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A Batch Arrival Queue with a Random Setup Time Under Bernoulli Vacation Schedule*

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ABSTRACT

We consider an $M^x/G/1$ queueing system with a random setup time under Bernoulli vacation schedule, where the service of the first unit at the completion of each busy period or a vacation period is preceded by a random setup time, on completion of which service starts. However, after each service completion, the server may take a vacation with probability p or remain in the system to provide next service, if any, with probability $(1-p)$. This generalizes both the $M^x/G/1$ queueing system with a random setup time as well as the Bernoulli vacation model. We carryout an extensive analysis for the queue size distributions at various epochs. Further, attempts have been made to unify the results of related batch arrival vacation models.

Keywords: $M^x/G/1$ Queue, Setup Time, Vacation Time, Bernoulli Schedule Vacation, Queue Size

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1. Introduction

Vacation models are characterized by the fact that the idle time of the server may be used for some other secondary job, for instance to serve the customers in other systems. Allowing the server to take vacations makes the queueing models more realistic and flexible in studying real world queueing situations; e.g. see [2, 15, 23]. Applications of such models arise naturally in call centers with multitask employees, telecommunication and computer networks, production and quality control problems, etc. Baba [3], Teghem [32], Rosenberg and Yechiali [27], and Choudhury [6] among others generalized this type of model for batch arrival queueing systems, and recently, Tadj [29] generalized for a bulk service queueing system. The literature on vacation models is growing very rapidly and it has become the subject matter of current research due to its numerous applications in many real life situations. Instead of reviewing it here again, it will be more convenient to refer the readers to survey papers by Doshi [13, 14], Teghem [31], Medhi [25], and also the monograph of Takagi [30] for more information as well as a complete set of references.

The classical vacation scheme with Bernoulli schedule discipline was introduced and studied by Keilson and Servi [17]. In their model of type *GI/G/1*, a single channel goes on vacation when the queue becomes empty. The server keeps taking vacations until at least one customer is present in the system upon completion of a vacation period. If on service completion the queue is not empty, the server goes on vacation with probability $p(p>0)$ and resumes service with probability $q=1-p$. Following the seminal work of Keilson and Servi [15], the queueing model with vacation under Bernoulli schedule received attention from many authors [9, 18, 19, 24, 26, 28].

Many queueing situations have the feature that the service of the first unit at the commencement of each busy period needs a random setup time, on completion of which service begins. This type of queueing system is known as queue with a random setup time and was investigated by Doshi [12] and Levy and Kleinrock [22], almost simultaneously. Recently, Choudhury [5, 7], Choudhury and Krishnamoorthy [8], Ke [16] and Lee and Park [21] among others studied this type of batch arrival queueing model without and with control operating policies. Although some aspects have been discussed separately on queueing systems with setup time, Bernoulli schedule, and multiple vacations, however, no works are found that combine these features together, even in most recent studies. The fundamental reason for analyzing such type of model is that its structure appears in many representations of computer and communication networks. Another reason is that we analyze the stationary system behavior in more depth. Most of the previous studies only give solutions in terms of generating functions; e.g. see [5, 7, 9]. However, in the present study, we develop a more detailed analysis which includes recursive completion of limiting probabilities. To this end, the mathematical methodology will be based on a combination of the embedded Markov chain technique and the theory of Markov regenerative processes.

In this paper, we first describe the mathematical model in section -2. Section -3 deals with queue size distribution at a busy period initiation epoch. The queue size distribution due to idle period process is discussed in section -4. The embedded Markov chain at a departure epoch of the queue size distribution is investigated in section -5. Finally, in section -6, we study the queue size distribution of the server's state.

2. Mathematical Model

We consider a single server queueing system in which arrival occurs according to a compound Poisson process with batches of random size X. The server is turned off as soon as the system becomes empty. The system becomes operative only when one or a batch of customers arrives to the system. At this point, the server does not offer proper service to the first customer immediately. Rather, it undertakes an additional amount of time called setup time (*SET*), during which no proper work is done, in order to bring the system to operative mode (setup period). The setup time random variable 'S' is assumed to follow a general law of probability with distribution function (d.f) $S(x)$, Laplace-Stieltjes transform (LST) $S^*(\theta)$, and finite moments $E(S^k)(k \geq 1)$. On completion of the setup period, the server starts the actual service (busy period) to the waiting units on an FCFS basis. We assume that the service time random variable 'B' follows a general law of probability with d.f $B(x)$, *LST* $B^*(\theta)$, and finite moments $E(B^k)$ ($k \geq 1$). As soon as the service of a unit is completed, the server may go for a vacation of random length 'V' (vacation period) with probability p ($0 \leq p \leq 1$) or may continue to serve the next unit, if any, with probability $q = (1-p)$. Otherwise, it turns off the system and the system will be turned on again if a batch of customers arrives to the system. This is the case of the '***Bernoulli vacation schedule with single vacation***'. Thus, the system remains idle during a turned off period and a random setup period and these two periods constitute a generalized idle period. Next, we assume that the vacation random variable 'V' follows a general probability distribution with d.f $V(x)$, LST $V^*(\theta)$, and finite moments $E(V^k)$ $(k \geq 1)$ and is independent of the setup time random variable.

Notationally, our model may be denoted as $M^X / G / V_S / 1(BS) / SET$ queue, where V_S represents single vacation, BS denotes Bernoulli schedule, and SET represents setup time. Thus the total service time required by a unit to complete the service cycle is given by

$$G = \begin{cases} B + V & \text{with probability } p \\ B & \text{with probability } q = (1-p) \end{cases}$$

3. Queue Size Distribution at Busy Period Initiation Epoch

In this section, we derive the probability generating function (PGF) of the queue size distribution at a busy period initiation, as a supplementary tool for deriving the queue size distribution at different epochs. To derive it, we define $\alpha_n (n \geq 1)$ as the steady state probability that an arbitrary (tagged) customer finds a batch of 'n' customers in the queue (including those that are in service, if any) at a busy period initiation epoch (or completion epoch of the idle period). Then, conditioning on the number of units within the arriving batches during the setup period, and utilizing the argument of $PASTA$ (see Wolff [34]), we may write following state equation

$$\alpha_n = \sum_{i=1}^{n} a_i \sum_{j=0}^{n-i} b_j a_{n-i}^{(j)}; \quad n \geq 1 \tag{3.1}$$

where $a_k = Prob\,[X = k]$; $k = 1, 2, \cdots$, X is the size of arrival batches (a random variable) with X_i's i.i.d. random variables having the same distribution as X, $B_n = X_1 + X_2 + \cdots + X_n$. Also,

$a_j^{(n)} = Prob\ \{B_n = j\}$ is the n-fold convolution of $\{a_j\}$ with itself and $a_j^{(0)} = 1$,

$b_k =\ Prob$ {'k' individual units arrive during a SET}

$= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^k}{k!} dS(t); k \geq 0$.

Let $\alpha(z) = \sum_{n=1}^{\infty} z^n \alpha_n$ and $X(z) = \sum_{n=1}^{\infty} z^n a_n$ be the PGFs of $\{\alpha_n\ n \geq 1\}$ and $\{a_n\ n \geq 1\}$, then from equation (3.1), we have

$$\alpha(z) = X(z) S^*(\lambda - \lambda X(z)) \tag{3.2}$$

which is the PGF of the queue size distribution at busy period initiation epoch.

The first two factorial moments are given by

$$E(\alpha) = \sum_{n=1}^{\infty} n\alpha_n = \alpha'(1) = E(X)[1 + \lambda E(S)] \qquad (3.3(a))$$

and $$E[\alpha(\alpha-1)] = \sum_{n=2}^{\infty} n(n-1)\alpha_n = \alpha''(1)$$

$$= E[X(X-1)][1 + \lambda E(S)] + \lambda E^2(X)[\lambda E(S^2) + 2E(S)] \qquad (3.3(b))$$

Note that expression (3.3(a)) represents the mean (expected) number of arrivals during the length of the idle period. Now utilizing Little's formula in (3.3(a)), we get

$$\frac{E(\alpha)}{\lambda E(X)} = \frac{1}{\lambda} + E(S) = E(T_0);$$

which is the expression for the expected length of the idle period process of this model.

4. Queue Size Distribution Due to Idle Period Process

In this section our objective is to obtain the stationary queue size distribution due to the idle period process. To obtain it, let us define $\{\psi_n;\, n \geq 0\}$ as the steady state probability that a batch of 'n' customers arrived before a tagged customer during the forward recurrence time (residual life) of the idle period in which the tagged customer is chosen randomly from the arriving batch that turns up at the busy period initiation epoch. Now since the batch of arriving customers is associated with the tagged customer which is chosen randomly from the arriving batches that turns up at the busy period initiation epoch, by virtue of *"stationary renewal process"* (see [20], page 94), we may write

$$\psi_n = \sum_{k=n+1}^{\infty} \frac{\gamma_k}{k};\ n = 0, 1, 2, \cdots \tag{4.1}$$

where $\{\gamma_k;\, k \geq 1\}$ is the probability that the k-th batch that starts a busy period to which the tagged arrival belongs is chosen randomly with probability $(1/k)$. This can be obtained directly from equation (3.1) by applying length biasing argument of renewal theory. Thus, we get

$$\gamma_n = \frac{n\alpha_n}{\sum_{n=1}^{\infty} n\alpha_n} = \frac{n\alpha_n}{E(X)[1+\lambda E(S)]};\ n = 1, 2, \cdots \tag{4.2}$$

Let $\psi(z)$ be the PGF of $\{\psi_n;\, n \geq 0\}$, then we have

$$\psi(z) = \frac{[1-\alpha(z)]}{E(X)[1+\lambda E(S)](1-z)} = \frac{[1-X(z)S^*(\lambda - \lambda X(z))]}{E(X)[1+\lambda E(S)](1-z)}; \tag{4.3}$$

which is the PGF of the number of customers arrived during the residual life of the idle period (i.e. during a turned off period plus a random setup period). Because of the *PASTA* property (see Wolff [34]) this is equivalent to the PGF of the number of customers that arrive during an interval from the beginning of the idle period to a random point in the idle period. More specifically, we may call it queue size distribution due to the idle period. Note that for single unit arrival case, our equation (4.3) is consistent with the result obtained in Takagi [30] (see page 131).

The mean queue size due to idle period is found to be

$$L_0 = \psi^{/}(1) = \frac{E[\alpha(\alpha-1)]}{2E(\alpha)}$$

$$= \frac{[E(X(X-1))\{1+\lambda E(S)\} + \lambda E^2(X)\{\lambda E(S^2) + 2E(S)\}]}{2E(X)[1+\lambda E(S)]}.$$

Further, from the utility point of view of the idle time, this can be considered as a generalized case of the multiple vacation model, where each vacation begins at the end of a service cycle. Define the following events

T_1 = length of the turned off period

and T_2 = length of the setup period.

Thus, we have

$$E(T_1) = \frac{1}{\lambda} \text{ and } E(T_2) = E(S).$$

Now $\frac{E(T_1)}{E(T_0)}$ is the proportion of the expected amount of time spent by the server in the turned off period to the expected amount of time spent by the server in the idle period. Hence, by the theory of regenerative processes it follows that

Prob {The server is in turned off period / the server is idle}

$$= [1+\lambda E(S)]^{-1} = \zeta \text{ (say)}$$

Similarly, it can be shown that

Prob {The server is in setup period / the server is idle}

$$= \frac{\lambda E(S)}{[1+\lambda E(S)]} = (1-\zeta).$$

Now after some algebraic manipulations with (4.3) and using these interpretations, we can put it in to the form

$$\psi(z)=\frac{\zeta[1-X(z)]S^*(\lambda-\lambda X(z))}{E(X)(1-z)}+\frac{(1-\zeta)[1-S^*(\lambda-\lambda X(z))]}{E(X)E(S)(\lambda-\lambda z)} \tag{4.4}$$

Taking the limit $\zeta \to 0$ (i.e. there is no turned off period in the system) in (4.4), we get

$$\lim_{\zeta\to 0}\ \psi(z)=\frac{[1-S^*(\lambda-\lambda X(z))]}{E(X)E(S)(\lambda-\lambda z)}=\psi_0(z)\ \ \text{(say)} \tag{4.5}$$

which is the PGF of the stationary queue size distribution due to the idle period process of the multiple vacation model, where the server takes a sequence of vacations until it finds at least one unit waiting in the system at end of a vacation, and this verifies the result obtained by Doshi [13] (for single unit arrival case). This is true because of the fact that, in the absence of a turned off period, the random setup time behaves like a vacation time. Thus, from this point on, the idle period process is identical with the multiple vacation models.

Now if we consider the *SET* as a vacation time and suppose that it is deterministic with a constant duration of length T (fixed), then it will be the case of the T-policy model (e.g. see Heyman [13]). Thus, for this model, we have $S^*(\lambda-\lambda X(z))\to \exp\{-\lambda T(1-X(z))\}$ and $E(S)=T$ and therefore (4.5) reduces to

$$\psi_0(z)=\frac{[1-e^{-\lambda T(1-X(z))}]}{TE(X)(\lambda-\lambda z)};$$

which is the PGF of the stationary queue size distribution due to the idle period process of the batch arrival queue under T-policy. Note that for *Prob* $[X=1]=1$, i.e. for the single unit arrival case, this is consistent with formula (18) of Tadj [29].

Remark 4.1: -

The LST of the distribution of the unfinished work can also be obtained from (4.4). Let $U_0^*(\theta)$ be the LST of the probability distribution function for the unfinished work at a stationary point of the idle period. Then, utilizing the standard argument of queueing (e.g. see [30], page 118) we can further obtain

$$U_0^*(\theta) = \psi[\beta^*(\theta)] = \frac{[1 - X(\beta^*(\theta))S^*(\lambda - \lambda X(\beta^*(\theta)))]}{E(X)[1 + \lambda E(S)][1 - \beta^*(\theta)]}$$

where $\beta^*(\theta) = [q + pV^*(\theta)]B^*(\theta)$ is the LST of G i.e., our modified service time random variable.

So that the LST of the probability distribution function of the unfinished work $U^*(\theta)$ (say) of this model can be obtained easily by using the following decomposition result (e.g. see [8])

$$U^*(\theta) = U^*(M^x / G(\text{mod}\,ified) / 1;\ \theta)U_0^*(\theta)$$

where $\rho^* = \rho + \lambda pE(X)E(V)$ and $\rho = \lambda E(X)E(B)$ is the utilization factor of the system and $U^*(M^x / G(\text{mod}\,ified) / 1\ ;\theta) = \dfrac{(1-\rho^*)\theta}{[\theta - \lambda + \lambda X(\beta^*(\theta))]}$ is the LST of the waiting time distribution of the first unit in a batch to which it belongs in the standard $M^x/G/1$ queue with our modified service time, and therefore we have

$$U^*(\theta) = \frac{\left(1-\rho^*\right)\theta[1 - X(\beta^*(\theta))S^*(\lambda - \lambda X(\beta^*(\theta)))]}{E(X)[1 + \lambda E(S)][1 - \beta^*(\theta)]\left[\theta - \lambda + \lambda X\left(\beta^*(\theta)\right)\right]}.$$

5. Queue Size Distribution at a Departure Epoch

Our next objective is to investigate the steady state queue size distribution at a departure epoch. To do this, we follow the argument of embedded Markov chain. Let τ_n be the time of n-th service completion epoch i.e., we are considering the epochs at which total service requested by a customer expires. The sequence $N_m = N(\tau_m + 0)$ forms a Markov chain, which is the embedded Markov renewal process of a continuous time Markov processes.

The sequence $\{N_m;\ m \geq 0\}$ is a homogeneous Markov chain and it is owing to the transition

$$N_{m+1} = \begin{cases} X_1 + V_{m+1} + W_{m+1} - 1; & N_m = 0, \\ N_m + V_{m+1} - 1; & N_m > 0. \end{cases}$$

where V_m is the number of units that arrived during the m-th service period, W_m is the number of units that arrived during the m-th setup period and X_1 is the size of the first batch. Then, the transition probability matrix $P = (P_{i,j})$ is readily seen to be a Δ_2-matrix, which is a special case of $\Delta_{m,n}$-matrices introduced and studied by Abolnikov and Dukhovny [1]. Our Δ_2-matrix differs from that of the $M^x/G/1$ queue in the first row only. Consequently, $\rho^* < 1$, where $\rho^* = \lambda E(X)[E(B) + pE(V)]$, is the necessary and sufficient condition for existence of the steady state solution, which we assume to be met throughout the paper.

Now, we assume $\rho^* < 1$ to guarantee that $\{N_m; m \geq 0\}$ is recurrent-positive. Thus, the limiting probabilities

$$\pi_j = \lim_{m \to \infty} \Pr ob\{N_m = j\}; \ j \geq 0,$$

exist and are positive.

This means that $\{\pi_j; j \geq 0\}$ is the steady state probability that 'j' customers are left behind by a departing customer. Then, the one step transition probability matrix $P = (P_{i,j})$ associated with $\{N_m; m \geq 0\}$ has the elements

$$P_{i,j} = \begin{cases} \sum_{m=1}^{j+1} \alpha_m (qh^1_{j-m+1} + pf_{j-m+1}) & if \ \ i = 0, j \geq 0 \\ (qh^1_{j-m+1} + pf_{j-m+1}) & if \ \ j \geq i-1 \\ 0 \ \ if \ \ i \geq 1, 0 \leq j < i-1 & \end{cases}$$

where $h^1_j = Prob$ {Several batches totaling 'j' customers during a service time 'B'}

$$= \sum_{k=0}^{j} \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^k}{k!} a_j^{(k)} dB(t); \ j \geq 0,$$

$h^2_j = Prob$ {Several batches totaling 'j' customers during a vacation time 'V'}

$$= \sum_{k=0}^{j} \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^k}{k!} a_j^{(k)} dV(t); \ j \geq 0,$$

and $f_k = \sum h_j^1 h_{k-j}^2;\ k \geq 0.$

Then, the Kolmogorov equation associated with the Markov chain $\{N_m;\ m \geq 0\}$ can be written as $\pi_j = \sum_{i=0}^{\infty} \pi_i P_{i,j}$. This implies that

$$\pi_j = \sum_{i=1}^{j+1} (\pi_0 \alpha_i + \pi_i)\{q h_{j-i+1}^1 + p f_{j-i+1}\};\ j \geq 0. \tag{5.1}$$

Because of the presence of convolutions, equation (5.1) can be transformed with the help of following PGF's

$$\pi(z) = \sum_{j=0}^{\infty} z^j \pi_j\,,\quad H_i(z) = \sum_{j=0}^{\infty} z^j h_j^i;\quad \text{for } i = 1, 2$$

$$\text{and}\quad F(z) = \sum_{j=0}^{\infty} z^j f_j = H_1(z) H_2(z).$$

Note that $H_1(z) = B^*(\lambda - \lambda X(z))$ and $H_2(z) = V^*(\lambda - \lambda X(z))$.

Now utilizing (3.2) in (5.1), it becomes

$$\pi(z) = \frac{\pi_0 [1 - X(z) S^*(\lambda - \lambda X(z))][q + p V^*(\lambda - \lambda X(z))] B^*(\lambda - \lambda X(z))}{[\{q + p V^*(\lambda - \lambda X(z))\} B^*(\lambda - \lambda X(z)) - z]}. \tag{5.2}$$

Since $\pi(1) = 1$, equation (5.2) yields

$$\pi_0 = \frac{(1 - \rho^*)}{E(X)[1 + \lambda E(S)]}; \tag{5.3}$$

which is the steady state probability that no unit is waiting at a departure epoch. Also, from equation (5.3), we have $\rho^* < 1$, which is the stability condition under which steady state solution exists.

Thus, we have

$$\pi(z)=\frac{(1-\rho^*)[1-X(z)S^*(\lambda-\lambda X(z))][q+pV^*(\lambda-\lambda X(z))]B^*(\lambda-\lambda X(z))}{E(X)[1+\lambda E(S)][\{q+pV^*(\lambda-\lambda X(z))\}B^*(\lambda-\lambda X(z))-z]}. \tag{5.4}$$

Note that for $p = 0$ (i.e., there is no vacation in the system) and *Prob* [$X = 1$] = 1 (i.e., for single unit arrival case), the above equation (5.4) is consistent with the result obtained by Takagi [30] .

The stochastic decomposition property for this model can be demonstrated easily by showing

$$\begin{aligned}\pi(z)&=\left[\frac{1-X(z)S^*(\lambda-\lambda X(z))}{E(X)[1+\lambda E(S)](1-z)}\right]\left[\frac{(1-\rho^*)(1-z)[q+pV^*(\lambda-\lambda X(z))]B^*(\lambda-\lambda X(z))}{\{q+pV^*(\lambda-\lambda X(z))\}B^*(\lambda-\lambda X(z))-z}\right]\\&=\psi(z)\pi(M^x/G(\text{mod}\,\mathit{ified})/1;\,z)\end{aligned} \tag{5.5}$$

where $\pi(M^x/G(\text{mod}\,\mathit{ified})/1;\,z)$, the second factor in the right hand side of (5.5), is the PGF of the stationary queue size distribution of an $M^x/G/1$ queue with a vacation time under Bernoulli schedule. This can be obtained easily from Pollaczek-Khinchine formula by replacing the original service time distribution by our modified service time distribution, i.e. $\beta^*(\theta)=[q+pV^*(\theta)]B^*(\theta)$ (in terms of LST) and thus we have

$$\pi(M^x/G(\text{mod}\,\mathit{ified})/1;\,z)=\frac{(1-\rho^*)(1-z)[q+pV^*(\lambda-\lambda X(z))]B^*(\lambda-\lambda X(z))}{\{q+pV^*(\lambda-\lambda X(z))\}B^*(\lambda-\lambda X(z))-z}. \tag{5.6}$$

Further utilizing (4.4) in (5.5), we may write

$$\pi(z)=[\zeta A(z)L(z)+(1-\zeta)\psi_0(z)]\pi(M^x/G(\text{mod}\,\mathit{ified})/1;\,z)$$

where

$L(z)=\dfrac{\alpha(z)}{X(z)}=S^*(\lambda-\lambda X(z))$ is the PGF of the number of units arrived during a setup time;

$A(z)=\dfrac{[1-X(z)]}{E(X)(1-z)}$ is the PGF of the number of units placed before a tagged customer in a batch in which the tagged customer arrives;

and $\psi_0(z)=\dfrac{[1-L(z)]}{L'(1)(1-z)}$.

Now, taking the limit $\zeta\to 0$ and $V\approx S$ (i.e., setup time is equivalent to vacation time) in (5.6), we get

$$\lim_{\substack{\zeta\to 0\\ V\approx S}}\pi(z)=\frac{(1-\rho^*)[1-V^*(\lambda-\lambda X(z))][q+pV^*(\lambda-\lambda X(z))]B^*(\lambda-\lambda X(z))}{\lambda E(X)E(V)[\{q+pV^*(\lambda-\lambda X(z))\}B^*(\lambda-\lambda X(z))-z]}$$

$$=\pi_0(z)\quad\text{(say)}\tag{5.7}$$

which is consistent with equation (3.21) of Choudhury and Madan [9] and with the result obtained by Servi [27] for single unit arrival case of Bernoulli schedule vacation under multiple vacation policy. Thus our equation (5.7) represents the PGF of the departure point queue size distribution of an $M^x/G/1$ queue with Bernoulli schedule vacation under multiple vacation policy. Let L_D be the mean queue size at the departure point of time, then

$$L_D=\pi'(1)=\frac{\lambda^2E^2(X)[E(B^2)+pE(V^2)+2pE(B)E(V)]}{2(1-\rho^*)}$$
$$+\frac{\lambda E(X)\rho^*\left[\lambda E(S^2)+2E(S)\right]}{2\left[1+\lambda E(S)\right]}+\frac{\rho^*(2-\rho^*)E(X_R)}{(1-\rho^*)};$$

where $E(X_R)=\dfrac{E\{X(X-1)\}}{2E(X)}$ is the mean residual batch size.

6. Servers State Queue Size Distribution

Finally, in this section an attempt has been made to obtain the queue size distributions due to a setup period, a busy period and a vacation period. To obtain them, we follow the argument of regenerative processes. In this case the state of the system at time 't' can be described by means of a Markov process $X(t) = \{Y(t), N(t), \xi(t)\}$, where $Y(t)$ = 0, 1 or 2 according to whether the server is in setup period, in busy period or server is in vacation period at time 't', $N(t)$ represents the number of customers in the queue (including the customer in the service, if any) at time t, and if $Y(t) \in \{0, 1, 2\}$, then $\xi(t)$ represents the corresponding elapsed setup time, service time and vacation time in progress.

We neglect the elapsed times $\xi(t)$, then the theory of Markov regenerative processes guaranteees that for $(i, j) \in \Omega$, where $\Omega = \{i = 0, j \geq 1$ and $i = 1, 2, \; j \geq 0\}$, the limiting (time average) probabilities

$$Q_{i,j} = \lim_{t\to\infty} \Pr ob\{Y(t), N(t) = (i, j)\}$$

exist.

Again, since the arrival process is compound Poisson process, it follows from Burke's theorem (see Cooper [11], pages 187-188) that the stationary probabilities $\{Q_{i,j};\ (i,j)\in\Omega\}$ are positive under the same conditions of limiting probabilities $\{\pi_j;\ j\geq 0\}$ of the embedded Markov chain $\{N_m;\ m\geq 0\}$ i.e., if and only if $\rho^* < 1$.

Then $\{X(t);\ t>0\}$ is a Markov regenerative process with embedded Markov renewal process $\{N_m;\ m\geq 0\}$, hence we may use classical limiting theorems established by Cinlar [10] to obtain

$$Q_{i,j} = \frac{\sum_{n=0}^{\infty} \pi_n \tau_n(i,j)}{\sum_{n=0}^{\infty} \pi_n m_n};\ (i,j)\in\Omega \tag{6.1}$$

where

- $\tau_n(i,j)$ is the expected amount of time spent by the process $\{X(t);\ t>0\}$ in the state (i, j) during an interval of time between two successive total completion epochs i.e. service given that at the beginning of this interval the number of customers in the queue was 'n'

- m_n is the expectation length of the service cycle given that at the beginning of this interval the number of customers in the queue was 'n'.

We first note that

$$m_n = \begin{cases} \frac{1}{\lambda} + E(S) + E(B) + pE(V); \mathit{for} \quad n = 0 \\ E(B) + pE(V); \qquad \mathit{for} \quad n > 1 \end{cases}$$

and therefore

$$\sum_{n=0}^{\infty} \pi_n m_n = [\lambda E(X)]^{-1} \tag{6.2}$$

First of all let us consider the case of stationary queue size distribution due to setup period. For this case, substituting (6.2) in equation (6.1) and a simple probability argument leads to

$$Q_{0,j} = \lambda E(X) \sum_{i=1}^{j} a_i \sum_{k=0}^{j-i} b_k^1 a_{j-i}^{(k)}; \; j \geq 1, \tag{6.3}$$

where $\lambda b_k^1 = (1 - b_k); \; k \geq 0$.

However, for the cases of a busy period and a vacation period need a little more care.

Now let us suppose that the service time ends (i.e., busy period expires) leaving 'n' customers in the queue. We may distinguish two cases according to the origin of the customer who receives the next service. Let us assume that this customer is a primary one then its service starts at time (say) t=0. Then we observe that the time interval (t, t+Δt) contributes to $\tau_n(1, j)$ if:

(i) the service has not been completed before time t (with probability [1-$B(t)$], and

(ii) (j-n+1) primary customers arrive during (0, t] and therefore we have

$$\tau_n(1, j) = k_{j-n+1}^1; \; \mathit{for} \quad j \geq \max(0, n-1) \tag{6.4}$$

where $k_j^1 = \sum_{i=0}^{j} \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^i}{i!} a_j^{(i)} (1 - B(t)) dt;\ j \geq 0$.

Now, utilizing the argument of regenerative processes (see Tijms [33], page 288) and using (6.2) in (6.1), we get on simplification

$$Q_{1,j} = \lambda E(X) \left[\pi_0 \sum_{n=1}^{j+1} \alpha_n k_{j-n+1}^1 + \sum_{n=1}^{j+1} \pi_n k_{j-n+1}^1 \right];\ j \geq 0 \tag{6.5}$$

Proceeding in similar manner for the case of vacation period it can be shown that

$$Q_{2,j} = \lambda p E(X) \left[\pi_0 \sum_{n=1}^{j+1} \alpha_n \sum_{i=1}^{j-n+1} h_i^1 k_{j-n+1-i}^2 + \sum_{n=1}^{j+1} \pi_n \sum_{i=1}^{j-n+1} h_i^1 k_{j-n+1-i}^2 \right];\ j \geq 0 \tag{6.6}$$

where $k_j^2 = \sum_{i=0}^{j} \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^i}{i!} a_j^{(i)} (1 - V(t)) dt;\ j \geq 0$.

A stable recursive scheme for the computation of the limiting probabilities $\{Q_{i,j};\ i = 1, 2\ \mathit{and}\ j \geq 0\}$ in terms of $\{\pi_j;\ j \geq 0\}$ follows by combining (5.1) with (6.5) and (6.6).

The stationary probabilities of server states are given by

Prob {the server is in setup period} = $Q_0(1) = \frac{(1-\rho^*)\lambda E(X) E(S)}{[1 + \lambda E(S)]}$

Prob {the server is in busy period} = $Q_1(1) = \lambda\, E(X)\, E(B)$

Prob {the server is in vacation period} = $Q_2(1) = \lambda\, p E(X) E(V)$.

Now, let us consider the *PGF*'s of $\{Q_{0,j};\ j \geq 1\}$ and $\{Q_{i,j};\ j \geq 0\}$ for i = 1, 2 which follows from (6.4) and for i = 1, 2 from equations (6.5), (6.6) on utilizing (5.3) and (5.4). Then routine algebraic manipulation of equations (6.4), (6.5) and (6.6) leads to

$$Q_0(z) = \frac{(1-\rho^*) X(z) \left[1 - S^*(\lambda - \lambda X(z))\right]}{\left[1 + \lambda E(S)\right]\left[1 - X(z)\right]}$$

$$Q_1(z)=\frac{(1-\rho^*)\left[1-X(z)S^*(\lambda-\lambda X(z))\right]\left[1-B^*(\lambda-\lambda X(z))\right]}{\left[1+\lambda E(S)\right]\left[1-X(z)\right]\left[\left\{q+pV^*(\lambda-\lambda X(z))\right\}B^*(\lambda-\lambda X(z))-z\right]}$$

$$Q_2(z)=\frac{p(1-\rho^*)\left[1-X(z)S^*(\lambda-\lambda X(z))\right]\left[1-V^*(\lambda-\lambda X(z))\right]B^*(\lambda-\lambda X(z))}{\left[1+\lambda E(S)\right]\left[1-X(z)\right]\left[\left\{q+pV^*(\lambda-\lambda X(z))\right\}B^*(\lambda-\lambda X(z))-z\right]}$$

which are the *PGF*'s of the queue size distribution due to setup period, busy period and vacation period respectively.

Further if we denote P_0 as the stationary probability that the server is idle, then from the observer point of view (see Chaudhry [4]), we may write

$$P_0=\pi_0 E(X)=\frac{(1-\rho^*)}{[1+\lambda E(S)]}.$$

Next, the stationary queue size distribution at a random epoch is given by

$$\begin{aligned}P(z)&=P_0+Q_0(z)+z\left[Q_1(z)+Q_2(z)\right]\\&=\frac{(1-\rho^*)(1-z)\left[1-X(z)S^*(\lambda-\lambda X(z))\right]\left[q+pV^*(\lambda-\lambda X(z))\right]B^*(\lambda-\lambda X(z))}{\left[1+\lambda E(S)\right]\left[1-X(z)\right]\left[\left\{q+pV^*(\lambda-\lambda X(z))\right\}B^*(\lambda-\lambda X(z))-z\right]}.\end{aligned}$$

Thus, the relationship between the stationary queue size distribution at a random epoch and at a post departure epoch is given by

$$P(z)=\frac{(1-z)E(X)}{[1-X(z)]}\pi(z)$$

as expected.

Next we derive following system characteristics as follows

The mean busy period is given by $E(T_b)=\frac{1-P_0}{P_0}E(T_0)$

$$=\frac{[1+\lambda E(S)]}{\lambda P_0}(1-P_0)$$

and the mean busy cycle is given by $E(T_C) = E(T_b) + E(T_0)$

$$= \frac{[1+\lambda E(S)]}{\lambda P_0}$$

The expected number of units in the queue during the busy period, vacation period and setup period are given as follows

$$E(N_B) = Q_1'(1) = \frac{\lambda^2 E^2(X)}{2}\left[E(B^2) + \frac{E(B)\left[\lambda E(S^2) + 2E(S)\right]}{[1+\lambda E(S)]}\right]$$

$$+\lambda E(X)E(B)\left[\frac{\lambda^2 E^2(X)\left\{E(B^2) + pE(V^2) + 2pE(B)E(V)\right\}}{2(1-\rho^*)} + \frac{E(X_R)}{(1-\rho^*)}\right]$$

$$E(N_V) = Q_2'(1) = p\left[\frac{\lambda^2 E^2(X)}{2}\left\{\frac{E(V)\left[\lambda E(S^2) + 2E(S)\right]}{[1+\lambda E(S)]} + E(V^2) + 2E(B)E(V)\right\}\right.$$

$$\left. +\lambda E(X)E(V)\left\{\frac{\lambda^2 E^2(X)\left\{E(B^2) + pE(V^2) + 2pE(B)E(V)\right\}}{2(1-\rho^*)} + \frac{E(X_R)}{(1-\rho^*)}\right\}\right]$$

and $E(N_S) = Q_0'(1) = \dfrac{\lambda E(X)(1-\rho^*)\left\{\lambda E(S^2) + 2E(S)\right\}}{2[1+\lambda E(S)]}$.

So that the mean queue size at the stationary point of time is given by

$$L_Q = E(N_B) + E(N_V) + E(N_S)$$

$$= \frac{\lambda^2 E^2(X)\left\{E(B^2) + pE(V^2) + 2pE(B)E(V)\right\}}{2(1-\rho^*)}$$

$$+ \frac{\lambda E(X)\left[\lambda E(S^2) + 2E(S)\right]}{2[1+\lambda E(S)]} + \frac{E(X_R)\rho^*}{(1-\rho^*)} .$$

For $p = 0$, we obtain

$$L_Q = \frac{\lambda^2 E^2(X)E(B^2)}{2(1-\rho)} + \frac{\lambda E(X)\left[\lambda E(S^2) + 2E(S)\right]}{2[1+\lambda E(S)]} + \frac{E(X_R)\rho}{(1-\rho)} .$$

which is the mean queue size at a stationary point of time for an $M^X/G/1/SET$ queue (e.g., .see [5]).

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