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On the Convex Hull of Multicuts on a Cycle*

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ABSTRACT

The minimum multicut problem on a cycle is to find a multicut on an undirected cycle such that the sum of weights is minimized, which is known to be polynomially solvable. This paper shows that there exists a compact polyhedral description of the set of feasible solutions to the problem whose number of variables and constraints is $O(v\kappa)$.

Keywords: Multicuts, Cycles, Convex Hull

1. Introduction

Given an undirected graph $G=(V,\ E)$ with a weight $c_e \in \mathbb{R}$ for each edge $e \in E$ and a set of node pairs $(o_k,\ d_k)$ for each $k \in K$, where $K=\{1,\ 2,\cdots,\ \kappa\}$ and $o_k \neq d_k$, a *multicut* is a set of edges $C \subseteq E$ whose removal from G disconnects o_k from d_k for each $k \in K$. The *minimum multicut problem* is to find a multicut $C \subseteq E$ in G such that $\sum_{e \in C} C_e$ is minimized. The problem is known to be NP-hard and MAX-SNP-hard even for $\kappa=3$ Dahlhaus *et al.* [5]. However, it is tractble for trees if the number of node pairs is bounded Bartholdi *et al.* [4]. Moreover, there exists a polynomial-time algorithm for the problem in planar graphs where all the nodes, o_k and d_k for all $k \in K$, are on the outer face and the number of node pairs is bounded Bartholdi *et al.* [4]. For more detailed discussion on the problem, we refer the readers to Schrijver [6].

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This paper considers the minimum multicut problem defined on a cycle (MMC). MMC is a special case of the minimum multicut problem defined on planar graphs where all the nodes, o_k and d_k for all $k \in K$, are on the outer face and the number of node pairs is bounded, so MMC is polynomially solvable Bartholdi *et al.* [4]. Since there exists a polynomial-time algorithm for MMC, there also may exist an explicit description of the convex hull of the set of the feasible solutions. This paper shows that there exists a compact formulation of the convex hull of the feasible solutions to MMC.

Let $G = (V, E)$ be an undirected cycle with $V = \{1, 2, \cdots, v\}$ and $L = \{(1, 2), (2, 3), \cdots, (v-1, v), (v, 1)\}$. Note that $|V| = |E| = v$ in a cycle G. For each $e \in E$, let us define a binary variable x_e which is equal to 1 if edge e is included in a multicut, 0, otherwise. Let $P_k^+ = \{(i, i+1) \in E \mid o_k \le i < d_k\}$ and $P_k^- = E \setminus P_k^+$, for each $k \in K$. Then P_k^+ (P_k^-) is the set of edges which are used by the clockwise (counter-clockwise) path between o_k and d_k.

Now, let $X \subseteq \mathbb{B}^v$ be the set of binary vectors that satisfy the following inequalities (1) and (2):

$$\sum_{\{e \in E \mid e \in P_k^+\}} x_e \ge 1, \quad \text{for all } k \in K, \tag{1}$$

$$\sum_{\{e \in E \mid e \in P_k^-\}} x_e \ge 1, \quad \text{for all } k \in K. \tag{2}$$

Then, X is the set of incidence vectors of multicuts on a cycle.

In the next section, we show that there exists a compact polyhedral description of $\operatorname{conv}(X)$, i.e., the number of variables and constraints is bounded by a polynomial function in v and κ. This compact description is extended in the sense that it gives a polyhedral description of $\operatorname{conv}(X)$ in a higher dimensional space that includes the original space, so that $\operatorname{conv}(X)$ can be obtained by projecting down the description onto the original space.

2. A compact Formulation of conv(X)

Let $P \subseteq \mathbb{R}^v$ be the polytope defined by inequalities (1) and (2) along with (3), which is the linear relaxation of X.

$$0 \le x_e \le 1, \text{ for all } e \in E, \tag{3}$$

Let $P(n) := P \cap \{x \in \mathbb{R}^v \mid \sum_{e \in E} x_e = n\}$, and let $X(n) := X \cap P(n)$ for each $n = 2, \cdots, v$. Then, the following lemma can be readily shown.

Lemma 1. $X = \bigcap_{n=2}^{v} X(n)$ and $P(n)$ is integral for each $n = 2, \cdots, v$.

Proof. Since $2 \le \sum_{e \in E} x_e \le v$ for every feasible point in X, it is clear that $X = \bigcup_{i=2}^{v} X(n)$. Now, consider $P(n)$ for some nonnegative integer n. Let A be the $(2\kappa+1) \times v$ constraint matrix of $P(n)$ other than the bound constraints (3), in which the last row corresponds to the left-hand-side vector of the equation, $\sum_{e \in E} x_e = n$. Observe that A is a circular 1's matrix, i.e., the columns of A can be permuted in such a way that the 1's appear consecutively in each row. Note that the last and first elements of a row are also considered to be consecutive. Bartholdi *et al.* [3] showed that a polyhedron whose constraint matrix is a circular 1's matrix is integral if the right-hand-side vector is integral and the sum of values of all the variables is fixed to some integer, which means $P(n)$ is integral for any integer n. Therefore, the result follows. □

The following lemma 2 is helpful in our subsequent analysis, which is given by Balas (Balas [1, 2]).

Lemma 2. Let $\Pi_i := \{x \in \mathbb{R}^n : A^i x \le b^i\}, i \in M$, be a finite set of polyhedra, and let $C^i := \{x \in \mathbb{R}^n : A^i x \le 0\}$ be the recession cone of P^i for each $i \in M$, where $M := \{1, 2, \cdots, m\}$. If $C^i = C^j$ for each pair of $i, j \in M$ such that $i \ne j$, then $\text{conv}(\bigcup_{i \in M} \Pi_i)$ is the projection of the following polyhedron Ω onto x-space:

$$\Omega := \left\{ x \in \mathbb{R}^n,\ (y^i,\ y_0^i) \in \mathbb{R}^m \times \mathbb{R} : \begin{array}{c} x - \sum_{i \in M} y^i = 0, \\ A^i y^i \le b^i y_0^i, \ \text{for all } i \in M, \\ \sum_{i \in M} y_0^i = 1, \\ y_0^i \ge 0, \ \text{for all } i \in M. \end{array} \right\}$$

Now, let Q be a polyhedron defined by the following set of equalities and inequalities (4)~(9), whose number of variables and constraints of is $O(v\kappa)$.

$$x_e = \sum_{i=0}^{v} y_e^i, \text{for all } e \in E, \tag{4}$$

$$\sum_{\{e \in E | e \in P_k^+\}} y_e^i \ge z_i, \text{ for all } k \in K,\ i \in \{2, \cdots, v\}, \tag{5}$$

$$\sum_{\{e \in E | e \in P_k^-\}} y_e^i \ge z_i, \text{ for all } k \in K,\ i \in \{2, \cdots, v\}, \tag{6}$$

$$\sum_{e \in E} y_e^i = iz_i,\ i \in \{2, \cdots, v\}, \tag{7}$$

$$0 \le y_e^i \le 1,\ e \in E,\ i \in \{2, \cdots, v\}, \tag{8}$$

$$\sum_{i=0}^{v} z_i = 1,\ z_i \ge 0,\ i \in \{2, \cdots, v\}. \tag{9}$$

Then, the following theorem shows that there exists a compact polyhedral description of conv(X).

Theorem 1. Q is an extended formulation of conv(X).

Proof. By lemma 1, conv(X) = conv$(\bigcup_{n=2}^{v} X(n))$ and conv$(X(n))$ = $P(n)$ for each $n \in \{2, \cdots, v\}$. Hence, conv$(\bigcup_{n=2}^{v} X(n))$ = conv$(\bigcup_{n=2}^{v} P(n))$. Observe that $P(n)$ for all $n = 2, \cdots, v$ have the same recession cones. Therefore, the projection of Q onto x-space is conv$(\bigcup_{n=2}^{v} P(n))$ by lemma 2, which means Q is an extended formulation of conv(X). □

3. Concluding remarks

The convex hull of X in the original space can be obtained by the projection of Q onto the original space, which would be needed to characterize the facets of conv(X). We also think that it is worthwhile to apply the disjunctive approach used in this paper to more general polynomially solvable cases such as the minimum multicut problem defined on planar graphs.

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