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Characterization of Radial Stress in Curved Beams[*1]

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ABSTRACT

Curved glued laminated timber (glulam) is rapidly coming into the domestic modern timber frame buildings and predominant in building construction. The radial stress is frequently occurred in curved beams and is a critical design parameter in curved glulam. Three models, Wilson equation, Exact solution and Approximation equation were introduced to determine the radial stress of curved glulam under pure bending condition. It is obvious that radial stress distribution between small radius and large radius was different due to slight change of neutral plane location to center line. If the beam design with extremely small radius, it should be considered to determine the exact location of maximum radial stress. The current standard KSF 3021 was reviewed and would be considered some adjustment determining the optimum radius in curved glulam.

Current design principle is that the stress factor is given by the curvature term only in constant depth of the beam, but like tapered or small radius of beams, the stress factor by Wilson equation was underestimated. So current design formula should be considered to improvement for characterizing the radial stress factor under pure bending condition.

Keywords : curved glued laminated timber beams (glulam), radial stress, stress factor, Wilson equation, Exact solution, Approximation equation

1. INTRODUCTION

Glued laminated timber beam (glulam) is an engineered wood product and commonly used in long span structure with large cross section. Glulam is rapidly coming into the domestic modern timber frame buildings. The most common type of glulam is a straight member as a simple beam, but the curved members are predominant in building construction to construct roof structure. Also, architectural point of view, curved members give an excellent aesthetic features to the buildings.

For most cases, the glulam is made of several wood laminations using adhesives, and the radial stress is introduced as the radius of curvature changes. The induced radial stress by bending, tension perpendicular to grain, can cause the failure of the structure by seperation of the fibers (Foschi and Fox, 1970). Therefore, the radial stress which is tension perpendicular to grain or adhesive layers, must be considered in

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the design of curved glulam. The curvature of the beam causes the extreme fibers to be stressed differently to a prismatic straight beam.

As wood is the weakest in the direction of perpendicular to grain in tension, the failure due to radial stress is critical in curved glulam member. This is because the low radial tension stress limit often results in this stress governing the desing load of the member (Gopu, 1980). The curvature factor is applied to curved member design to set and modify the allowable stress. In general, the mathematical theory of elasticity (Timoshenko and Goodier, 1970) is applied to a curved members to determine the radial and circumferential stresses. Wilson (1939) presented approximation equation to calculate maximum stress subjected to bending moment. Also, Noris (1963) studied stresses within curved laminated beams of Douglas-fir lumber using the mathematical theory of elasticity. The basic formula $\sigma_r = \frac{3M}{2Rth}$ is sufficiently accurate only curved beams of constant depth, but analysis shows that this formula underestimates the maximum radial stress in the case of variable depth and design method should be improved (Foschi and Fox, 1970).

In most practical design of curved beams, the estimation of the radial stress is essential. In spite of importance of curved beam design formula, radial stress determination formulas and their comparison were not critically studied. Moreover, the radial stress design of curved beam with tapered cross section pays a little attention in current domestic design standard, KSF 3021.

This paper compared the radial stress equations derived for curved beams, and determined the radial stress distribution according to curvature and tapered section. Furthermore, this presents a discussion to KSF 3021 and a method of estimating stress factor for the radial stress determination under pure bending condition.

2. FORMULAS for DETERMINATION of RADIAL STRESS in CURVED BEAM

2.1. Wilson Equation

When a curved members are subjected to a bending moment, a stress is induced in a radial direction, that is perpendicular to the grain in wood members. If the moment increases the radius of curvature, the stress is tension; if the moment decreases the radius of curvature, the stress is compression. The formula for computing maximum radial stress (σ_r) with constant cross section is given by Wilson (1939).

$$\sigma_r = \frac{3M}{2Rth} \tag{1}$$

where
M = bending moment
t = width of the member
h = depth of the member
R = radius of center line curvature

This stress equation is used to calculate the maximum radial stress to design the curved glulam, but it is difficult to get the radial stress distribution along the beam depth and that of glulam with tapered section or small radius.

2.2. Exact Solution

In as much as to get the radial stress distribution along the beam depth, the exact solution is useful (Ugural and Fenster, 1981). Consider a beam subjected to equal end couples M such that bending take place in the plane of curvature as shown in Fig. 1. The appropriate polar equation of equlibrium under absence of body forces,

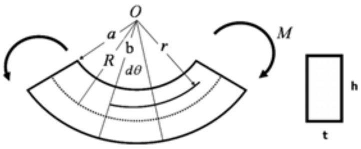


Fig. 1. Curved beam element with applied moment *M.*

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma\theta}{r} = 0 \quad (2)$$

and the compatibility equation for plane stress,

$$\frac{d(\sigma_r + \sigma_\theta)}{dr^2} + \frac{1}{r}\frac{d(\sigma_r + \sigma_\theta)}{dr} = 0 \quad (3)$$

must also be satisfied. Solving the equation by direct integration using the boundary condition as $\sigma_r = 0$ when $r = a$, $r = b$. Also, there is no force acting at the ends, $t\int_a^b \sigma_\theta dr = 0$ and the normal force at the ends produce a couple *M*, leads to $t\int_a^b r\sigma_\theta dr = M$. The radial stress equation from the above each condition, is obtained as follows.

$$\sigma_r = \frac{4M}{tb^2N}\left[(1-\frac{a^2}{b^2})\ln(\frac{r}{a}) - (1-(\frac{a^2}{r^2})\ln(\frac{b}{a})\right] \quad (4)$$

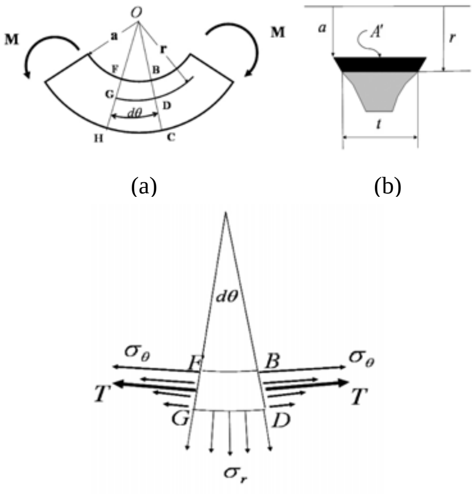
where,

$$N = (1-\frac{a^2}{b^2})^2 - 4\frac{a^2}{b^2}ln^2\frac{b}{a} \quad (5)$$

$a =$ distance from *O* to inner radii of boundary in the cross-section
$b =$ distance from *O* to outer radii of boundary in the cross-section
$r =$ distance from *O* to any point of interest in the cross-section

2.3. Approximation Equation

Although the radial stress is very small, it may be significant relative to radial strength when anisotropic materials such as wood are formed into curved form (Boresi, etc., 1993). From Fig. 2, the F_r represents the radial body force per unit volume, the equilibrium in radial direction is



(a) (b)

(c)

Fig. 2. Determination of radial stress in curved beam. (a) Side view. (b) Cross sectional shape, (c) Element BDGF (Boresi, etc., 1993).

$$\sum F_r = 0 = \sigma_r trd\theta - 2Tsin(\frac{d\theta}{2}) \quad (6)$$

As the term $\sin(\frac{d\theta}{2})$ is about the same to $\frac{d\theta}{2}$, it is obtained as following equation.

$$\sum F_r = 0 = \sigma_r trd\theta - 2T(\frac{d\theta}{2}) \quad (7)$$

Therefore, the radial stress equation is as follows.

$$\sigma_r = \frac{T}{rt} \quad (8)$$

Also *T* represents the circumferenial body

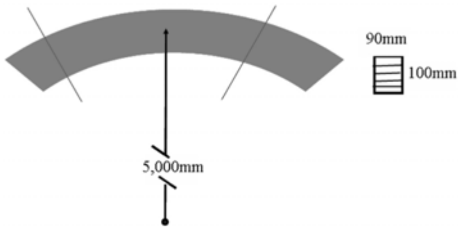


Fig. 3. Curved beam geometry - virtual section.

force per unit volume,

$$T=\int_a^r \sigma_\theta dA, \tag{9}$$

The circumferenial stress (σ_θ) is expressed as

$$\sigma_\theta=\frac{P}{A}+\frac{M_x(A-rA_m)}{rA(RA_m-A)} \tag{10}$$

Direct integration then leads to

$$T=\frac{P}{A}\int_a^r dA+\frac{M_xA}{A(RA_m-A)}\int_a^r\frac{dA}{r}-\frac{M_xA}{A(RA_m-A)}\int_a^r\frac{dA}{r}, \tag{11}$$

$$T=\frac{A'}{A}P+\frac{(AA'_m-A_mA')M_x}{A(RA_m-A)} \tag{12}$$

where

$$A_m=\int_a^c\frac{dA}{r}\ \ (c=a+h)\ ,\ A'_m=\int_a^r\frac{dA}{r},$$
$$A'=\int_a^r dA \tag{13}$$

The effect of axial load is small (Chung and Harold, 1983), and the normal force term is ignored.

From the equation (8) and (12), the radial stress is expressed as

$$\sigma_r=\frac{(AA'_m-A'A_m)}{trA(RA_m-A)}M_x \tag{14}$$

Table 1. Comparison of maximum radial stress determined by each model equation

Model equation	Maximum radial stress (N/mm^2)	Difference[2)]
Wilson equation	$3.33333\times10^{-8}\times M$[1)]	-
Exact solution	$3.33329\times10^{-8}\times M$	0.99998
Approximation equation	$3.33353\times10^{-8}\times M$	1.00005

1) Applied moment
2) ratio of exact/approximation equation to wilson equation

3. RESULTS of APPLICATION of CURVED BEAM FORMULAS to VIRTUAL GLULAM

3.1. Comparison among Radial Stress Equation

In the all case, the curved glulam beam composed of five laminations and has a width *t* and depth *h* of 90 mm × 100 mm and the *R* is 5,000 mm as shown in Fig. 3. The radial stress was calculated using 3 model equations and obtained results are in Table 1. There is no differences among the calculated radial stresses, and the Wilson equation was considered as a approximated procedure to determine the radial stress of curved glulam beam under pure bending conditions with constant depth.

The stress distribution within the beam is approximately parabolic through the cross section as shown in Fig. 4 and is assumed constant throughout the width. As the maximum radial stress occurred in the near middle position, the critical parameter is considered the tensile stress perpendicular the grain in the middle portion of the glulam beam and glue bond strength among the lay-up laminations (Jonsson, 2005). For instance, in Western Hemlock species group, the design properties of tension parallel to grain is 86,000 kPa and tension perpendicular to grain is

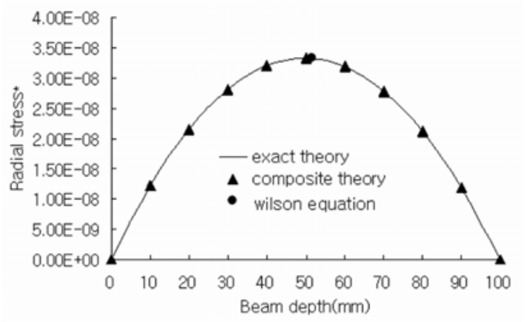


Fig. 4. Radial Stress distribution through the cross section of the curved beam.

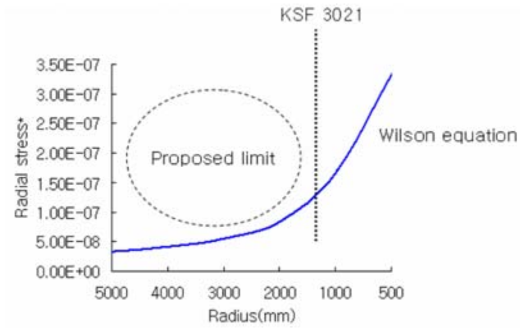


Fig. 5. Radius of curvature limitation KSF 3021 and proposed radius limit (* × M, M is applied moment).

2,300 kPa (Woodhandbook, 1999). Also the allowable radial stress perpendicular to grain is limited to 1/3 the allowable unit stress in horizontal shear stress (AFPA, 2005). It is emphasized that the weakness of wood stressed perpendicular the grain. The radial stress distribution is also different according to the curvature.

The virtual beam in this study is classified as glulam with medium cross section in KSF 3021 - 'Structural Glued Laminated Timber'. In this standard, the minium radius of curvature is 1,300 mm for lamination with thickness 10 mm. The limitation is somewhat too short, because the radial stress is rapidly increasing less than the 2,000 mm radius (Fig. 5). This means that

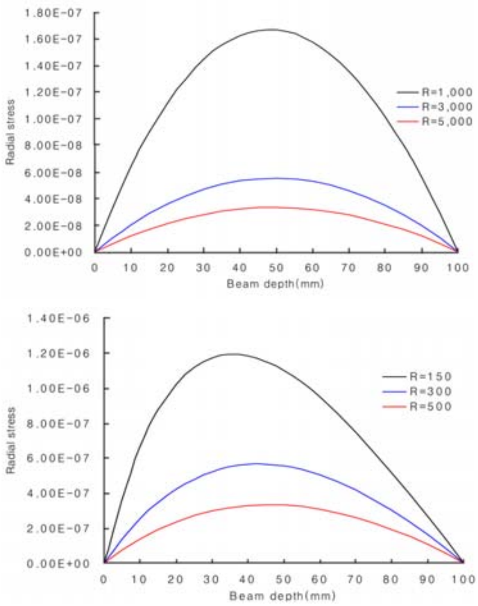


Fig. 6. Radial Stress distribution through the cross section with various radius.

applied moment grows larger, the radial stress is rapidly increasing and leads to uncertain failure between laminations under small radial stress. So it would be discussed to determine the radius limitation in curved glulam beams.

3.2. Radial Stress of Curved Glulam with Various Radius

The radial stress distribution through the beam depth was configurated using the approximation equation. From the equation, the radial stress is always positive for the direction of bending and increases toward the neutral plane and reached maximum near the plane in large radius of curvature. In the small radius of curvature, the maximum radial stress is somewhat displaced from the neutral plane and moves toward to the inner position of the beam as shown in Fig. 6.

As the radial stress is a critical design issue in glulam beam design, it needs to be de-

Table 2. Change of neutral axis location according to radius of curvature

Radius of curvature (mm)	Neutral axis location (mm)	Difference[1)]	Stress ratio[2)]
150	144.269	0.961797	1.022392
300	297.201	0.990671	0.987622
500	498.328	0.996658	0.995551
1,000	999.161	0.999166	0.998889
3,000	2999.722	0.999907	0.999877
5,000	4999.833	0.999967	0.999956

1) ratio to radius of curvature (centroid of the cross-section)
2) ratio of radial stress in neutral plane to radial stress in centroid

Table 3. Comparison of maxium radial stresses by Wilson and Approximation equation

Radius of curvature (mm)	Maxium radial stress (N/mm^2)		Stress ratio[2)]
	Wilson Equation	Approximation Equation	
150	$3.3333 \times 10^{-8} \times M$[1)]	$3.3335 \times 10^{-8} \times M$	1.072781
300	$5.5556 \times 10^{-8} \times M$	$5.5565 \times 10^{-8} \times M$	1.016901
500	$1.6667 \times 10^{-7} \times M$	$1.6691 \times 10^{-7} \times M$	1.005994
1,000	$3.3333 \times 10^{-7} \times M$	$3.3533 \times 10^{-7} \times M$	1.001489
3,000	$5.5556 \times 10^{-7} \times M$	$5.6494 \times 10^{-7} \times M$	1.000165
5,000	$1.1111 \times 10^{-6} \times M$	$1.1920 \times 10^{-6} \times M$	1.000059

1) Applied moment
2) ratio of to Approximation Equation to Wilson Equation.

termined the neutral axis that reaches the maximum radial stress. Let $\overline{R}$ is a distance from O to the neutral axis of the cross-section, $\overline{R}$ is determined by following equation

$$\overline{R} = \frac{A}{\int_A \frac{dA}{r}} \tag{15}$$

The closed form equation for $\overline{R}$ for rectangular cross section is given by the inner and outer radii, a and b respectively, and thickness of the beam t, $dA = tdr$. Therefore,

$$\int_A \frac{dA}{r} = \int_a^b t\frac{dr}{r} = t\ln\frac{b}{a} \tag{16}$$

The distance from center of the curvature of the member to the neutral axis is therefore given by following equation.

$$\overline{R} = \frac{t(b-a)}{t\ln\frac{b}{a}} = \frac{(b-a)}{\ln\frac{b}{a}} \tag{17}$$

It is obvious that greatest radial stress occurred at the inner radius and the smaller region below the neutral axis and showed a different radial stress distribution in small radius of curvature to large radius of curvature. The difference of radial stress in neutral plane to radial stress in centroid was greater as the radius of curvature was decreased. If the beam design with extremely small radius like 150 mm radius, the neutal axis location is somewhat different about 4%, so it should be considered to determine the maxium radial stress configuration. If the radial stress is a critical issue in small radius beams, it

Table 4. Polynominal approximation to K_r*

β	A	B	C
0.0	0.0000	0.2500	0.0000
2.5	0.0079	0.1747	0.1284
5.0	0.0174	0.1251	0.1939
10.0	0.0391	0.0754	0.2119
15.0	0.0629	0.0619	0.1722
20.0	0.0893	0.0608	0.1393
25.0	0.1214	0.0605	0.1238
30.0	0.1649	0.0603	0.1115

* Foschi and Fox (1970)

should be needed the radial reinforcement of curved beams (Kasal and Heiduschke, 2004).

Consquently, Boresi, etc. (1993) pointed out that the approximated radial stress equation is conservative with R/d > 3, the error between Wilson equation and approximation equation is less than 3%. This study's results also showed similar results that the error was less than 1% to R/d > 3, but the error increased 7% in R/d = 1.5 and Wilson equation was considered unconservative. Therefore, it would be necessary to apply the modification to Wilson equation in the design of curved beams with small radius of curvature.

3.3. Stress Factor for Curved Beam with Tapered Section

When analyzing curved member subject to bending moment *M*, the stress factor is used in the design formula. The maximum allowable radial stress is given by the greater of the two expressions:

$\sigma_r = K_r \frac{6M}{th^2}$, or $\sigma_r = \frac{3}{2}\frac{M}{Rth}$. The stress factor, *Kr*, for curved member under pure bending condition can be obtained as follows (Foschi and Fox, 1970).

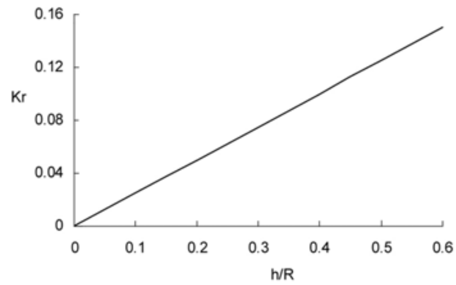


Fig. 7. Radial stress factor for constant depth of the curved beam.

$$K_r = \frac{\sigma_r}{\sigma_o} = \frac{\frac{3M}{2Rth}}{\frac{6M}{th^2}} = \frac{1}{4}\frac{h}{R} \quad (18)$$

where

σ_o = Nominal stress

From the equation, the stress factor is dependent only *h/R* ratio and confirmed Wilson's equation. This equation is adoptable in pure bending condition with constant depth beam design but most common design condition is uniformly distributed loading and partially tapered beam. In the case of partially tapered beam, the stress factor is dependent noy only *h/R* ratio but also the tapered angle β must be considered. The stress factor, K_r for the tapered beam is expressed as a function of *h/R* under pure bending condition;

$$K_r = A + B\frac{h}{R} + C(\frac{h}{R})^2 \quad (19)$$

where *A*, *B*, *C* are constant and calculated results by finite element analysis are in Table 4.

For the pure bending condition with constant depth, the values of *A*, *B*, *C* is 0, 1/4, 0 respectively and K_r showed linear distribution according to *h/R* as shown in Fig. 7. This leads to the same to Wilson equation, that is

Table 5. Comparison of radial stress according to radius and tapered angle

R (mm)	h/R	β (°)							
		0	2.5	5.0	10.0	15.0	20.0	25.0	30.0
150	0.67	1.00*	1.09	1.12	1.10	1.08	1.15	1.30	1.53
1000	0.10	1.00	1.07	1.27	1.95	2.76	3.87	5.15	6.88
5000	0.02	1.00	2.29	4.00	8.12	12.83	18.11	24.53	33.23

*All values are the ratios of calculated radial stress with stress factor to Wilson equation

$$\sigma_r = K_r \frac{6M}{th^2} = \left(\frac{1}{4}\frac{h}{R}\right)\frac{6M}{th^2} = \frac{3}{2}\frac{M}{Rth} \quad (20)$$

From the equation (20), the stress factor is given by the curvature term only in constant depth of the beam. But like pitch cambered beams or small radius of beams, the stress factor by the equation should be considered to improvement. If the beam has small angles of β, the K_r shows parabolic curves from the calculated results using the determined values in Table 4. The K_r values were all over the β = 0, constant depth of the beam, and this was the lower bound. And the K_r values increased rapidly over the h/R > 0.33, that is in the case of very small radius. Also, the radial stress factor under the β = 10, is increased rapidly but this tendency was not over the β = 15. The radial stress factor increased steadily up to 0.33 of h/R ratio and increased rapidly over the former ratio. AITC (2005) proposed C_r as a reduction factor, a function of the shape of the member uniformly distributed member, but not in the pure bendng condition. Thus, It was considered that maxium radial stress equation (Wilson equation) induced by pure bending was needed somewhat to be improved. The radial stress according to radius and tapered angle was compared to the radial stress with constant depth of the beam. The calculated results are in Table 5.

If the beam has constant depth, there is no difference between the two equations. As the β was increased, the ratios was increased, and was also increased with increasing the radius.

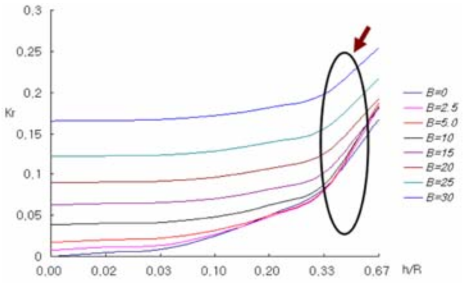


Fig. 8. Radial stress factor for the tapered curved beam (The arrows indicate that rapid increasing zone in stress factor).

The beam which has tapered angle β smaller than 30 and h/R ratio 0.1, the stress factor is 0.17, and that given by Wilson equation 0.025. This lead to 3.4 times difference and overestimation the current design formula. Gopu (1980) pointed out the radial tension stress factor was overestimated the current procedure in the case of uniformly distributed loading condition and needs to be revised to correct the present conservation design of curved beams for practical loading condition. Also the comparison of radial stress between the two equations indicated that the difference became larger as the increasing the radii and tapered angle (Table 5). For the beam that has R = 1,000 and β smaller than 10, the difference was 95%, which means radial stress with stress factor of Wilson equation was underestimated. The β becomes greater than 10, the difference was more larger and when β is 30, the difference was 6～7 times than stress factor of Wilson equation. Also, the stress factor is rapidly increased over the *h/R* > 0.33 and it should be considered to design the curved beam with small radius. So current design formula should be discussed further to determine the radial stress factor under pure bending condition.

4. CONCLUSIONS

The radial stress under pure bending condition was estimated by Wilson equation, exact solution and approximation equation. As their comparison indicated that three equations are identical, this lead to the verification of Wilson equation. Through the virtual beam study, KSF 3021 - 'Structural Glued Laminated Timber' the limitation for curvature is somewhat too short, and it is possible to occur uncertain failure between laminations under small radial stress.

The maximum radial stress occurred in the near middle position, similar to the neutral plane, but the difference between neutral plane and centroid becomes larger as the radii becomes smaller and stress distribution is eccentric in small radii.

Current design principle is that the stress factor is given by the curvature term only in constant depth of the beam. But like tapered beams or small radius of beams, the stress factor by the equation should be considered to improvement. K_r values increased rapidly over the *h/R* > 0.33, that is in the case of very small radius. The difference between calculated radial stress determined by stress factor to Wilson equation was increased as the β was increased, this lead to high difference and overestimation the current design formula. So the further studies to determine the radius limitation in curved glued laminated beams and the radial stress factor under pure bending condition would be needed to optimum design of curved glulam.

REFERENCES

1. American F. and P. Association. 2005. National design specification for wood construction. ANSI/ AF&PA NDS-2005 American Wood Council.
2. American Institute of Timber Construction. 2005. Timber construction manual. American Institute of Timber Construction. John Wiley and Sons, Inc.
3. Boresi, A. P., R. J. Schmidt, and O. J. Sidebottom. 1993. Advanced mechanics of materials, 5th edition. John Wiely & Sons, Inc.
4. Cheung, C. K. and H. C. Sorensen. 1983. Effect of loads on radial stress in curved beams. Wood and Fiber Science 15(3): 263～275.
5. Forest Product Laboratory. 1999. Wood handbook. FPL General Tech. Report FPL-GTR-113.
6. Foschi, R. O. and S. P. Fox. (1970) Radial stress in curved timber beams. J. of Structural Div. ST10: 1997～2008.
7. Gopu, V. K. A. (1980) Radial stresses in curved glulam beams. J. of Structural Div. 106 (ST11): 15797～15801.
8. Jonsson, J. 2005. Load carrying capacity of curved glulam members reinforced with self-tapping screws. Holz als Roh- und Werkstoff 63: 342～346.
9. Kasal, B. and A. Heiduschke. 2004. Radial reinforcement of curved glue laminated wood beams with composite materials. Forest Prod. J. 54(1): 74～79.
10. Korea Standard Association. 2005. Structural glued laminated timber. KS F 3021.
11. Norris, C. B. (1963) Stresses within curved laminated beams of Douglas-fir. USDA. Forest Service. Forest Product Laboratory. FPL-020.
12. Timonshenko, S. P. and J. N. Goodier. (1970) Theory of elasticity. 3rd version. Mcgraw-Hill.
13. Wilson, T. R. C. 1939. The glued laminated wooden arch. USDA Techincal Bulletin No.691.
14. Ugural A. C. and S. K. Fenster. 1981. Advanced strength and applied elasticity-The SI version. Elsevier North Holand, New York.