DECOMPOSITION OF DIRICHLET FORMS ASSOCIATED TO UNBOUNDED DIRICHLET OPERATORS

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ABSTRACT. In [8], the author decomposed the Dirichlet form associated to a bounded generator G of a weakly*-continuous, completely positive, KMS-symmetric Markovian semigroup on a von Neumann algebra $\mathcal{M}$. The aim of this paper is to extend G to the unbounded generator using the bimodule structure and derivations.

1. Introduction

Noncommutative Dirichlet forms and Markovian semigroups on C^*-algebras or von Neumann algebras, pioneered by Albeverio and Hoegh-Krohn [2] in the tracial case and later extensively developed to the nontracial situations by Cipriani [4], were well understood at an analytic level [7, 11] and many constructions and applications were made [5, 7, 9].

Many works were established on the structure of the generators of Markovian semigroups on von Neumann algebras. In [1], the authors showed that the generator of a symmetric conservative quantum dynamical semigroup with norm-bounded generator on a von Neumann algebra equipped with a faithful semi-finite trace is expressed as a sum of square of self-adjoint derivations. Cipriani and Sauvageot analyzed the structure of a tracially symmetric Dirichlet form on C^*-algebras or von Neumann algebras, in terms of a closable derivation taking values in a Hilbert space with a bimodule structure [6]. In [10], for a general Lindblad type generator G of a KMS-symmetric quantum Markovian semigroup on a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$, the author gave the sufficient condition so that the operator H induced by G via the symmetric embedding of $\mathcal{M}$ into $\mathcal{H}$ is symmetric and the type of a Dirichlet operator given in [9]. Recently, in [8], the Dirichlet form associated to a bounded generator G of a weakly*-continuous, completely positive, KMS-symmetric Markovian semigroup on a von Neumann algebra $\mathcal{M}$ was decomposed.

The aim of this paper is to extend the generator G studied in [8] to the unbounded operator. We study a weakly*-continuous, completely positive, KMS-symmetric Markovian semigroup $\{S_t\}_{t \geq 0}$, $S_t = e^{-tG}$ with an unbounded generator G on a von Neumann algebra $\mathcal{M}$ acting on a separable Hilbert space $\mathcal{H}$. Modifying the bimodule structure used in [6], we construct a vector valued derivation on the algebra $\mathcal{M}$. Applying this derivation and the method used in [8], we decompose the Dirichlet form associated to the operator induced by G via the symmetric embedding into an integral of forms whose associated generators are the divergence of derivations.

We organize the paper as follows: In Section 2, for a weakly*-continuous, completely positive, KMS-symmetric Markovian semigroup with unbounded generator on the von Neumann algebra, we give the associated Dirichlet form in the sense of Cipriani [4], and state the main result. Section 3 is devoted to the proofs of Proposition 2.1, Theorem 2.2 and Theorem 2.3. Using the similar method used in [6] and [8] (and also Section 1 of [11]), we construct a bimodule structure and a derivation, and decompose the associated Dirichlet form.

2. Preliminaries and the main result

In this section, we give the Dirichlet form in the sense of Cipriani [4] associated to a weakly*-continuous, completely positive, KMS-symmetric Markovian semigroup with unbounded generator on a von Neumann algebra, and state the main result.

Let $L(\mathfrak{h})$ be the von Neumann algebra of all bounded linear operators on a separable Hilbert space $\mathfrak{h}$ and the faithful, normal, semi-finite trace on $L(\mathfrak{h})$ is denoted Tr. Let $L^2(\mathfrak{h})$ be the Hilbert-Schmidt class, its inner product is given by $\langle \xi, \eta \rangle = \mathrm{Tr}(\xi^*\eta)$. Consider two maps

$$\pi_L : L(\mathfrak{h}) \to B(L^2(\mathfrak{h})), \quad \pi_L(a) = L_a,$$
$$\pi_R : L(\mathfrak{h}) \to B(L^2(\mathfrak{h})), \quad \pi_R(a) = R_a,$$

where L_a and R_a are the left and right multiplication operators on $L^2(\mathfrak{h})$, $L_a(\xi) = a\xi$ and $R_a(\xi) = \xi a$, $\xi \in L^2(\mathfrak{h})$, respectively. Put $\mathcal{M} = \pi_L(L(\mathfrak{h}))$, $\mathcal{H} = L^2(\mathfrak{h})$ and let $\mathcal{M}_+$ be the set of positive elements in $\mathcal{M}$. Then the commutant $\mathcal{M}'$ of $\mathcal{M}$ is $\pi_R(L(\mathfrak{h}))$.

Let ρ be a (fixed) strictly positive density matrix, i.e., a strictly positive trace class operator on $\mathfrak{h}$ with $\mathrm{Tr}(\rho) = 1$. Let ω be a faithful normal state on $\mathcal{M}$ defined by

(2.1) $\omega(L_a) = \mathrm{Tr}(\rho a) = \langle \xi_0, a\xi_0 \rangle, \quad a \in L(\mathfrak{h}),$

where $\xi_0 = \rho^{1/2} \in \mathcal{H}$. Since ρ is a strictly positive operator, ξ_0 is a cyclic and separating vector for $\mathcal{M}$. The associated modular operator Δ and modular automorphism group $\{\sigma_t\}_{t \in \mathbb{R}}$ are given by

$$\Delta^{1/2}\xi = \rho^{1/2}\xi\rho^{-1/2}, \quad \xi \in L(\mathfrak{h})\rho^{1/2} = L(\mathfrak{h})\xi_0,$$
$$\sigma_t(L_a) = \Delta^{it} L_a \Delta^{-it} = L_{\rho^{it} a \rho^{-it}}, \quad a \in L(\mathfrak{h}).$$

The element $a \in L(\mathfrak{h})$ may be identified as the left multiplicative operator $L_a \in \mathcal{M}$ on $\mathcal{H}$. We write $a := L_a$, $\Delta^{it} a \Delta^{-it} := \Delta^{it} L_a \Delta^{-it}$. Let $\mathcal{M}_0$ be the $*$-subalgebra of the σ_t-entire analytic elements in $\mathcal{M}$[3]. The state ω in (2.1) satisfies the σ-KMS condition, $\omega(ab) = \omega(b\sigma_{-i}(a))$, $a, b \in \mathcal{M}_0$ and σ-invariant, for all $t \in \mathbb{R}$ $\omega(\sigma_t(a)) = \omega(a)$, $a \in \mathcal{M}$.

The closed convex cone $\mathcal{P}$ in $\mathcal{H}$ of nonnegative Hilbert-Schmidt operators is self-dual in the sense that $\{\xi \in \mathcal{H} : \langle \xi, \eta \rangle \geq 0, \ \forall \eta \in \mathcal{P}\}$. It is the closure of the set $\{\Delta^{1/4} a^* a \xi_0 = \rho^{1/4} a^* a \rho^{1/4} : a \in \mathcal{M}_0\}$. The associated modular conjugation J is an adjoint map on $\mathcal{H}$, $J\xi = \xi^*$ for $\xi \in \mathcal{H}$. $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ is a standard form for $L(\mathfrak{h})$.

The map $j : \mathcal{M} \rightarrow \mathcal{M}'$ is the anti-linear $*$-isomorphism defined by $j(a) = JaJ$, $a \in \mathcal{M}$. The Hilbert space $\mathcal{H}$ is the complexification of the real subspace $\mathcal{H}^J := \{\xi \in \mathcal{H} : \langle \xi, \eta \rangle \in \mathbb{R}, \ \forall \eta \in \mathcal{P}\}$, $\mathcal{H} = \mathcal{H}^J \oplus i\mathcal{H}^J$. For any two elements $\xi, \eta \in \mathcal{H}^J$, $\xi \leq \eta$ means $\eta - \xi \in \mathcal{P}$, thus $\mathcal{H}^J$ has an ordered structure. The order interval $I = \{\xi \in \mathcal{H}^J : 0 \leq \xi \leq \xi_0\}$, denoted by $[0, \xi_0]$, is a closed convex subset of $\mathcal{H}$. We denote by ξ_I the nearest point projection of $\xi \in \mathcal{H}^J$ onto $I = [0, \xi_0]$. For the details we refer [4].

From now on we consider a (fixed) weakly*-continuous semigroup $\{S_t\}_{t \geq 0}$, $S_t = e^{-tG}$ on $\mathcal{M}$ with (unbounded) generator G satisfying

$$S_t(\mathbf{1}) = \mathbf{1} \text{ and } 0 \leq S_t(a) \leq \|a\|\mathbf{1}, \ \ \forall \, a \in \mathcal{M}_+ \text{ (Markovian)},$$

$$\omega(S_t(a)\sigma_{-i/2}(b)) = \omega(\sigma_{i/2}(a)S_t(b)), \ \ \forall a, b \in \mathcal{M}_0 \text{ (KMS-symmetric)}$$

for all $t \geq 0$ and

$$(2.2) \quad \sum_{j,k=1}^{n} b_j^* S_t(a_j^* a_k) b_k \geq 0, \ \ \forall a_j, b_j \in \mathcal{M}, \ j = 1, \dots, n \text{ (completely positive)}$$

for all $t \geq 0$ and $n \in \mathbb{N}$. $S_t(\mathbf{1}) = \mathbf{1}$ for all $t \geq 0$ implies $G(\mathbf{1}) = 0$, and the KMS-symmetric condition is written as

$$(2.3) \qquad \omega\big(G(a)\sigma_{-i/2}(b)\big) = \omega\big(\sigma_{i/2}(a)G(b)\big), \ \ \forall a, b \in D(G) \cap \mathcal{M}_0.$$

For the weakly*-continuous, completely positive, KMS-symmetric Markovian semigroup $\{S_t\}_{t \geq 0}$, define the semigroup $\{T_t\}_{t \geq 0}$ on $\mathcal{H}$ with generator H, $T_t = e^{-tH}$ by

$$(2.4) \qquad\qquad T_t \Delta^{1/4} a \xi_0 = \Delta^{1/4} S_t(a) \xi_0, \ \forall a \in \mathcal{M}_0,$$

and the quadratic form $\mathcal{E}$ by

$$(2.5) \quad \mathcal{E}(\Delta^{1/4} a \xi_0, \Delta^{1/4} b \xi_0) = \langle \Delta^{1/4} a \xi_0, H \Delta^{1/4} b \xi_0 \rangle, \ \ \forall a, b \in D(G) \cap \mathcal{M}_0.$$

The closure of the quadratic form $\mathcal{E}$ is denoted by $(\mathcal{E}, D(\mathcal{E}))$ again. Then $\{T_t\}_{t \geq 0}$ is a strongly continuous, symmetric, contraction semigroup on $\mathcal{H}$ and the generator H is symmetric, positive on $\mathcal{H}$. Also the form $(\mathcal{E}, D(\mathcal{E}))$ satisfies

for any $\xi, \eta \in D(\mathcal{E})$, $J\xi, J\eta \in D(\mathcal{E})$ and $\mathcal{E}(J\xi, J\eta) = \mathcal{E}(\eta, \xi)$ (J-real),

and for any $\xi \in D(\mathcal{E}) \cap \mathcal{H}^J$, $\xi_I \in D(\mathcal{E})$ and $\mathcal{E}(\xi_I, \xi_I)$ (sub-Markovian).

See Lemma 2.2, Theorem 2.11 and Theorem 4.11 of [4]. (2.4) implies

$$(2.6) \qquad H\Delta^{1/4}a\xi_0 = \Delta^{1/4}G(a)\xi_0, \ \forall a \in D(G) \cap \mathcal{M}_0.$$

In this case, H and $(\mathcal{E}, D(\mathcal{E}))$ are called a Dirichlet operator and a Dirichlet form, respectively.

Assume that B_0 is a weakly*-dense subalgebra of $D(G)$ satisfying

  (i) $\mathbf{1} \in B_0 \subset \mathcal{M}_0$ and $G(B_0) \subset B_0$,
  (ii) for any $a \in B_0$, $\sigma_{t+is}(a) \in B_0$, $\forall t, s \in \mathbb{R}$,
  (iii) for any $a \in B_0 \cap \mathcal{M}_+$, $a^{1/2} \in B_0$.

The above condition (iii) corresponds to one of sufficient conditions in order that the derivation on C*-algebra is dissipative (Proposition 3.2.22 of [3]). When the density matrix ρ is expressed as $\rho = \sum_{n=0}^{\infty} \lambda_n \langle \xi_n, \cdot \rangle \xi_n$, $\lambda_n > 0$, where $\{\xi_n\}_{n=0}^{\infty}$ is an orthonormal basis of $\mathfrak{h}$, the class spanned by the elements expressed as the sum of the identity and finite rank operators with respect to the orthonormal basis satisfies the assumption (ii) and (iii).

Now we state our results, and their proofs are given in Section 3. We first introduce a Hilbert space $\mathcal{H}_0$ deduced from the algebraic tensor product space $B_0 \otimes B_0\xi_0$ (Lemma 3.1) and construct a B_0-B_0 bimodule structure on $\mathcal{H}_0$.

Proposition 2.1. *There exists a B_0-B_0 bimodule structure on $\mathcal{H}_0$ by*

$$a(b \otimes c\xi_0) = ab \otimes c\xi_0 - a \otimes bc\xi_0,$$
$$(b \otimes c\xi_0)a = b \otimes ca\xi_0$$

for any $a, b, c \in B_0$.

Let ∂ be a map defined by

$$(2.7) \qquad \partial : B_0 \rightarrow \mathcal{H}_0, \ \partial(a) = a \otimes \xi_0.$$

By Proposition 2.1, $a\partial(b) = \partial(ab) - a \otimes b\xi_0$ and $\partial(a)b = a \otimes b\xi_0$ for all $a, b \in B_0$.

Theorem 2.2. (a) *∂ defined as in (2.7) is an $\mathcal{H}_0$-valued B_0-B_0 bimodule derivation:*

$$\partial(ab) = \partial(a)b + a\partial(b), \ a, b \in B_0.$$

(b) *There exists an antilinear isometric involution $\mathcal{J}$ of $\mathcal{H}_0$ satisfying*

$$\mathcal{J}(\partial(a)) = \partial(\sigma_{-i/2}(a^*)), \ a \in B_0$$

and

$$\mathcal{J}(a\xi b) = \sigma_{-i/2}(b^*)\mathcal{J}(\xi)\sigma_{-i/2}(a^*), \ a, b \in B_0, \xi \in \mathcal{H}_0.$$

Now we state the main result of the paper and its proof is given in Section 3. The inner product and the induced norm of the Hilbert space $\mathcal{H}_0$ will be given in (3.5) and (3.6) respectively. The following is the main result of the paper:

Theorem 2.3. *The Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ defined as in (2.5) is expressed by*

$$\mathcal{E}(a\xi_0, a\xi_0) = \int \|\partial(\sigma_t(a))\|_{\mathcal{H}_0}^2 f_0(t)dt, \ a \in B_0,$$

where the function $f_0 : \mathbb{R} \to \mathbb{R}$ is given by

$$(2.8) \qquad f_0(t) = 2(e^{2\pi t} + e^{-2\pi t})^{-1}.$$

Remark 2.4. In case ω in (2.1) is a tracial state, the similar structure of bimodule in Proposition 2.1 and bimodule derivation in Theorem 2.2 have been introduced in [6], and the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ has been decomposed. For a nontracial state and a bounded generator G, the decomposition of the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ was established in [8].

3. Proof of the main result

In this section, we produce the proofs of Proposition 2.1, Theorems 2.2 and 2.3. Using the similar method used in [6] and [8], we decompose the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ in the sense of Cipriani [4] associated to a weakly*-continuous, completely positive, KMS-symmetric Makovian semigroup $\{S_t\}_{t\geq 0}$ with the unbounded generator G on $\mathcal{M}$. See also Section 1 of [11].

For each $\varepsilon > 0$, the resolvent maps defined by

$$(3.1) \qquad R_\varepsilon := (\mathbf{1} + \varepsilon G)^{-1} = \int_0^\infty e^{-t} S_{\varepsilon t} \, dt$$

are bounded, completely positive, normal contractions on $\mathcal{M}$, and the operators $G_\varepsilon := G(\mathbf{1} + \varepsilon G)^{-1} = \varepsilon^{-1}(\mathbf{1} - R_\varepsilon)$, $\varepsilon > 0$, are bounded, positive preserving on $\mathcal{M}$. Then there exist a Hilbert space $\mathcal{H}_\varepsilon$, a (*-homomorphism) representation π_ε of $\mathcal{M}$ into $L(\mathcal{H}_\varepsilon)$ and a linear operator W_ε of $\mathcal{H}$ into $\mathcal{H}_\varepsilon$ with norm less than 1, such that $R_\varepsilon(a) = W_\varepsilon^* \pi_\varepsilon(a) W_\varepsilon$, $a \in \mathcal{M}$.

Lemma 3.1. (a) *For any $n \in \mathbb{N}$, $\varepsilon > 0$ and $a_1, \ldots, a_n \in \mathcal{M}$, the $n \times n$ matrix $\left[\Lambda_\varepsilon(a_j^*, a_k)\right]_{j,k=1,\ldots,n}$ is positive, where $\Lambda_\varepsilon(a^*, b) = G_\varepsilon(a^*)b + a^* G_\varepsilon(b) - G_\varepsilon(a^*b)$, $a, b \in \mathcal{M}$.*
    (b) *For any $a, b, c, d \in B_0$, the limit $\lim_{\varepsilon \downarrow 0} \omega(b^* \Lambda_\varepsilon(a^*, c)d)$ exists and equals*

$$
\begin{aligned}
(3.2) \quad \lim_{\varepsilon \downarrow 0} \omega(b^* \Lambda_\varepsilon(a^*, c)d) &= \mathcal{E}(\sigma_{-i/4}(a)\xi_0, \sigma_{i/4}(cd)\sigma_{-3i/4}(b^*)\xi_0) \\
&\quad + \mathcal{E}(\sigma_{i/4}(ab)\sigma_{-3i/4}(d^*)\xi_0, \sigma_{-i/4}(c)\xi_0) \\
&\quad - \mathcal{E}(\sigma_{i/4}(b)\sigma_{-3i/4}(d^*)\xi_0, \sigma_{-i/4}(a^*c)\xi_0).
\end{aligned}
$$

Proof. (a) Let $a_j, b_j \in \mathcal{M}$, $j = 1, \ldots, n$. Using $G_\varepsilon = \varepsilon^{-1}(\mathbf{1} - R_\varepsilon)$ and $R_\varepsilon(a) = W_\varepsilon^* \pi_\varepsilon(a) W_\varepsilon$, $\varepsilon > 0$ in the above, we have

$$
\begin{aligned}
&\Lambda_\varepsilon(a_j^*, a_k) \\
&= \varepsilon^{-1}\left(a_j^* a_k - R_\varepsilon(a_j^*)a_k - a_j^* R_\varepsilon(a_k) + R_\varepsilon(a_j^* a_k)\right) \\
(3.3) \quad &= \varepsilon^{-1}\left[\left(W_\varepsilon a_j - \pi_\varepsilon(a_j)W_\varepsilon\right)^*\left(W_\varepsilon a_k - \pi_\varepsilon(a_k)W_\varepsilon\right) + a_j^*(\mathbf{1} - W_\varepsilon^* W_\varepsilon)a_k\right].
\end{aligned}
$$

Thus we get from (3.3) that

$$(3.4) \qquad \sum_{j,k=1}^{n} b_j^* \Lambda_\varepsilon(a_j^*, a_k) b_k$$

$$= \varepsilon^{-1} \left(\sum_{j=1}^{n} (W_\varepsilon a_j - \pi_\varepsilon(a_j) W_\varepsilon) b_j \right)^* \left(\sum_{j=1}^{n} (W_\varepsilon a_j - \pi_\varepsilon(a_j) W_\varepsilon) b_j \right)$$

$$+ \varepsilon^{-1} \left(\sum_{j=1}^{n} (1 - W_\varepsilon^* W_\varepsilon)^{1/2} a_j b_j \right)^* \left(\sum_{j=1}^{n} (1 - W_\varepsilon^* W_\varepsilon)^{1/2} a_j b_j \right)$$

$$\geq 0,$$

which implies (a).

(b) By the definition of G_ε, the limit $\lim_{\varepsilon \downarrow 0} \omega(b^* \Lambda_\varepsilon(a^*, c) d)$ exists and equals $\omega(b^* \Lambda(a^*, c) d)$, where $\Lambda(a^*, c) = G(a^*) c + a^* G(c) - G(a^* c)$. Applying the definition of the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ and KMS-symmetric condition of G, we easily obtain the relation (3.2). $\qquad \square$

Due to Lemma 3.1, we can define a Hilbert space $\mathcal{H}_0$ deduced from the algebraic tensor product space $B_0 \otimes B_0 \xi_0$ with respect to the inner product $(\cdot, \cdot)$ defined by

$$(3.5) \qquad (a \otimes b\xi_0, c \otimes d\xi_0) = \mathcal{E}(\sigma_{-i/4}(a)\xi_0, \sigma_{i/4}(cd)\sigma_{-3i/4}(b^*)\xi_0)$$
$$+ \mathcal{E}(\sigma_{i/4}(ab)\sigma_{-3i/4}(d^*)\xi_0, \sigma_{-i/4}(c)\xi_0)$$
$$- \mathcal{E}(\sigma_{i/4}(b)\sigma_{-3i/4}(d^*)\xi_0, \sigma_{-i/4}(a^*c)\xi_0)$$

for any $a, b, c, d \in B_0$, and the associated norm $\| \cdot \|_{\mathcal{H}_0}$

$$(3.6) \qquad \| a \otimes b\xi_0 \|_{\mathcal{H}_0}^2 = \mathcal{E}(\sigma_{-i/4}(a)\xi_0, \sigma_{i/4}(ab)\sigma_{-3i/4}(b^*)\xi_0)$$
$$+ \mathcal{E}(\sigma_{i/4}(ab)\sigma_{-3i/4}(b^*)\xi_0, \sigma_{-i/4}(a)\xi_0)$$
$$- \mathcal{E}(\sigma_{i/4}(b)\sigma_{-3i/4}(b^*)\xi_0, \sigma_{-i/4}(a^*a)\xi_0)$$

for any $a, b \in B_0$.

By (2.3) and $G(\mathbf{1}) = 0$, we have $\omega(G(a^*a)) = 0$ for any $a \in B_0$, and so

$$(3.7) \qquad E(a) := \| a \otimes \xi_0 \|_{\mathcal{H}_0}^2$$
$$= \omega(G(a^*)a + a^* G(a)).$$

Lemma 3.2. *For any $n \in \mathbb{N}$, $a, a_1, \ldots, a_n, b_1, \ldots, b_n \in B_0$, two inequalities*

$$(3.8) \qquad \left\| \sum_{j=1}^{n} a_j \otimes b_j a\xi_0 \right\|_{\mathcal{H}_0} \leq \left\| \sigma_{-i/2}(a^*) \right\| \left\| \sum_{j=1}^{n} a_j \otimes b_j \xi_0 \right\|_{\mathcal{H}_0},$$

$$(3.9) \qquad \left\| \sum_{j=1}^{n} \left(a a_j \otimes b_j \xi_0 - a \otimes a_j b_j \xi_0 \right) \right\|_{\mathcal{H}_0} \leq \|a\| \left\| \sum_{j=1}^{n} a_j \otimes b_j \xi_0 \right\|_{\mathcal{H}_0}$$

hold. Moreover, $\| a \otimes b\xi_0 \|_{\mathcal{H}_0} \leq \| \sigma_{-i/2}(b^*) \| \sqrt{E(a)}$, $a, b \in B_0$.

Proof. Notice that by σ-KMS condition, the inequality

$$(3.10) \qquad \omega(b^*a^*ab) \leq \|\sigma_{-i/2}(b^*)\|^2 \omega(a^*a), \quad a,b \in B_0$$

holds.

Let $a, a_j, b_j \in B_0$, $j = 1,\dots,n$. Due to Lemma 3.1 (a), for any $\varepsilon > 0$, $\sum_{j,k=1}^{n} b_j^* \Lambda_\varepsilon(a_j^*, a_k) b_k$ is positive. We get from the assumption of B_0 and (3.10) that

$$
\begin{aligned}
\left\| \sum_{j=1}^{n} a_j \otimes b_j a \xi_0 \right\|_{\mathcal{H}_0}^2
&= \lim_{\varepsilon \downarrow 0} \omega\left(a^*\big(\sum_{j,k=1}^{n} b_j^* \Lambda_\varepsilon(a_j^*, a_k) b_k \big) a \right) \\
&\leq \|\sigma_{-i/2}(a^*)\|^2 \lim_{\varepsilon \downarrow 0} \omega\left(\sum_{j,k=1}^{n} b_j^* \Lambda_\varepsilon(a_j^*, a_k) b_k \right) \\
&= \|\sigma_{-i/2}(a^*)\|^2 \left\| \sum_{j=1}^{n} a_j \otimes b_j \xi_0 \right\|_{\mathcal{H}_0}^2,
\end{aligned}
$$

which implies (3.8).

Next, to produce the inequality (3.9), we estimate

$$
(3.11) \qquad \left\| \sum_{j=1}^{n} \big(a a_j \otimes b_j \xi_0 - a \otimes a_j b_j \xi_0 \big) \right\|_{\mathcal{H}_0}^2
$$

$$
= \sum_{j,k=1}^{n} \big(a a_j \otimes b_j \xi_0 - a \otimes a_j b_j \xi_0, \, a a_k \otimes b_k \xi_0 - a \otimes a_k b_k \xi_0 \big)
$$

$$
= \sum_{j,k=1}^{n} \lim_{\varepsilon \downarrow 0} \omega\big(b_j^* \big[G_\varepsilon(a_j^* a^* a) a_k + a_j^* G_\varepsilon(a^* a a_k)
$$

$$
- a_j^* G_\varepsilon(a^* a) a_k - G_\varepsilon(a_j^* a^* a a_k) \big] b_k \big).
$$

By the similar calculation used in Lemma 3.1, for $\varepsilon > 0$ we have

$$
(3.12) \quad \sum_{j,k=1}^{n} b_j^* \left[G_\varepsilon(a_j^* a^* a) a_k + a_j^* G_\varepsilon(a^* a a_k) - a_j^* G_\varepsilon(a^* a) a_k - G_\varepsilon(a_j^* a^* a a_k) \right] b_k
$$

$$
= \varepsilon^{-1} \sum_{j,k=1}^{n} b_j^* \big(W_\varepsilon a_j - \pi_\varepsilon(a_j) W_\varepsilon \big)^* \pi_\varepsilon(a^* a) \big(W_\varepsilon a_k - \pi_\varepsilon(a_k) W_\varepsilon \big) b_k
$$

$$
\leq \varepsilon^{-1} \|a\|^2 \sum_{j,k=1}^{n} b_j^* \big(W_\varepsilon a_j - \pi_\varepsilon(a_j) W_\varepsilon \big)^* \big(W_\varepsilon a_k - \pi_\varepsilon(a_k) W_\varepsilon \big) b_k
$$

$$
= \|a\|^2 \sum_{j,k=1}^{n} b_j^* \big[G_\varepsilon(a_j^*) a_k + a_j^* G_\varepsilon(a_k) - G_\varepsilon(a_j^* a_k) \big] b_k
$$

$$
- \varepsilon^{-1} \|a\|^2 \sum_{j,k=1}^{n} b_j^* a_j^* (1 - W_\varepsilon^* W_\varepsilon) a_k b_k.
$$

Here we have used $\|\pi_\varepsilon(a^*a)\| \leq \|a\|^2$ in the inequality, and (3.4) in second equality. Substituting (3.12) into (3.11), and using $W_\varepsilon^* W_\varepsilon \leq \mathbf{1}$, we get

$$\left\| \sum_{j=1}^n aa_j \otimes b_j\xi_0 - a \otimes \sum_{j=1}^n a_jb_j\xi_0 \right\|_{\mathcal{H}_0}^2$$

$$\leq \quad \|a\|^2 \lim_{\varepsilon\downarrow 0} \omega\left(\sum_{j,k=1}^n b_j^* \left[G_\varepsilon(a_j^*)a_k + a_j^* G_\varepsilon(a_k) - G_\varepsilon(a_j^*a_k) \right] b_k \right)$$

$$= \quad \|a\|^2 \| \sum_{j=1}^n a_j \otimes b_j\xi_0 \|_{\mathcal{H}_0}^2.$$

Thus (b) holds. Finally the rest follows from (3.8) and $\|a \otimes \xi_0\| = \sqrt{E(a)}$. $\quad\square$

Due to Lemma 3.2, we construct structures of a right B_0-module and a left B_0-module on $\mathcal{H}_0$.

Proposition 3.3. *There exist structures of a right B_0-module on $\mathcal{H}_0$ by*

$$\left(\sum_{j=1}^n a_j \otimes b_j\xi_0 \right)a = \sum_{j=1}^n a_j \otimes b_ja\xi_0$$

and a left B_0-module on $\mathcal{H}_0$ by

$$a\left(\sum_{j=1}^n a_j \otimes b_j\xi_0 \right) = \sum_{j=1}^n (aa_j \otimes b_j\xi_0 - a \otimes a_jb_j\xi_0)$$

for all $n \in \mathbb{N}$, $a, a_j, b_j \in B_0, j = 1, \ldots, n$. The involutions

$$\left(a \otimes b\xi_0, (c \otimes d\xi_0)e \right) = \left((a \otimes b\xi_0)\sigma_{-i}(e^*), c \otimes d\xi_0 \right)$$

and

$$\left(e(a \otimes b\xi_0), c \otimes d\xi_0 \right) = \left(a \otimes b\xi_0, e^*(c \otimes d\xi_0) \right)$$

hold for any $a, b, c, d, e \in B_0$.

Proof. By Lemma 3.2 and continuity, we have the right multiplication by $(\sum_{j=1}^n a_j \otimes b_j\xi_0)a = \sum_{j=1}^n a_j \otimes b_ja\xi_0$ and the left multiplication by $a(\sum_{j=1}^n a_j \otimes b_j\xi_0) = \sum_{j=1}^n (aa_j \otimes b_j\xi_0 - a \otimes a_jb_j\xi_0)$ for all $n, a, a_j, b_j \in B_0, j = 1, \ldots, n$.

By the direct calculation we have the involutions: for any $a, b, c, d, e \in B_0$

$$
\begin{aligned}
(a \otimes b\xi_0, (c \otimes d\xi_0)e) &= \lim_{\varepsilon\downarrow 0} \omega(b^* \Lambda_\varepsilon(a^*, c)de) \\
&= \lim_{\varepsilon\downarrow 0} \omega(\sigma_i(e)b^* \Lambda_\varepsilon(a^*, c)d) \\
&= \left(a \otimes b\sigma_{-i}(e^*)\xi_0, c \otimes d\xi_0 \right) \\
&= \left((a \otimes b\xi_0)\sigma_{-i}(e^*), c \otimes d\xi_0 \right)
\end{aligned}
$$

and

$$
\begin{aligned}
(e(a \otimes b\xi_0), c \otimes d\xi_0) &= (ea \otimes b\xi_0 - e \otimes ab\xi_0, c \otimes d\xi_0) \\
&= \lim_{\varepsilon \downarrow 0} \omega(b^*[G_\varepsilon(a^*e^*)c + a^*e^*G_\varepsilon(c) - G_\varepsilon(a^*e^*c)]d \\
&\qquad - b^*a^*[G_\varepsilon(e^*)c + e^*G_\varepsilon(c) - G(e^*c)]d) \\
&= \lim_{\varepsilon \downarrow 0} \omega\big(b^*[G_\varepsilon(a^*)e^*c + a^*G_\varepsilon(e^*c) - G_\varepsilon(a^*e^*c)]d \\
&\qquad - b^*[G_\varepsilon(a^*)e^* + a^*G_\varepsilon(e^*) - G(a^*e^*)]cd\big) \\
&= \big(a \otimes b\xi_0, e^*c \otimes d\xi_0\big) - \big(a \otimes b\xi_0, e^* \otimes cd\xi_0\big) \\
&= \big(a \otimes b\xi_0, e^*(c \otimes d\xi_0)\big).
\end{aligned}
$$

The proof is completed. $\qquad\square$

Proof of Proposition 2.1. The proof is directly obtained from Proposition 3.3.
$\qquad\square$

Recall that $\partial(a) = a \otimes \xi_0$ and $\partial(a)b = a \otimes b\xi_0$ for all $a, b \in B_0$.

Lemma 3.4. *For any $a, b, c, d \in B_0$, the following relations hold:*

(a) $(\partial(a), c\partial(b)) = (\partial(\sigma_{-i/2}(b^*)), \partial(\sigma_{-i/2}(a^*))\sigma_{-i/2}(c))$,
(b) $(\partial(a), \partial(b)) = (\partial(\sigma_{-i/2}(b^*)), \partial(\sigma_{-i/2}(a^*)))$,
(c) $(\partial(a), c\partial(b)d) = (\sigma_{-i/2}(d^*)\partial(\sigma_{-i/2}(b^*))\sigma_{-i/2}(c^*), \partial(\sigma_{-i/2}(a^*)))$.

Proof. (a) We rewrite (2.3) as

$$(3.13) \qquad \omega(G(a)b) = \omega(\sigma_{i/2}(a)G(\sigma_{i/2}(b))), \quad a, b \in B_0.$$

Let $a, b, c \in B_0$. By $\omega(G(a^*cb)) = 0$ and σ-KMS condition, one has

$$
\begin{aligned}
(\partial(a), c\partial(b)) &= (a \otimes \xi_0, cb \otimes \xi_0 - c \otimes b\xi_0) \\
&= \omega\big(G(a^*c)b + a^*G(cb) - a^*G(c)b\big) \\
(3.14) \qquad &= \omega\big(G(a^*c)b + G(cb)\sigma_{-i}(a^*) - G(c)b\sigma_{-i}(a^*)\big).
\end{aligned}
$$

Applying (3.13) and σ-KMS condition to (3.14), we get

$$
\begin{aligned}
(\partial(a), c\partial(b)) &= \omega\big(\sigma_{i/2}(a^*c)G(\sigma_{i/2}(b)) + \sigma_{i/2}(cb)G(\sigma_{-i/2}(a^*)) \\
&\qquad - \sigma_{i/2}(c)G(\sigma_{i/2}(b)\sigma_{-i/2}(a^*))\big) \\
&= \omega\big(G(\sigma_{i/2}(b))\sigma_{-i/2}(a^*c) + \sigma_{i/2}(b)G(\sigma_{-i/2}(a^*))\sigma_{-i/2}(c) \\
&\qquad - G(\sigma_{i/2}(b)\sigma_{-i/2}(a^*))\sigma_{-i/2}(c)\big) \\
&= \big(\sigma_{-i/2}(b^*) \otimes \xi_0, \sigma_{-i/2}(a^*) \otimes \sigma_{-i/2}(c)\xi_0\big) \\
&= \big(\partial(\sigma_{-i/2}(b^*)), \partial(\sigma_{-i/2}(a^*))\sigma_{-i/2}(c)\big).
\end{aligned}
$$

(b) Putting $c = 1$ in (a), we obtain the relation.
(c) Let $a, b, c, d \in B_0$. Notice that

$$(\partial(a), c\partial(b)d) = (\partial(a), c\partial(bd)) - (\partial(a), cb\partial(d)).$$

Applying (a) to the above, and using the involutions in Proposition 3.3, we
have

$$
\begin{aligned}
(\partial(a), c\partial(b)d) &= (\partial(\sigma_{-i/2}(d^*b^*)), \partial(\sigma_{-i/2}(a^*))\sigma_{-i/2}(c)) \\
&\quad -(\partial(\sigma_{-i/2}(d^*)), \partial(\sigma_{-i/2}(a^*))\sigma_{-i/2}(cb)) \\
&= (\partial(\sigma_{-i/2}(d^*b^*))\sigma_{-i/2}(c^*), \partial(\sigma_{-i/2}(a^*))) \\
&\quad -(\partial(\sigma_{-i/2}(d^*))\sigma_{-i/2}(b^*c^*), \partial(\sigma_{-i/2}(a^*))) \\
&= (\sigma_{-i/2}(d^*)\partial(\sigma_{-i/2}(b^*))\sigma_{-i/2}(c^*), \partial(\sigma_{-i/2}(a^*))).
\end{aligned}
$$

The proof is completed.   □

Proof of Theorem 2.2. (a) The definition of ∂ and the bimodule structure di-
rectly produce (a).

(b) Notice that $B_0 \otimes B_0 \xi_0$ is dense in $\mathcal{H}_0$. By Lemma 3.4, there exists an
antilinear isometric involution $\mathcal{J}$ of $\mathcal{H}_0$ satisfying $\mathcal{J}(\partial(a)) = \partial(\sigma_{-i/2}(a^*))$, $a \in$
B_0 and $\mathcal{J}(a\xi b) = \sigma_{-i/2}(b^*)\mathcal{J}(\xi)\sigma_{-i/2}(a^*)$, $a, b \in B_0$, $\xi \in \mathcal{H}_0$.   □

To characterize the structure of Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$, we introduce a
lemma.

Lemma 3.5. *Let C be a bounded operator on $\mathcal{H}$. For any $a, b \in \mathcal{M}_0$, the
relation*

$$
\begin{aligned}
&\int [\langle \Delta^{it+1/4} a\xi_0, C\Delta^{it-1/4} b\xi_0 \rangle + \langle \Delta^{it-1/4} a\xi_0, C\Delta^{it+1/4} b\xi_0 \rangle] f_0(t)dt \\
&= \langle a\xi_0, Cb\xi_0 \rangle
\end{aligned}
$$

holds, where $f_0(t)$ is defined as in (2.8).

Proof. See the proof of Proposition 4.1 of [8]. See also the proof of Lemma 3.1
of [10].   □

Proof of Theorem 2.3. Let $a \in B_0$. By assumption (ii) of B_0, $\sigma_t(a) \in B_0$ for
all $t \in \mathbb{R}$. Since $\Delta^{1/4}$ is closed and $G_\varepsilon(a)\xi_0 \in D(\Delta^{1/4})$ for any $\varepsilon > 0$, we
get from (2.4) that $\Delta^{1/4}(1+\varepsilon G)^{-1}(a)\xi_0 = H_\varepsilon \Delta^{1/4} a\xi_0$, and so $\Delta^{1/4} G_\varepsilon(a)\xi_0 =$
$H_\varepsilon \Delta^{1/4} a\xi_0$, where $H_\varepsilon = H(\mathbf{1}+\varepsilon H)^{-1}$.

By (3.7) and the dominated convergence theorem, we have

$$
\begin{aligned}
&\int \|\partial(\sigma_t(a))\|^2_{\mathcal{H}_0} f_0(t)dt \\
&= \int \lim_{\varepsilon \downarrow 0} \omega\big(G_\varepsilon(\sigma_t(a^*))\sigma_t(a) + \sigma_t(a^*)G_\varepsilon(\sigma_t(a))\big) f_0(t)dt \\
&= \lim_{\varepsilon \downarrow 0} \int \omega\big(G_\varepsilon(\sigma_t(a^*))\sigma_t(a) + \sigma_t(a^*)G_\varepsilon(\sigma_t(a))\big) f_0(t)dt.
\end{aligned}
$$

Using the relation $\Delta^{1/4}G_\varepsilon(a)\xi_0 = H_\varepsilon\Delta^{1/4}a\xi_0$ and $\Delta^{it}\xi_0 = \xi_0$ for all $t \in \mathbb{R}$, we have

$$\int \omega(G_\varepsilon(\sigma_t(a^*))\sigma_t(a) + \sigma_t(a^*)G_\varepsilon(\sigma_t(a)))f_0(t)dt$$

$$= \int [\langle G_\varepsilon(\sigma_t(a))\xi_0, \sigma_t(a)\xi_0\rangle + \langle \sigma_t(a)\xi_0, G_\varepsilon(\sigma_t(a))\xi_0\rangle]f_0(t)dt$$

$$= \int [\langle \Delta^{it+1/4}a\xi_0, H_\varepsilon\Delta^{it-1/4}a\xi_0\rangle + \langle \Delta^{it-1/4}a\xi_0, H_\varepsilon\Delta^{it+1/4}a\xi_0\rangle]f_0(t)dt$$

$$= \langle a\xi_0, H_\varepsilon a\xi_0\rangle.$$

Since for any $\xi \in D(H)$ $H_\varepsilon\xi \to H\xi$ as $\varepsilon \downarrow 0$, we get

$$\int \|\partial(\sigma_t(a))\|^2_{\mathcal{H}_0}f_0(t)dt = \langle a\xi_0, Ha\xi_0\rangle$$
$$= \mathcal{E}(a\xi_0, a\xi_0).$$

The proof is completed. $\qquad\square$

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