# Robust Model Predictive Control Using Polytopic Description of Input Constraints

**Sang-Moon Lee**†

**Abstract** – In this paper, we propose a less conservative a linear matrix inequality (LMI) condition for the constrained robust model predictive control of systems with input constraints and polytopic uncertainty. Systems with input constraints are represented as perturbed systems with sector bounded conditions. For the infinite horizon control, closed-loop stability conditions are obtained by using a parameter dependent Lyapunov function. The effectiveness of the proposed method is shown by an example.

**Keywords:** Model predictive control, Input constraint, Parameter dependent Laypunov function, LMI

## 1. Introduction

Saturation nonlinearities are shown in virtually all physical systems even though the degree of importance may vary. Input saturation often causes a stability problem or aggravation of the performance of the control systems. Therefore, the model predictive control (MPC) technique for an uncertain linear system subject to input saturation has been studied by many researchers in past decades [1-7]. The model predictive control scheme is a very useful technique to handle time varying systems, input constraints and tracking problems. Recently, Kothare et al. [1] presented an MPC algorithm for a time varying uncertain system with input constraint using a linear matrix inequality (LMI) condition. Cuzzola et al. [2] proposed a less conservative LMI condition for a robust constrained MPC by using a parameter dependent Lyapunov function. And Ding et al. [3] improved performance by adding *N* free control moves before the linear feedback law. The controller design problem is formulated as a minimization of the upper bound of a finite or infinite horizon cost function subject to cost monotonicity. For cost monotonicity, an LMI condition for terminal inequality is derived. In order to derive a less conservative condition, relaxation matrices or a parameter dependent Lyapunov function is used [10-12].

In this paper, we represent a constrained system as a perturbed system with sector bounded conditions. Using a polytopic description of the system with input constraint, we propose a new LMI condition for the constrained robust MPC algorithm. The stability condition is obtained using a parameter dependent Lyapunov function which is dependent on the saturation of the control input. The control inputs are obtained by solving the min-max problem subject to cost monotonicity, which is expressed in terms of the linear matrix inequality (LMI) [8-9]. A numerical example shows the effectiveness of the proposed approach.

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Received : July 17, 2009; Accepted : October 21, 2009

## 2. Problem Statement and Preliminaries

Consider the following time varying uncertain discrete time system:

$$x(k+1) = A(k)x(k) + B(k)u_s(k) \tag{1}$$

with input constraints

$$u_s(k) = \mathrm{sat}(u(k)),$$
$$\mathrm{sat}(u(k)) = \begin{cases} u_i(k), & |u_i(k)| \le u_i^{max}, \\ \mathrm{sign}(u_i(k))u_i^{max} & |u_i(k)| > u_i^{max}, \end{cases} \tag{2}$$
$$k \ge 0, \quad i = 1,2,\dots,m.$$

where $x \in R^n$ is the state vector, $u_s \in R^m$ is the control input vector. Moreover, we assume that

$$[A(k)|B(k)] \in \Omega, \ \forall \tag{3}$$

where

$$\Omega := \mathrm{Co}\{[A_1|B_1],[A_2|B_2],\dots,[A_L|B_L]\} \tag{4}$$

Then, the system matrices of (1) can be expressed as follows, if L nonnegative coefficients $\lambda_i(k)$ (with $i$ = 1, 2, ...,$L$) exist, such that

$$\textstyle\sum_{i=1}^{L}\lambda_i(k) = 1, [A(k)|B(k)] = \sum_{i=1}^{L}\lambda_i(k)\,[A_i|B_i]. \tag{5}$$

The goal of this paper is to design a robust state feedback controller

$$u(k) = F(k)x(k) \tag{6}$$

for (1) using the model predictive control strategy. To find such a control input, we consider the following infinite horizon quadratic cost index

$$J_\infty(k) = \sum_{i=0}^{\infty} x(k+i|k)^T \mathcal{Q} x(k+i|k) + u_s(k+i|k)^T \mathcal{R} u_s(k+i|k) \tag{7}$$

In order to reduce the conservatism of the input constraints (2), we define the following deadzone nonlinearities:

$$\phi(u(k)) = u(k) - sat(u(k)) \tag{8}$$

then the system (1) can be represented to the following equation with deadzone nonlinearities such as in Fig 1:

$$x(k+1) = \{A(k) + B(k)F(k)\}x(k) - B(k)\phi(u(k))$$
$$u(k) = F(k)x(k) \tag{9}$$

where $\phi(\cdot)$ is memoryless time-invariant nonlinearities with sector bound such as:

$$b_i \le \frac{\phi_i(u_i(k))}{u_i(k)} \le a_i \tag{10}$$

where $b_i = 0$ and $a_i$ is pre-defined parameter with $0 \le a_i \le 1$.

Using this sector bounded condition of input constraint, the input nonlinearity can be translated into a constraint on the state vector $x$:

$$|F_i(k)x(k)| \le r_i,\ x(k) \in R_{ri} \tag{11}$$

where $F_i(k)$ is $i$-th row of matrix $F(k)$, $r_i = u_i^{max}/(1-a_i)$ and $R_{ri}$ is given by

$$R_{ri} \triangleq \{x(k) \,|\, |F_i(k)x(k)| \le r_i\} \tag{12}$$

## 3. Main Results

In this section, we propose new LMI conditions for the constrained robust MPC of the system (1) with the infinite-horizon cost function defined in Eq. 7.

In order to represent the input nonlinearity to the polytopic description, the nonlinear function $\phi(\cdot)$ represents the following equation

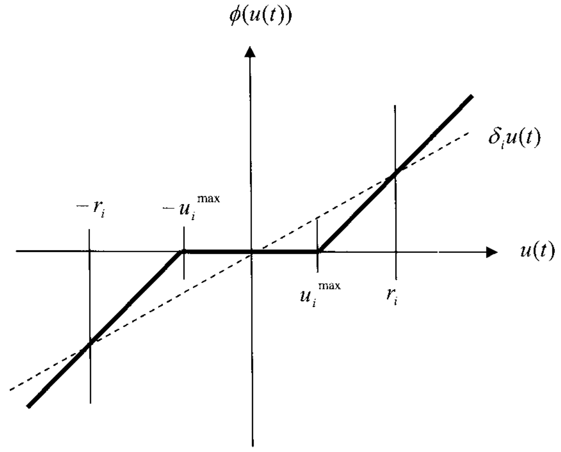


**Fig. 1.** Input Nonlinearity

$$\phi(\cdot) \triangleq h(u)u = diag\{h_1(u_1), \dots, h_m(u_m)\}u$$
$$= \begin{bmatrix} h_1(u_1)u_1 \\ \vdots \\ h_m(u_m)u_m \end{bmatrix} = \begin{bmatrix} \{h_{11}(u_1)a_1 + h_{12}(u_1)b_1\}u_1 \\ \vdots \\ \{h_{m1}(u_m)a_m + h_{m2}(u_m)b_m\}u_m \end{bmatrix} \tag{13}$$

where

$$h_{i1}(u_i) \triangleq \frac{\phi_i(u_i) - b_i u_i}{(a_i - b_i)u_i},\ h_{i2}(u_i) \triangleq \frac{a_i u_i - \phi_i(u_i)}{(a_i - b_i)u_i},$$
$$h_{i1}(u_i) + h_{i2}(u_i) = 1 \text{ and } h_{i1}(u_i), h_{i2}(u_i) \ge 0 \tag{14}$$

Let $\Delta i$ be a vertex matrix of the diagonal matrix $h(u)$, respectively. Then the function $h(u)$ can be represented by:

$$h(u) = \sum_{i=1}^{2^m} \alpha_i(u)\Delta_i, \tag{15}$$

where

$$\sum_{i=1}^{2^m} \alpha_i(u) = 1,\ 0 \le \alpha_i(u) \le 1.$$

For example, if m=2, then:

$$h(u) \in Co\left\{\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} a_1 & 0 \\ 0 & b_2 \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}\right\}$$

Using this polytopic description of input constraint, the closed-loop system with a state feedback controller (6) is given by:

$$x(k+1) = (A(k) + \bar{B}(k)F(k))x(k) \tag{16}$$

where $\bar{B}(k) = B(k)(1 - \sum_{i=1}^{2^m} \alpha_i(k)\Delta_i)$ and the feedback gain $F(k)$ should be satisfied by Eq. 11.

Our proposed LMI conditions are formulated by defining a parameter dependent quadratic Lyapunov function

$$V(i,k) = x(k+i)^T P(i,k) x(k+i),\ i \ge 0, \tag{17}$$

where $\forall k \ge 0$ and $P(i,k) = \sum_{j=1}^{2^m} \sum_{l=1}^{L} \alpha_j(k+i)\lambda_l(k+i)P_{jl}, P_{jl} > 0$.

In the following, we denote the ellipsoid set:

$$\mathcal{E}(\gamma) = \{x \in \mathbb{R}^n : V(i,k) \le \gamma\}. \tag{18}$$

The ellipsoid $\mathcal{E}(\gamma)$ is an invariant set if $\Delta V\ (i,\ k) < 0$ and $\mathcal{E}(\gamma) \subset \mathcal{R}_r$ and the cost index is bounded by the following design conditions:

$$V(i+1,k) - V(i,k) \le -\{x(k+i|k)^T \mathcal{Q} x(k+i|k) + u(k+i|k)^T \mathcal{R} u(k+i|k)\}, i \ge 0. \tag{19}$$

Summing (19) from $i = 0$ to $\infty$ obtains

$$\max_{[A(k+i)|B(k+i)] \in \Omega, i \ge 0} J_\infty(k) \le V(0,k) = x(k|k)^T P(0,k) x(k|k). \tag{20}$$

Define $Q_{ij} = \gamma P_{ij}$, $i = 1, \dots, 2m, j = 1, \dots, L$, $F(k) = YG^{-1}$, then the following theorem gives us the stability condition of the system (1) with the cost index (7).

**Theorem 1 :** Consider the system (1) with an input constraint (2) at time instant $k$. If $2m \times L$ symmetric matrices $Q_{ij}$ exist with $i = 1, 2, ..., 2m, j = 1, 2, ..., L$ and $Y$, $G$ subject to

$$\begin{bmatrix} G + G^T - Q_{ij} & * & * & * \\ A_j G + B_j(1-\Delta_i)Y & Q_{kl} & * & * \\ \mathcal{Q}^{1/2}G & 0 & \gamma I & * \\ \mathcal{R}^{1/2}Y & 0 & 0 & \gamma I \end{bmatrix} > 0, \quad (21)$$

$$\begin{bmatrix} 1 & x(k)^T \\ x(k) & Q_{ij} \end{bmatrix} > 0, \quad (22)$$

$$\begin{bmatrix} Z & Y \\ Y^T & G + G^T - Q_{ij} \end{bmatrix} > 0,\ Z_{ii} \le r_i^2, \quad (23)$$

for $i = 1,\dots,2m$, $j = 1,\dots,L$, $k = 1,\dots,2m.$, $l = 1,\dots,L$, then the state feedback controller $u(k) = Y\,G^{-1}x(k)$ exponentially stabilizes the system in the invariant set $\mathcal{E}(\gamma)$and the upper bound of $J_\infty(k)$, that is, $\gamma$ is minimized by solving the following LMI optimization problem:

$$\min_{Y,G,Q_{ij}} \gamma \quad (24)$$

subject to (21), (22), (23).

**Proof:** Eq. 19 is satisfied with Eq. 6 if and only if, $P(i,k), i \ge 0$ exists, such that:

$$[A(k) + \bar{B}(k)F(k)]^T P(i+1,k)[A(k) + \bar{B}(k)F(k)] - P(i,k) + Q + F(k)^T R F(k) \le 0, \quad (25)$$

Assume that conditions (21), (22), (23) are feasible, then (21) leads to:

$$G + G^T - Q_{ij} \ge 0. \quad (26)$$

for all $i$ and $j$. If $Q_{ij} > 0$ the matrix $(G^T - Q_{ij})Q_{ij}^{-1}(G^T - Q_{ij})$ is nonnegative definite and consequently

$$0 < G + G^T - Q_{ij} \le G^T Q_{ij}^{-1} G. \quad (27)$$

With the inequality (27), using a similar multiplication and summing procedure as proof of Cuzzola et al., the inequality (21) leads to:

$$\begin{bmatrix} P(i,k) & * & * & * \\ P(i+1,k)(A(k+i) + \bar{B}(k+i)F) & P(i+1,k) & * & * \\ \mathcal{Q}^{\frac{1}{2}} & 0 & I & * \\ \mathcal{R}^{\frac{1}{2}}F & 0 & 0 & I \end{bmatrix} > 0, \quad (28)$$

Using the Schur complement, the condition (25) is derived from Eq. 28.

Also, the following upper bound of the cost index is obtained from the condition (25):

$$V(0,k) = x(k|k)^T P(0,k) x(k|k) < \gamma. \quad (29)$$

For the ellipsoid $\mathcal{E}(\gamma)$ and the matrix $F$, the constraint $\mathcal{E}(\gamma) \subset \mathcal{R}_r$ is equivalent to

$$\gamma F_l P(i,k)^{-1} F_l^T \le r_l^2,\ l = 1,\dots,m. \iff \begin{bmatrix} r_l^2 & F_l \\ F_l^T & \frac{P_{ij}}{\gamma} \end{bmatrix} \ge 0,$$

$$\iff \begin{bmatrix} r_l^2 & Y_l \\ Y_l^T & G^T Q_{ij}^{-1} G \end{bmatrix} \ge 0. \quad (30)$$

Finally, the system (1) is exponentially stable at the origin in the invariant set $\mathcal{E}(\gamma)$ if the inequality (25) holds and $\mathcal{E}(\gamma) \subset \mathcal{R}_r$ ∎

**Remark 1:** If we set the parameter $r_i = u_i^{max}$, $i = 1,\dots,m$, it becomes that of Cuzzola et al.(2002).

**Remark 2:** With the parameter $a_i = 1$, i.e., $r_i = 1, i = 1,\dots,m$, if conditions (21),(22), and (23) are feasible at the same time as k, then the state feedback controller (6) globally stabilizes the system (1).

## 4. Numerical Example

Consider the following system

$$x(k+1) = \begin{bmatrix} 1 & 0 \\ K(k) & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k),$$
$$y(k) = [0 \quad 1] x(k), \quad (31)$$

with the input constraint $|u(k)| \le 1$, where $K(k) \in [0.5, 2.5]$ is a time-varying parameter. The objective is to design a constrained robust model predictive controller for the output $y(k)$ to track the set point $y_r = 10$. By using the origin to the steady state $x_s = [0 \quad 10]^T$, we reduce the problem to the regulation problem with the initial condition $x(0) = [0 \quad -10]^T$ . And we set the sector condition $a_1 = 2.5$, the weighting matrices $Q = I$ and $R = 1$.

Fig. 2 shows the simulation results. The proposed algorithm is better than that of Cuzzola et al. [2], because our condition relaxes the input utilized maximally in the invariant set.

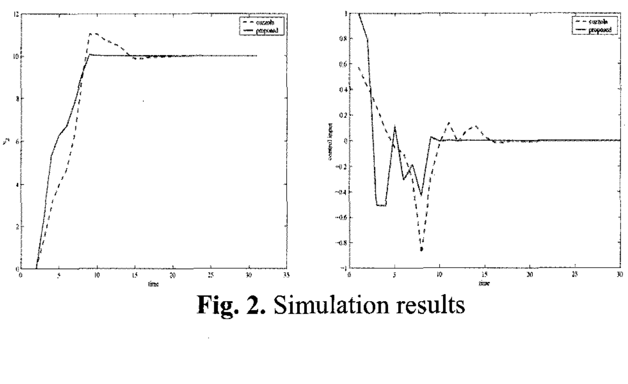

**Fig. 2.** Simulation results

## 5. Conclusion

In this paper, we propose a constrained robust model predictive controller for the polytopic time varying uncertain system with input constraint. Input constraints are also represented as a polytopic uncertainty. In the infinite horizon, the cost index is minimized using the fully utilizable control input at every instant time $k$. A numerical example shows the effectiveness of the proposed method.

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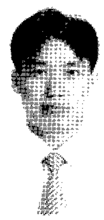

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