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ON HÖLDER CONTINUOUS UNIVERSAL PRIMITIVES

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Abstract. We prove a universality theorem from which we deduce the existence of Hölder continuous universal primitives in the sense of Marcinkiewicz.

1. Introduction

For $p \in (0,1)$ let $L^p([0,1])$ denote the F-space of all measurable functions $g : [0,1] \to \mathbb{R}$ with

$$\int_0^1 |g(x)|^p \, dx < \infty$$

(modulo sets of Lebesgue measure zero), endowed with the metric

$$d(g_1, g_2) = \int_0^1 |g_1(x) - g_2(x)|^p \, dx.$$

In generalization of Marcinkiewicz's very famous result on the existence of universal primitives [10] it is known [6], [7] that to each sequence (λ_n) of positive numbers with limit 0, there exists a continuous function f, such that to each function $g \in L^p([0,1])$ there is a subsequence of

$$\left(\frac{f(x+\lambda_n) - f(x)}{\lambda_n} \right)$$

with limit g in $L^p([0,1])$ (there is no such function f if $p \geq 1$, see [1], [3]). In this paper we will prove a universality theorem from which we will deduce that f may be chosen to be Hölder continuous for each exponent $\alpha \in (0,1)$. Of course there are no Lipschitz continuous universal primitives since each Lipschitz continuous function is differentiable almost everywhere. On the other hand there are universal primitives in the sense of Marcinkiewicz with an amazing amount of points of smoothness [8].

For a comprehensive presentation of generalizations of Marcinkiewicz's result we refer to [5] and the references given there.

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2. Universal elements

We will make use of the Universality Criterion of Grosse-Erdmann [5, Th. 1]:

Let X, Y be topological spaces with X a Baire space and Y second countable. Let $T_j : X \to Y$ $(j \in J)$ be a family of continuous mappings. An element $x \in X$ is called universal for this family if $\{T_j x : j \in J\}$ is dense in Y. Let U denote the set of all universal elements.

Proposition 1 (Universality Criterion)**.** *Equivalent are*:

(1) *The set U is a dense G_δ-subset of X.*
(2) *The set U is dense in X.*
(3) *The set $\{(x, T_j x) : x \in X, j \in J\}$ is dense in $X \times Y$.*

Now, assume that $(X, (p_k)_{k\in\mathbb{N}})$ is a Fréchet space with an increasing sequence of semi norms $(p_k)_{k\in\mathbb{N}}$, and that (Y, d) is a separable topological vector space with invariant metric d. Let $T_n : X \to Y$ $(n \in \mathbb{N})$ be sequence of continuous linear operators. Let $A : D(A) \to Y$ be the linear operator defined by

$$Ax = \lim_{n\to\infty} T_n x$$

on

$$D(A) = \{x \in X : (T_n x) \text{ is convergent}\}.$$

The following criterion will turn out to be useful to prove that U is not empty.

Proposition 2. *Under the settings above assume that*

$$\{Ax : x \in D(A),\ p_k(x) \le 1\} \tag{1}$$

is dense in Y for each $k \in \mathbb{N}$. Then $U \cap \overline{D(A)}$ is a dense G_δ-subset of $\overline{D(A)}$.

Proof. Fix $k \in \mathbb{N}$. Since $D(A)$ is a subspace of X and A is linear, (1) implies that

$$\{Ax : x \in D(A),\ p_k(x) \le \varepsilon\}$$

is dense in Y for each $\varepsilon > 0$. Next we prove that

$$\{Ax : x \in D(A),\ p_k(x - x_0) \le \varepsilon\}$$

is dense in Y for each $\varepsilon > 0$ and each $x_0 \in \overline{D(A)}$:

Fix $y \in Y$ and let $\delta > 0$. Choose $x_1 \in D(A)$ with $p_k(x_1 - x_0) \le \varepsilon/2$, and $x \in D(A)$ with $p_k(x) \le \varepsilon/2$ and $d(Ax, y - Ax_1) \le \delta$. Then

$$p_k((x + x_1) - x_0) \le p_k(x) + p_k(x_1 - x_0) \le \varepsilon,$$

and

$$d(A(x + x_1), y) = d(Ax, y - Ax_1) \le \delta.$$

Now, let $x_0 \in \overline{D(A)}$, $y_0 \in Y$, $k \in \mathbb{N}$ and $\varepsilon > 0$. We find $x \in D(A)$ such that

$$p_k(x - x_0) \le \varepsilon, \quad d(Ax, y_0) \le \varepsilon/2.$$

By choosing $n \in \mathbb{N}$ such that $d(T_n x, Ax) \le \varepsilon/2$ we obtain $d(T_n x, y_0) \le \varepsilon$. Thus

$$\{(x, T_n x) : x \in \overline{D(A)}, n \in \mathbb{N}\}$$

is dense in $\overline{D(A)} \times Y$. Application of Proposition 1 completes the proof. □

3. The Gelfand space of Hölder continuous functions

For $\alpha \in (0,1)$ let $C^\alpha([0,1])$ denote the Banach space of all continuous functions $f : [0,1] \to \mathbb{R}$ with

$$n_\alpha(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$$

endowed with the norm $|f(0)| + n_\alpha(f)$, that is, the space of Hölder continuous functions with exponent α. By $C^1([0,1])$ we denote the space of all continuously differentiable functions $f : [0,1] \to \mathbb{R}$, and $C([0,1])$ is the Banach space of all continuous functions $f : [0,1] \to \mathbb{R}$ endowed with the maximum norm $\| \cdot \|_\infty$.

By $\Lambda^\alpha([0,1])$, $\alpha \in (0,1)$, we denote the space of all $f \in C^\alpha([0,1])$ with the property

$$\forall \varepsilon > 0 \ \exists \delta > 0 : |x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \varepsilon |x - y|^\alpha.$$

It is known that $\Lambda^\alpha([0,1])$ is a closed subspace of $C^\alpha([0,1])$, and

$$C^1([0,1]) \subseteq C^\beta([0,1]) \subseteq \Lambda^\alpha([0,1]) \subseteq C^\alpha([0,1]) \subseteq C([0,1]) \tag{2}$$

for $0 < \alpha < \beta < 1$, compare [2, Ch.IV-23]. Moreover $C^1([0,1])$ is a dense subset of $\Lambda^\alpha([0,1])$ for each $\alpha \in (0,1)$. This can easily be checked for example by means of Friedrich's mollifiers.

Let $(\alpha_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence in $(0,1)$ with limit 1. We consider the Gelfand space

$$C^{1-}([0,1]) := \bigcap_{\alpha \in (0,1)} C^\alpha([0,1]) = \bigcap_{k \in \mathbb{N}} C^{\alpha_k}([0,1])$$

endowed with the sequence of norms $p_k(f) = |f(0)| + n_{\alpha_k}(f)$. Clearly $C^{1-}([0,1])$ is a Fréchet space, and by means of (2)

$$C^{1-}([0,1]) = \bigcap_{\alpha \in (0,1)} \Lambda^\alpha([0,1]),$$

which proves that $C^1([0,1])$ is a dense subset of $C^{1-}([0,1])$.

4. A universality theorem

The following result is a devolvement of [6, Th. 1] to Hölder continuous functions.

Theorem 1. *Let $p \in (0,1)$ and let $T_n : C^{1-}([0,1]) \to L^p([0,1])$ be a sequence of continuous linear operators such that*

$$T_n f \to f' \ (n \to \infty)$$

in $L^p([0,1])$ for each $f \in C^1([0,1])$. Then the set of functions $f \in C^{1-}([0,1])$ such that

$$\{T_n(f) : n \in \mathbb{N}\} \text{ is dense in } L^p([0,1])$$

is a dense G_δ subset of $C^{1-}([0,1])$.

Proof. We will apply Proposition 2. Note that under the assumptions of Theorem 1

$$Af = f' \ (f \in C^1([0,1]) \subseteq D(A)), \quad \overline{D(A)} = C^{1-}([0,1]).$$

Thus, we are done if we can prove that for each $\alpha \in (0,1)$

$$\{f' : f \in C^1([0,1]), |f(0)| + n_\alpha(f) \leq 1\}$$

is dense in $L^p([0,1])$. Since $C([0,1])$ is dense in $L^p([0,1])$, see [4], it is sufficient to approximate continuous functions.

Fix $\alpha \in (0,1)$, and let $g \in C([0,1])$ and $\varepsilon > 0$.

First, we choose $\varphi \in C^\infty(\mathbb{R}, [0,\infty))$ with $\mathrm{supp}(\varphi) \subseteq [0,1]$ and

$$\int_0^1 \varphi(t)\ dt = 1.$$

Since $p \in (0,1)$ we can arrange in addition

$$(\|g\|_\infty)^p \int_0^1 (\varphi(t))^p\ dt \leq \varepsilon.$$

Next, choose $m \in \mathbb{N}$ such that

$$\left(\frac{4}{m}\right)^{1-\alpha} \|g\|_\infty (1 + \|\varphi\|_\infty)^\alpha \leq 1.$$

Set

$$\gamma_k := m \int_{k/m}^{(k+1)/m} g(s)\ ds \quad (k = 0, \dots, m-1).$$

We have

$$\begin{aligned}\beta_k &:= \int_{k/m}^{(k+1)/m} \left| g(t) - \gamma_k \varphi\left(m\left(t - \frac{k}{m}\right)\right)\right|\ dt \\ &\leq \frac{1}{m}(\|g\|_\infty + |\gamma_k|) \leq \frac{2\|g\|_\infty}{m} \quad (k = 0, \dots, m-1).\end{aligned}$$

Define $v, w : [0,1] \to \mathbb{R}$ by

$$v(x) = -\gamma_k \varphi\left(m\left(x - \frac{k}{m}\right)\right)\ (x \in [k/m, (k+1)/m],\ k = 0, \dots, m-1),$$

and

$$w(x) = \int_0^x g(t) + v(t)\ dt \quad (x \in [0,1]).$$

Note that $\mathrm{supp}(\varphi) \subseteq [0,1]$ implies that v is continuous (even in C^∞), hence $w \in C^1([0,1])$, and also

$$\|v\|_\infty \leq \|g\|_\infty \|\varphi\|_\infty.$$

Next we show that $w(k/m) = 0$ $(k = 0, \dots, m-1)$. We have $w(0) = 0$, and

$$\begin{aligned} w((k+1)/m) - w(k/m) &= \int_{k/m}^{(k+1)/m} g(t) + v(t)\ dt \\ &= \frac{\gamma_k}{m} - \frac{\gamma_k}{m} \int_{k/m}^{(k+1)/m} m\varphi\left(m\left(t - \frac{k}{m}\right)\right)\ dt = 0, \end{aligned}$$

as

$$\int_{k/m}^{(k+1)/m} m\varphi\left(m\left(t - \frac{k}{m}\right)\right)\ dt = 1.$$

Let $x \in [k/m, (k+1)/m]$. Then

$$|w(x)| = |w(x) - w(k/m)| \le \int_{k/m}^{(k+1)/m} |g(t) + v(t)|\ dt = \beta_k \le \frac{2\|g\|_\infty}{m}.$$

Thus, for $x, y \in [0,1]$, $x \ne y$

$$\begin{aligned} \frac{|w(x) - w(y)|}{|x-y|^\alpha} &\le \ \min\left\{\frac{4\|g\|_\infty}{m}\frac{1}{|x-y|^\alpha},\ (\|g\|_\infty + \|v\|_\infty)|x-y|^{1-\alpha}\right\} \\ &\le \ \min\left\{\frac{4\|g\|_\infty}{m}\frac{1}{|x-y|^\alpha},\ \|g\|_\infty(1 + \|\varphi\|_\infty)|x-y|^{1-\alpha}\right\}. \end{aligned}$$

For each choice of $c_1, c_2 \in [0, \infty)$ we have the inequality

$$\min\left\{\frac{c_1}{|x-y|^\alpha},\ c_2|x-y|^{1-\alpha}\right\} \le c_1^{1-\alpha} c_2^\alpha\ (x, y \in [0,1],\ x \ne y).$$

We conclude

$$\begin{aligned} \frac{|w(x) - w(y)|}{|x-y|^\alpha} &\le \left(\frac{4\|g\|_\infty}{m}\right)^{1-\alpha} \Big(\|g\|_\infty(1 + \|\varphi\|_\infty)\Big)^\alpha \\ &= \left(\frac{4}{m}\right)^{1-\alpha} \|g\|_\infty \Big(1 + \|\varphi\|_\infty\Big)^\alpha \le 1. \end{aligned}$$

Since $w(0) = 0$ we have $|w(0)| + n_\alpha(w) \le 1$.

Next,

$$\begin{aligned} d(g, w') &= \int_0^1 |g(t) - w'(t)|^p\ dt = \int_0^1 |v(t)|^p\ dt \\ &= \sum_{k=0}^{m-1} |\gamma_k|^p \int_{k/m}^{(k+1)/m} \left|\varphi\left(m\left(t - \frac{k}{m}\right)\right)\right|^p\ dt \\ &= \left(\sum_{k=0}^{m-1} \frac{|\gamma_k|^p}{m}\right) \int_0^1 |\varphi(t)|^p\ dt \\ &\le \left(\sum_{k=0}^{m-1} \frac{(\|g\|_\infty)^p}{m}\right) \int_0^1 |\varphi(t)|^p\ dt = (\|g\|_\infty)^p \int_0^1 |\varphi(t)|^p\ dt \le \varepsilon. \end{aligned}$$

□

5. Universal primitives in $C^{1-}([0,1])$

Let $(\lambda_n)_{n=1}^\infty$ be any sequence with $|\lambda_n| \in (0,1]$ and limit 0. Theorem 1 applies to difference quotients: For $f \in C^{1-}([0,1])$ let $f_e : [-1,2] \to \mathbb{R}$ be the extension of f defined by

$$f_e(x) = \begin{cases} 2f(1) - f(2-x) & (x \in (1,2]), \\ f(x) & (x \in [0,1]), \\ 2f(0) - f(-x) & (x \in [-1,0)), \end{cases}$$

and let $T_n : C^{1-}([0,1]) \to L^p([0,1])$ be defined by

$$(T_n f)(x) = \frac{f_e(x+\lambda_n) - f_e(x)}{\lambda_n}.$$

By standard reasoning each T_n is continuous and $T_n f \to f'$ in $L^p([0,1])$ (even in $C([0,1])$) for each $f \in C^1([0,1])$. Thus, the set of functions $f \in C^{1-}([0,1])$ such that

$$\{T_n f : n \in \mathbb{N}\} \text{ is dense in } L^p([0,1])$$

is a dense G_δ subset of $C^{1-}([0,1])$.

6. Further applications

As in [6] we can apply Theorem 1 to the derivatives of Bernstein or Lagrange polynomials.

For example, consider

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}.$$

Theorem 1 applies to the operators $T_n : C^{1-}([0,1]) \to L^p([0,1])$, $T_n f = (B_n f)'$ ($n \in \mathbb{N}$), compare [9, Sec.1.8]. Thus, the set of functions $f \in C^{1-}([0,1])$ such that

$$\{(B_n f)' : n \in \mathbb{N}\} \text{ is dense in } L^p([0,1])$$

is a dense G_δ subset of $C^{1-}([0,1])$.

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