)$ (since $U(x_0, r)$ is a closed set). By letting $m \to \infty$ in (12) we get (9). In particular for $n = 0$, and $m = n + 1$ (12) gives $x_{n+1} \in U(x_0, r)$. That is $x_n \in U(x_0, r)$ for all $n \ge 0$.

Furthermore by letting $n \to \infty$ in (7) we get $x^* = Q(x^*)$ since operator Q is continuous by (5).

To show uniqueness, let $y^* \in U(x_0, r)$ be a fixed point of Q then by (5) we get for $x^* \ne y^*$

$$\begin{aligned}
\|x^* - y^*\| &= \|Q(x^*) - Q(y^*)\| \le q\|x^* - y^*\|^p \le d\|x^* - y^*\| \\
&< \|x^* - y^*\|,
\end{aligned}$$

which is a contradiction.

Hence we deduce:

$$x^* = y^*.$$

That completes the proof of Theorem 1.

Remark 1. Conditions can be given to guarantee the existence and uniqueness of r. Indeed define scalar function h by

$$h(t) = 2^{p-1}qt^p - t + \eta, \quad \|x_0 - Q(x_0)\| \leq \eta. \tag{13}$$

By the intermediate value theorem (6) has a solution r in I if

$$h\left(\frac{1}{2}q^{\frac{1}{1-p}}\right) \leq 0 \tag{14}$$

or if

$$\eta \leq \frac{1}{2}q^{\frac{1}{1-p}}(1 - q^{-1}) = \eta_0 \tag{15}$$

and

$$q \geq 1. \tag{16}$$

This solution is unique if

$$\eta \leq \eta_1 \tag{17}$$

where,

$$\eta_1 = \min\left\{\eta_0, \frac{1}{2}(pq)^{\frac{1}{1-p}}\right\}. \tag{18}$$

Indeed if (16) and (17) hold, it follows that

$$h'(t) \leq 0 \ \text{ on } \ I_1 = [\eta, \eta_1] \subset I.$$

Therefore h crosses the t-axis only once (since h is nonincreasing on I_1).

Set

$$\overline{\eta} = \min\{\eta_1, \eta_2\} \tag{19}$$

where,

$$\eta_2 = (2^p q)^{\frac{1}{1-p}}. \tag{20}$$

It follows from (13) and (19) that if

$$\eta \leq \overline{\eta}, \tag{21}$$

then

$$r \leq 2\eta = r_0. \tag{22}$$

We can state and prove the main semilocal convergence theorem for Newton's method involving a p-(PBA) ($p > 1$) approximation for f.

Theorem 2. *Let X and Y be Banach spaces, D a closed convex subset of X, $x_0 \in D$, and F a continuous operator from D into Y. Suppose that F has a p-(PBA) approximation at x_0. Moreover assume:*

$\delta(A(x_0, \cdot), D) \geq \delta_0 > 0$;

$$2\ell_0 < \delta_0, \quad (1 - 2\ell_0\delta_1^{-1}r_0)\delta_0\eta - \ell\eta^p \geq 0 \tag{23}$$

where,

$$\delta_1 = \delta_0(A(x_0, \cdot), D);$$

(9) *in* [2, Lemma 1] *and conditions* (16), (21) *in Remark 1 hold for* $\alpha = \delta_0\eta$ *and* $q = \ell(\delta_0 - 2\ell_0)^{-1}$;

for each $y \in U(0, \delta_0\eta)$ *the equation* $A(x_0, x) = y$ *has a solution* x;

the solution $T(x_0)$ *of* $A(x_0, T(x_0)) = 0$ *satisfies* $\|x_0 - T(x_0)\| \leq \eta$, *and*

$$U(x_0, r_0) \subseteq D,$$

where r_0 *is given by* (25).

Then the Newton iteration defining x_{n+1} *by*

$$A(x_n, x_{n+1}) = 0$$

remains in $U(x_0, r_0)$, *and converges to a solution* $x^* \in U(x_0, r_0)$ *of equation* $F(x) = 0$, *so that estimates* (8) *and* (9) *hold.*

Proof. It is identical to the proof of Theorem 2 in [3].

Remark 2. For the study of the uniqueness of solution x^* we refer the reader to the corresponding Remark 2 in [3].

Remark 3. Our Theorem 2 compares favorably with Theorem 3.2 in [6, p. 298]. First of all the latter theorem cannot be used when e.g. $p \in [1, 2)$ (see the example that follows). In the case $p = 2$ our condition (23) becomes for $\delta_0 = \delta_1$

$$h_0 = \delta_0^{-1}(\ell + 4\ell_0)\eta \leq 1 \tag{24}$$

where as the corresponding one in [5] becomes

$$h = 4\delta_0^{-1}\ell\eta \leq 1. \tag{25}$$

Clearly (24) is weaker than (25) if

$$\frac{\ell}{\ell_0} > \frac{4}{3}. \tag{26}$$

But $\frac{\ell}{\ell_0}$ can be arbitrarily large [2]. Therefore our Theorem 2 can be used in cases when Theorem 3.2 in [6] cannot when $p = 2$.

Example 1. Let $X = \mathbf{R}^{m-1}$, $m \geq 2$ an integer and define matrix operator Q on X by

$$Q(z) = M + M_1(z), \quad z = (z_1, z_2, \dots, z_{m-1}), \tag{27}$$

where M is a real $(m-1) \times (m-1)$ matrix,

$$M_1(z) = m_{ij} = \begin{cases} 0 & i \neq j, \\ z_i^p, & i = j \end{cases}$$

and e.g. $p \in [1, 2)$.

Operators Q of the form (27) appear in many discretization studies in connection with the solution of two boundary value problems [1]. Clearly no matter how operator A is chosen the conditions in Definition 2.1 in [6, p. 293] cannot hold, whereas our result can apply.

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CAMERON UNIVERSITY, DEPARTMENT OF MATHEMATICS SCIENCES, LAWTON, OK 73505, USA
Email address: `iargyros@cameron.edu`