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EVALUATING SOME DETERMINANTS OF MATRICES WITH RECURSIVE ENTRIES

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Abstract. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ be two sequences with $\alpha_1 = \beta_1$ and k and n be natural numbers. We denote by $A^{(k,\pm)}_{\alpha,\beta}(n)$ the matrix of order n with coefficients $a_{i,j}$ by setting $a_{1,i} = \alpha_i$, $a_{i,1} = \beta_i$ for $1 \leq i \leq n$ and

$$a_{i,j} = \begin{cases} a_{i-1,j-1} + a_{i-1,j} & \text{if} \quad j \equiv 2, 3, 4, \ldots, k+1 \pmod{2k} \\ a_{i-1,j-1} - a_{i-1,j} & \text{if} \quad j \equiv k+2, \ldots, 2k+1 \pmod{2k} \end{cases}$$

for $2 \leq i, j \leq n$. The aim of this paper is to study the determinants of such matrices related to certain sequences α and β, and some natural numbers k.

1. Introduction

In [1], R. Bacher considered the determinants of matrices associated to the Pascal triangle. Furthermore, he introduced the generalized Pascal triangles as follows. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ be two sequences starting with a common first term $\alpha_1 = \beta_1$. Define a matrix $P_{\alpha,\beta}(n)$ of order n with coefficients $p_{i,j}$ by setting $p_{i,1} = \beta_i$, $p_{1,i} = \alpha_i$ for $1 \leq i \leq n$ and $p_{i,j} = p_{i-1,j} + p_{i,j-1}$ for $2 \leq i, j \leq n$. The infinite matrix $P_{\alpha,\beta}(\infty)$ is called the *generalized Pascal triangle* associated to the sequences α and β. In addition he investigated some other similar constructions and made many interesting observations and posed some conjectures. Some of his conjectures were thoroughly investigated in [3] with positive answers.

In constructing the generalized Pascal triangles or the other similar constructions in which the coefficients, except for the first row and column, are determined by a recursive relation, only *one* recursive relation is used. Here we are willing to construct some similar arrangements associated to two arbitrary sequences α and β being in the first row and column, respectively, and the remaining coefficients are determined by *two* different recursive relations. Let us define this more precisely as follows.

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Definition. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ be two sequences starting with a common first term $\alpha_1 = \beta_1$ and k be a natural number. Define a matrix $A^{(k,\pm)}_{\alpha,\beta}(n)$ of order n with coefficients $a_{i,j}$ by setting $a_{i,1} = \beta_i$, $a_{1,i} = \alpha_i$ for $1 \leq i \leq n$ and

$$a_{i,j} = \begin{cases} a_{i-1,j-1} + a_{i-1,j} & \text{if} \quad j \equiv 2, 3, 4, \ldots, k+1 \pmod{2k} \\ a_{i-1,j-1} - a_{i-1,j} & \text{if} \quad j \equiv k+2, \ldots, 2k+1 \pmod{2k} \end{cases}$$

for $2 \leq i, j \leq n$. When $k = 1$, we put $A^{\pm}_{\alpha,\beta}(n) = A^{(1,\pm)}_{\alpha,\beta}(n)$.

In general, we are interested in the sequence of the determinants

$$(\det A^{(k,\pm)}_{\alpha,\beta}(1), \det A^{(k,\pm)}_{\alpha,\beta}(2), \ldots, \det A^{(k,\pm)}_{\alpha,\beta}(n), \ldots),$$

where α and β are certain sequences having a common first entry.

On the other hand, when we consider the constant sequence $\alpha = (1, 1, 1, \ldots)$, we notice that the generalized Pascal triangle $P_{\alpha,\alpha}(\infty)$ is, in fact, the classical Pascal triangle. Hence, in the early studies, we restrict our investigation to this sequence $\alpha = (1, 1, 1, \ldots)$ only, and we consider the principal minors of infinite matrices $A^{(k,\pm)}_{\alpha,\alpha}(\infty)$.

In this research, it has been tried to prove three theorems.

Theorem 1.1. *The matrices $A^{\pm}_{\alpha,\alpha}(n)$ associated to the sequence $\alpha = (1, 1, \ldots)$ have the determinant $3^{[\frac{n-1}{2}]}$ for every natural number n. In other words, we have*

$$\det A^{\pm}_{\alpha,\alpha}(n) = \begin{cases} 3^{l-1} & \text{if} \quad n = 2l, \quad (l = 1, 2, \ldots) \\ 3^{l} & \text{if} \quad n = 2l+1. \ (l = 0, 1, 2, \ldots) \end{cases}$$

Theorem 1.2. *The sequence $(\det A^{(2,\pm)}_{\alpha,\alpha}(n))$ of determinants associated to the sequence $\alpha = (1, 1, 1, \ldots)$ satisfies the following*

$$\det A^{(2,\pm)}_{\alpha,\alpha}(n) = \begin{cases} (-5)^{2l-1} & \text{if} \quad n = 4l, \quad (l = 1, 2, \ldots) \\ (-5)^{2l} & \text{if} \quad n = 4l + r. \ (r = 1, 2, 3; \ l = 0, 1, 2, \ldots) \end{cases}$$

Theorem 1.3. *The sequence $(\det A^{(3,\pm)}_{\alpha,\alpha}(n))$ of determinants associated to the sequence $\alpha = (1, 1, 1, \ldots)$ satisfies the following*

$$\det A^{(3,\pm)}_{\alpha,\alpha}(n) = \begin{cases} 11^{3l-1} & \text{if} \quad n = 6l, \quad (l = 1, 2, \ldots) \\ 11^{3l} & \text{if} \quad n = 6l + r, \ (r = 1, 2, 3, 4; \ l = 0, 1, 2, \ldots) \\ 11^{3l+1} & \text{if} \quad n = 6l + 5. \ (l = 0, 1, 2, \ldots) \end{cases}$$

Here, we have the following conjecture:

Conjecture. *Let k and n be natural numbers and $n - 1 = rk + s$ for some r, s with $r \geq 0$ and $0 \leq s < k$. Let $\alpha = (1, 1, 1, \ldots)$. Then we have*

$$\det A^{(k,\pm)}_{\alpha,\alpha}(n) = \begin{cases} \omega^{rk/2} & \text{if } r \text{ is even}, \\ \\ \omega^{k(r-1)/2+s} & \text{if } r \text{ is odd}, \end{cases}$$

where $\omega = [1 - (-2)^{k+2}]/3$.

2. Main results

As we mentioned before, we should concentrate on the sequence of determinants

$$(\det A^{(k,\pm)}_{\alpha,\alpha}(1), \det A^{(k,\pm)}_{\alpha,\alpha}(2), \ldots, \det A^{(k,\pm)}_{\alpha,\alpha}(n), \ldots)$$

for certain k. Therefore, in order to start, we consider the case $k = 1$, and prove the following theorem.

Theorem 1. *The matrices $A^{\pm}_{\alpha,\alpha}(n)$ associated to the sequence $\alpha = (1, 1, 1, \ldots)$ have the determinant $3^{[\frac{n-1}{2}]}$ for every natural number n.*

Proof. We apply LU-factorization method (see [2]). We claim that

$$A^{\pm}_{\alpha,\alpha}(n) = L \cdot U,$$

where $L = A^{\pm}_{\beta,\alpha}(n)$ with $\beta = (1, 0, 0, 0, \ldots)$, and where

$$U = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix},$$

with

$$U_i = \begin{cases} (\underbrace{1, 1, 1, \ldots, 1}_{n \text{ times}}) & \text{if } i = 1, \\ (\underbrace{0, 0, \ldots, 0}_{i-1 \text{ times}}, \underbrace{1, -1, 1, -1, \ldots, u_{i,n-1}, u_{i,n}}_{n-i+1 \text{ times } (2-\text{periodic})}) & \text{if } i \overset{2}{\equiv} 0, \\ (\underbrace{0, 0, \ldots, 0}_{i-1 \text{ times}}, \underbrace{3, -1, 3, -1, 3, \ldots, u_{i,n-1}, u_{i,n}}_{n-i+1 \text{ times } (2-\text{periodic})}) & \text{if } i > 1 \text{ and } i \overset{2}{\equiv} 1, \end{cases}$$

and $(u_{i,n-1}, u_{i,n})$ is satisfied in Table 1.

Table 1.

$i \backslash n$	$n \overset{2}{\equiv} 0$	$n \overset{2}{\equiv} 1$
$i \overset{2}{\equiv} 0$	$(-1, 1)$	$(1, -1)$
$i \overset{2}{\equiv} 1$	$(3, -1)$	$(-1, 3)$

The matrix L is a lower triangular matrix with 1's on the diagonal, whereas U is an upper triangular matrix with diagonal entries

$$\begin{array}{ll} 1 & \text{if } n=1,\\ 1,1 & \text{if } n=2,\\ 1,\underbrace{1,3,1,3,1,3,\ldots,1,3}_{n-1 \text{ times } (2-\text{periodic})} & \text{if } n>1 \text{ and } n\overset{2}{\equiv}1,\\ 1,\underbrace{1,3,1,3,1,3,\ldots,3,1}_{n-1 \text{ times } (2-\text{periodic})} & \text{if } n>2 \text{ and } n\overset{2}{\equiv}0. \end{array}$$

Since $\det L = 1$ and $\det U = 3^{[\frac{n-1}{2}]}$, it is obvious that the claimed factorization of $A^{\pm}_{\alpha,\alpha}(n)$ immediately implies the validity of the theorem.

Suppose that

$$L = (l_{i,j})_{1\leq i,j\leq n} \quad \text{and} \quad U = (u_{i,j})_{1\leq i,j\leq n}.$$

Then by definition, we have $l_{1,1} = 1$, $l_{1,j} = 0$, $l_{i,1} = 1$ for $2 \leq i, j \leq n$ and

$$l_{i,j} = \begin{cases} l_{i-1,j-1} + l_{i-1,j} & \text{if } j \overset{2}{\equiv} 0 \\ l_{i-1,j-1} - l_{i-1,j} & \text{if } j \overset{2}{\equiv} 1 \end{cases} \tag{1}$$

for $2 \leq i, j \leq n$. Also we have

$$(u_{1,j}, u_{2,j}, \ldots, u_{n,j})^T = \begin{cases} (1,0,0,\ldots,0)^T & j=1,\\ (1,\underbrace{1,-1,1,-1,\ldots,-1,1}_{j-1 \text{ times } (2-\text{periodic})},\underbrace{0,\ldots,0}_{n-j})^T & j\overset{2}{\equiv}0,\\ (1,\underbrace{-1,3,-1,3,\ldots,-1,3}_{j-1 \text{ times } (2-\text{periodic})},\underbrace{0,\ldots,0}_{n-j})^T & j\overset{2}{\equiv}1. \end{cases} \tag{2}$$

For the proof of the claimed factorization we compute the (i, j)-entry of $L\cdot U$, that is

$$(L \cdot U)_{i,j} = \sum_{k=1}^{n} l_{i,k} u_{k,j}.$$

It is easy to see that it is enough to show that $(L \cdot U)_{1,j} = 1$, $(L \cdot U)_{i,1} = 1$ for $1 \leq i, j \leq n$ and

$$(L \cdot U)_{i,j} = \begin{cases} (L\cdot U)_{i-1,j-1} + (L\cdot U)_{i-1,j} & j \overset{2}{\equiv} 0 \\ (L\cdot U)_{i-1,j-1} - (L\cdot U)_{i-1,j} & j \overset{2}{\equiv} 1 \end{cases}$$

for $2 \leq i, j \leq n$, in order to prove the theorem.

Let us do the required calculations. First, suppose that $i = 1$. Then

$$(L \cdot U)_{1,j} = \sum_{k=1}^{n} l_{1,k} u_{k,j} = l_{1,1} u_{1,j} = 1. \tag{3}$$

Next, suppose that $j = 1$. In this case we obtain

$$(L \cdot U)_{i,1} = \sum_{k=1}^{n} l_{i,k} u_{k,1} = l_{i,1} u_{1,1} = 1. \tag{4}$$

Finally, we assume that $2 \leq i, j \leq n$. We split the proof into two cases, according to the following possibilities for j.

Case 1. $j \stackrel{2}{\equiv} 0$. In this case we claim that

$$(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j}. \tag{5}$$

Since $j - 1 \stackrel{2}{\equiv} 1$, by (2) we get

$$(L \cdot U)_{i-1,j-1} = \sum_{k=1}^{n} l_{i-1,k} u_{k,j-1} = 1 - \sum_{k=1}^{\frac{j-2}{2}} l_{i-1,2k} + 3 \sum_{k=1}^{\frac{j-2}{2}} l_{i-1,2k+1},$$

and since $j \stackrel{2}{\equiv} 0$ we obtain

$$(L \cdot U)_{i-1,j} = \sum_{k=1}^{n} l_{i-1,k} u_{k,j} = 1 + \sum_{k=1}^{\frac{j}{2}} l_{i-1,2k} - \sum_{k=1}^{\frac{j-2}{2}} l_{i-1,2k+1},$$

and ultimately

$$(L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} = 2 + 2 \sum_{k=1}^{\frac{j-2}{2}} l_{i-1,2k+1} + l_{i-1,j}. \tag{6}$$

Again, since $j \stackrel{2}{\equiv} 0$ we obtain

$$(L \cdot U)_{i,j} = \sum_{k=1}^{n} l_{i,k} u_{k,j} = l_{i,1} + \sum_{k=1}^{\frac{j}{2}} l_{i,2k} - \sum_{k=1}^{\frac{j-2}{2}} l_{i,2k+1},$$

and by (1) we get

$$(L \cdot U)_{i,j} = l_{i,1} + \sum_{k=1}^{\frac{j}{2}} (l_{i-1,2k-1} + l_{i-1,2k}) - \sum_{k=1}^{\frac{j-2}{2}} (l_{i-1,2k} - l_{i-1,2k+1}),$$

and after some further simplification we obtain

$$(L \cdot U)_{i,j} = 2 + 2 \sum_{k=1}^{\frac{j-2}{2}} l_{i-1,2k+1} + l_{i-1,j}. \tag{7}$$

Now, from (6) and (7) we obtain (5).

Case 2. $j \stackrel{2}{\equiv} 1$. In this case we claim that

$$(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} - (L \cdot U)_{i-1,j}. \tag{8}$$

Here, since $j - 1 \stackrel{2}{\equiv} 0$, by (2) we obtain

$$(L \cdot U)_{i-1,j-1} = \sum_{k=1}^{n} l_{i-1,k} u_{k,j-1} = 1 + \sum_{k=1}^{\frac{j-1}{2}} l_{i-1,2k} - \sum_{k=1}^{\frac{j-3}{2}} l_{i-1,2k+1}$$

and similarly we deduce that

$$(L\cdot U)_{i-1,j}=\sum_{k=1}^{n}l_{i-1,k}u_{k,j}=1-\sum_{k=1}^{\frac{j-1}{2}}l_{i-1,2k}+3\sum_{k=1}^{\frac{j-1}{2}}l_{i-1,2k+1},$$

because $j\overset{2}{\equiv}1$. Therefore by an easy calculation we conclude that

$$(L\cdot U)_{i-1,j-1}-(L\cdot U)_{i-1,j}=2\sum_{k=1}^{\frac{j-1}{2}}l_{i-1,2k}-4\sum_{k=1}^{\frac{j-3}{2}}l_{i-1,2k+1}-3l_{i-1,j}. \tag{9}$$

Again, since $j\overset{2}{\equiv}1$ we have

$$(L\cdot U)_{i,j}=\sum_{k=1}^{n}l_{i,k}u_{k,j}=l_{i,1}-\sum_{k=1}^{\frac{j-1}{2}}l_{i,2k}+3\sum_{k=1}^{\frac{j-1}{2}}l_{i,2k+1}.$$

Now, by (1) we obtain

$$(L\cdot U)_{i,j}=1-\sum_{k=1}^{\frac{j-1}{2}}(l_{i-1,2k-1}+l_{i-1,2k})+3\sum_{k=1}^{\frac{j-1}{2}}(l_{i-1,2k}-l_{i-1,2k+1}),$$

and simply we can observe that

$$(L\cdot U)_{i,j}=2\sum_{k=1}^{\frac{j-1}{2}}l_{i-1,2k}-4\sum_{k=1}^{\frac{j-3}{2}}l_{i-1,2k+1}-3l_{i-1,j}. \tag{10}$$

Now, from (9) and (10) we obtain (8).

Therefore, from (3), (4), (5) and (8) we conclude the theorem. □

Next, we focus on the sequence $(\det A^{(2,\pm)}_{\alpha,\alpha}(n))$ for $n\in\mathbb{N}$.

Theorem 2. *The sequence* $(\det A^{(2,\pm)}_{\alpha,\alpha}(n))$*of determinants associated to the sequence* $\alpha=(1,1,1,\ldots)$ *satisfies the following*

$$\det A^{(2,\pm)}_{\alpha,\alpha}(n)=\begin{cases}(-5)^{2l-1} & \text{if } \ n=4l, \quad (l=1,2,\ldots)\\ (-5)^{2l} & \text{if } \ n=4l+r. \ (r=1,2,3, \ l=0,1,2,\ldots)\end{cases}$$

Proof. Again, we use the LU-factorization method. Here, we claim that

$$A^{(2,\pm)}_{\alpha,\alpha}(n)=L\cdot U,$$

where $L=A^{(2,\pm)}_{\beta,\alpha}(n)$ with $\beta=(1,0,0,\ldots)$ and where

$$U=\begin{bmatrix}U_1\\U_2\\ \vdots\\U_n\end{bmatrix},$$

with

$$U_i = \begin{cases} (\underbrace{1,1,1,\dots,1}_{n \text{ times}}) & i=1, \\ (\underbrace{0,0,\dots,0}_{i-1 \text{ times}}, \underbrace{-5,1,1,-1,-5,1,1,-1,\dots,u_{i,n-1},u_{i,n}}_{n-i+1 \text{ times (4-periodic)}}) & i \overset{4}{\equiv} 0, \\ (\underbrace{0,0,\dots,0}_{i-1 \text{ times}}, \underbrace{-5,3,-1,-1,-5,3,-1,-1,\dots,u_{i,n-1},u_{i,n}}_{n-i+1 \text{ times (4-periodic)}}) & i \overset{4}{\equiv} 1, \\ (\underbrace{0,0,\dots,0}_{i-1 \text{ times}}, \underbrace{1,1,-1,-1,1,1,-1,-1,\dots,u_{i,n-1},u_{i,n}}_{n-i+1 \text{ times (4-periodic)}}) & i \overset{4}{\equiv} 2, \\ (\underbrace{0,0,\dots,0}_{i-1 \text{ times}}, \underbrace{1,3,1,-1,1,3,1,-1,\dots,u_{i,n-1},u_{i,n}}_{n-i+1 \text{ times (4-periodic)}}) & i \overset{4}{\equiv} 3, \end{cases}$$

and $(u_{i,n-1}, u_{i,n})$ is satisfied in Table 2.

Table 2.

$i \backslash n$	$n \overset{4}{\equiv} 0$	$n \overset{4}{\equiv} 1$	$n \overset{4}{\equiv} 2$	$n \overset{4}{\equiv} 3$
$i \overset{4}{\equiv} 0$	$(-1,-5)$	$(-5,1)$	$(1,1)$	$(1,-1)$
$i \overset{4}{\equiv} 1$	$(-1,-1)$	$(-1,-5)$	$(-5,3)$	$(3,-1)$
$i \overset{4}{\equiv} 2$	$(1,-1)$	$(-1,-1)$	$(-1,1)$	$(1,1)$
$i \overset{4}{\equiv} 3$	$(1,3)$	$(3,1)$	$(1,-1)$	$(-1,1)$

The matrix L is a lower triangular matrix with 1's on the diagonal, whereas U is an upper triangular matrix with diagonal entries

$$1, \underbrace{1,1,-5,-5,1,1,\dots,u_{n-1,n-1},u_{n,n}}_{4\text{-periodic}},$$

where

$$(u_{n-1,n-1}, u_{n,n}) = \begin{cases} (1,-5) & \text{if } n \overset{4}{\equiv} 0, \\ (-5,-5) & \text{if } n \overset{4}{\equiv} 1, \\ (-5,1) & \text{if } n \overset{4}{\equiv} 2, \\ (1,1) & \text{if } n \overset{4}{\equiv} 3. \end{cases}$$

Since $\det L = 1$ and

$$\det U = \begin{cases} (-5)^{2l-1} & \text{if} \quad n = 4l, \qquad (l = 1,2,\dots) \\ (-5)^{2l} & \text{if} \quad n = 4l + r. \ (r = 1,2,3, \ l = 0,1,2,\dots) \end{cases}$$

Again, it is immediately obvious that the claimed factorization of $A^{(2,\pm)}_{\alpha,\alpha}(n)$ implies the validity of the theorem.

Suppose that

$$L = (l_{i,j})_{1\leq i,j\leq n} \quad \text{and} \quad U = (u_{i,j})_{1\leq i,j\leq n}.$$

Then by definition, we have $l_{1,1} = 1$, $l_{1,j} = 0$, $l_{i,1} = 1$ for $2 \leq i, j \leq n$ and

$$l_{i,j} = \begin{cases} l_{i-1,j-1} + l_{i-1,j} & \text{if } j \overset{4}{\equiv} 2,3 \\ l_{i-1,j-1} - l_{i-1,j} & \text{if } j \overset{4}{\equiv} 0,1 \end{cases} \tag{11}$$

for $2 \leq i, j \leq n$. Moreover, the jth column of U can be considered as follows:

$$(u_{1,j},\ldots,u_{n,j})^T = \begin{cases} (1,0,0,\ldots,0)^T & j=1, \\ (1, \underbrace{-1,3,-5,-1,-1,3,-5,-1,\ldots,3,-5}_{j-1 \text{ times (4-periodic)}}, \underbrace{0,\ldots,0}_{n-j})^T & j \overset{4}{\equiv} 0, \\ (1, \underbrace{-1,1,1,-5,-1,1,1,-5,\ldots,1,-5}_{j-1 \text{ times (4-periodic)}}, \underbrace{0,\ldots,0}_{n-j})^T & j \overset{4}{\equiv} 1, \\ (1, \underbrace{1,-1,1,3,1,-1,1,3,\ldots,3,1}_{j-1 \text{ times (4-periodic)}}, \underbrace{0,\ldots,0}_{n-j})^T & j \overset{4}{\equiv} 2, \\ (1, \underbrace{1,1,-1,-1,1,1,-1,-1,\ldots,1,1}_{j-1 \text{ times (4-periodic)}}, \underbrace{0,\ldots,0}_{n-j})^T & j \overset{4}{\equiv} 3. \end{cases} \tag{12}$$

For the proof of the claimed factorization we need again some calculations. In fact, the (i, j)-entry of $L \cdot U$ is

$$(L \cdot U)_{i,j} = \sum_{k=1}^{n} l_{i,k} u_{k,j}.$$

It is easy to see that it is enough to show that $(L \cdot U)_{1,j} = 1$, $(L \cdot U)_{i,1} = 1$ for $1 \leq i, j \leq n$ and

$$(L \cdot U)_{i,j} = \begin{cases} (L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} & j \overset{4}{\equiv} 2,3 \\ (L \cdot U)_{i-1,j-1} - (L \cdot U)_{i-1,j} & j \overset{4}{\equiv} 0,1 \end{cases}$$

for $2 \leq i, j \leq n$, in order to prove the theorem.

Again, we verify the claim by a direct calculation. First, suppose that $i = 1$. Then

$$(L \cdot U)_{1,j} = \sum_{k=1}^{n} l_{1,k} u_{k,j} = l_{1,1} u_{1,j} = 1. \tag{13}$$

Next, suppose that $j = 1$. In this case we obtain

$$(L \cdot U)_{i,1} = \sum_{k=1}^{n} l_{i,k} u_{k,1} = l_{i,1} u_{1,1} = 1. \tag{14}$$

Finally, we assume that $2 \leq i, j \leq n$. We split the proof into four cases, according to the following possibilities for j.

Case 1. $j \overset{4}{\equiv} 0$. In this case we claim that

$$(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} - (L \cdot U)_{i-1,j}. \tag{15}$$

Since $j - 1 \overset{4}{\equiv} 3$, we obtain

$$(L\cdot U)_{i-1,j-1} = l_{i-1,1} - \sum_{k=1}^{\frac{j-4}{4}} l_{i-1,4k} - \sum_{k=1}^{\frac{j-4}{4}} l_{i-1,4k+1} + \sum_{k=0}^{\frac{j-4}{4}} l_{i-1,4k+2} + \sum_{k=0}^{\frac{j-4}{4}} l_{i-1,4k+3},$$

and since $j \overset{4}{\equiv} 0$, it follows that

$$(L\cdot U)_{i-1,j} = l_{i-1,1} - 5\sum_{k=1}^{\frac{j}{4}} l_{i-1,4k} - \sum_{k=1}^{\frac{j-4}{4}} l_{i-1,4k+1} - \sum_{k=0}^{\frac{j-4}{4}} l_{i-1,4k+2} + 3\sum_{k=0}^{\frac{j-4}{4}} l_{i-1,4k+3}.$$

Consequently, we obtain

$$(L \cdot U)_{i-1,j-1} - (L \cdot U)_{i-1,j} = 4\sum_{k=1}^{\frac{j-4}{4}} l_{i-1,4k} + 2\sum_{k=0}^{\frac{j-4}{4}} l_{i-1,4k+2} - 2\sum_{k=0}^{\frac{j-4}{4}} l_{i-1,4k+3} - 5l_{i-1,j}. \tag{16}$$

On the other hand since $j \overset{4}{\equiv} 0$, we get

$$(L \cdot U)_{i,j} = l_{i,1} - 5\sum_{k=1}^{\frac{j}{4}} l_{i,4k} - \sum_{k=1}^{\frac{j-4}{4}} l_{i,4k+1} - \sum_{k=0}^{\frac{j-4}{4}} l_{i,4k+2} + 3\sum_{k=0}^{\frac{j-4}{4}} l_{i,4k+3}.$$

Now by (11) we deduce that

$$\begin{aligned}(L \cdot U)_{i,j} = 1 &- 5\sum_{k=1}^{\frac{j}{4}}(l_{i-1,4k-1} - l_{i-1,4k}) - \sum_{k=1}^{\frac{j-4}{4}}(l_{i-1,4k} - l_{i-1,4k+1}) \\ &- \sum_{k=0}^{\frac{j-4}{4}}(l_{i-1,4k+1} + l_{i-1,4k+2}) + 3\sum_{k=0}^{\frac{j-4}{4}}(l_{i-1,4k+2} + l_{i-1,4k+3}),\end{aligned}$$

and after some further simplifications the expression reduces to

$$(L \cdot U)_{i,j} = 4\sum_{k=1}^{\frac{j-4}{4}} l_{i-1,4k} + 2\sum_{k=0}^{\frac{j-4}{4}} l_{i-1,4k+2} - 2\sum_{k=0}^{\frac{j-4}{4}} l_{i-1,4k+3} - 5l_{i-1,j}. \tag{17}$$

Now, from (16) and (17) we obtain (15).

Case 2. $j \overset{4}{\equiv} 1$. Here, we claim that

$$(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} - (L \cdot U)_{i-1,j}. \tag{18}$$

Since $j - 1 \overset{4}{\equiv} 0$, we obtain

$$(L \cdot U)_{i-1,j-1} = l_{i-1,1} - 5\sum_{k=1}^{\frac{j-1}{4}} l_{i-1,4k} - \sum_{k=1}^{\frac{j-5}{4}} l_{i-1,4k+1} - \sum_{k=0}^{\frac{j-5}{4}} l_{i-1,4k+2} + 3\sum_{k=0}^{\frac{j-5}{4}} l_{i-1,4k+3}.$$

Similarly, since $j \overset{4}{\equiv} 1$ it follows that

$$(L\cdot U)_{i-1,j} = l_{i-1,1} + \sum_{k=1}^{\frac{j-1}{4}} l_{i-1,4k} - 5\sum_{k=1}^{\frac{j-1}{4}} l_{i-1,4k+1} - \sum_{k=0}^{\frac{j-5}{4}} l_{i-1,4k+2} - \sum_{k=0}^{\frac{j-5}{4}} l_{i-1,4k+3}.$$

Therefore, we have

(19)

$$(L\cdot U)_{i-1,j-1} - (L\cdot U)_{i-1,j} = -6\sum_{k=1}^{\frac{j-4}{4}} l_{i-1,4k} + 4\sum_{k=1}^{\frac{j-5}{4}} l_{i-1,4k+1} + 4\sum_{k=0}^{\frac{j-5}{4}} l_{i-1,4k+3} - 5l_{i-1,j}.$$

Furthermore, since $j \overset{4}{\equiv} 1$ we obtain

$$(L\cdot U)_{i,j} = l_{i,1} + \sum_{k=1}^{\frac{j-1}{4}} l_{i,4k} - 5\sum_{k=1}^{\frac{j-1}{4}} l_{i,4k+1} - \sum_{k=0}^{\frac{j-5}{4}} l_{i,4k+2} + \sum_{k=0}^{\frac{j-5}{4}} l_{i,4k+3}.$$

Now we apply (11), to get

$$(L\cdot U)_{i,j} = 1 + \sum_{k=1}^{\frac{j-1}{4}} (l_{i-1,4k-1} - l_{i-1,4k}) - 5\sum_{k=1}^{\frac{j-4}{4}} (l_{i-1,4k} - l_{i-1,4k+1})$$
$$- \sum_{k=0}^{\frac{j-5}{4}} (l_{i-1,4k+1} + l_{i-1,4k+2}) + \sum_{k=0}^{\frac{j-5}{4}} (l_{i-1,4k+2} + l_{i-1,4k+3}).$$

After some simplifications this leads to

$$(L\cdot U)_{i,j} = -6\sum_{k=1}^{\frac{j-4}{4}} l_{i-1,4k} + 4\sum_{k=1}^{\frac{j-5}{4}} l_{i-1,4k+1} + 4\sum_{k=0}^{\frac{j-5}{4}} l_{i-1,4k+3} - 5l_{i-1,j}. \tag{20}$$

Through comparing (19) and (20), we can get (18).

Case 3. $j \overset{4}{\equiv} 2$. In this case we claim that

$$(L\cdot U)_{i,j} = (L\cdot U)_{i-1,j-1} + (L\cdot U)_{i-1,j}. \tag{21}$$

Here since $j-1 \overset{4}{\equiv} 1$, by (12) we obtain

$$(L\cdot U)_{i-1,j-1} = 1 + \sum_{k=1}^{\frac{j-2}{4}} l_{i-1,4k} - 5\sum_{k=1}^{\frac{j-2}{4}} l_{i-1,4k+1} - \sum_{k=0}^{\frac{j-6}{4}} l_{i-1,4k+2} + \sum_{k=0}^{\frac{j-6}{4}} l_{i-1,4k+3},$$

and since $j \overset{4}{\equiv} 2$ it follows that

$$(L\cdot U)_{i-1,j} = 1 + \sum_{k=1}^{\frac{j-2}{4}} l_{i-1,4k} + 3\sum_{k=1}^{\frac{j-2}{4}} l_{i-1,4k+1} + \sum_{k=0}^{\frac{j-2}{4}} l_{i-1,4k+2} - \sum_{k=0}^{\frac{j-6}{4}} l_{i-1,4k+3}.$$

Therefore we have

$$(L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} = 2 + 2\sum_{k=1}^{\frac{j-2}{4}} l_{i-1,4k} - 2\sum_{k=1}^{\frac{j-2}{4}} l_{i-1,4k+1} + l_{i-1,j}. \tag{22}$$

On the other hand, since $j \overset{4}{\equiv} 2$ we deduce that

$$(L \cdot U)_{i,j} = 1 + \sum_{k=1}^{\frac{j-2}{4}} l_{i,4k} + 3\sum_{k=1}^{\frac{j-2}{4}} l_{i,4k+1} + \sum_{k=0}^{\frac{j-2}{4}} l_{i,4k+2} - \sum_{k=0}^{\frac{j-6}{4}} l_{i,4k+3},$$

and by (11) we conclude that

$$(L \cdot U)_{i,j} = 1 + \sum_{k=1}^{\frac{j-2}{4}} (l_{i-1,4k-1} - l_{i-1,4k}) + 3\sum_{k=1}^{\frac{j-2}{4}} (l_{i-1,4k} - l_{i-1,4k+1})$$
$$+ \sum_{k=0}^{\frac{j-2}{4}} (l_{i-1,4k+1} + l_{i-1,4k+2}) - \sum_{k=0}^{\frac{j-6}{4}} (l_{i-1,4k+2} + l_{i-1,4k+3}).$$

Now, an easy calculation shows that

$$(L \cdot U)_{i,j} = 2 + 2\sum_{k=1}^{\frac{j-2}{4}} l_{i-1,4k} - 2\sum_{k=1}^{\frac{j-2}{4}} l_{i-1,4k+1} + l_{i-1,j}. \tag{23}$$

By comparing (22) and (23), we may obtain (21).

Case 4. $j \overset{4}{\equiv} 3$. In this case, we claim that

$$(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j}. \tag{24}$$

Since $j - 1 \overset{4}{\equiv} 2$, we obtain

$$(L \cdot U)_{i-1,j-1} = 1 + \sum_{k=1}^{\frac{j-3}{4}} l_{i-1,4k} + 3\sum_{k=1}^{\frac{j-3}{4}} l_{i-1,4k+1} + \sum_{k=0}^{\frac{j-3}{4}} l_{i-1,4k+2} - \sum_{k=0}^{\frac{j-7}{4}} l_{i-1,4k+3}.$$

Similarly, since $j \overset{4}{\equiv} 3$ it follows that

$$(L \cdot U)_{i-1,j} = 1 - \sum_{k=1}^{\frac{j-3}{4}} l_{i-1,4k} - \sum_{k=1}^{\frac{j-3}{4}} l_{i-1,4k+1} - \sum_{k=0}^{\frac{j-3}{4}} l_{i-1,4k+2} + \sum_{k=0}^{\frac{j-3}{4}} l_{i-1,4k+3}.$$

Therefore, we have

$$(L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} = 2 + 2\sum_{k=1}^{\frac{j-3}{4}} l_{i-1,4k+1} + 2\sum_{k=0}^{\frac{j-3}{4}} l_{i-1,4k+2} + l_{i-1,j}. \tag{25}$$

On the other hand, since $j \overset{4}{\equiv} 2$ we obtain

$$(L\cdot U)_{i,j} = 1 - \sum_{k=1}^{\frac{j-3}{4}} l_{i,4k} - \sum_{k=1}^{\frac{j-3}{4}} l_{i,4k+1} + \sum_{k=0}^{\frac{j-3}{4}} l_{i,4k+2} - \sum_{k=0}^{\frac{j-3}{4}} l_{i,4k+3}.$$

Again by (11) we conclude that

$$(L\cdot U)_{i,j} = 1 - \sum_{k=1}^{\frac{j-3}{4}} (l_{i-1,4k-1} - l_{i-1,4k}) - \sum_{k=1}^{\frac{j-3}{4}} (l_{i-1,4k} - l_{i-1,4k+1})$$

$$+ \sum_{k=0}^{\frac{j-3}{4}} (l_{i-1,4k+1} + l_{i-1,4k+2}) + \sum_{k=0}^{\frac{j-3}{4}} (l_{i-1,4k+2} + l_{i-1,4k+3}),$$

and we easily deduce that

$$(L\cdot U)_{i,j} = 2 + 2\sum_{k=1}^{\frac{j-3}{4}} l_{i-1,4k+1} + 2\sum_{k=0}^{\frac{j-3}{4}} l_{i-1,4k+2} + l_{i-1,j}. \tag{26}$$

By comparing (25) and (26), we can get (24).

Therefore, from (13), (14), (15), (18), (21) and (24) we conclude the theorem. □

In the end, we consider the sequence $(\det A^{(3,\pm)}_{\alpha,\alpha}(n))$ for $n \in \mathbb{N}$.

Theorem 3. *The sequence* $(\det A^{(3,\pm)}_{\alpha,\alpha}(n))$ *of determinants associated to the sequence* $\alpha = (1,1,1,\ldots)$ *satisfies the following*

$$\det A^{(3,\pm)}_{\alpha,\alpha}(n) = \begin{cases} 11^{3l-1} & \textit{if} \quad n = 6l, \quad (l = 1,2,\ldots) \\ 11^{3l} & \textit{if} \quad n = 6l + r, \ (r = 1,2,3,4, \ l = 0,1,2,\ldots) \\ 11^{3l+1} & \textit{if} \quad n = 6l + 5. \ (l = 0,1,2,\ldots) \end{cases}$$

Proof. The proof is similar to the proof of Theorem 1.1 and Theorem 1.2, and we avoid presenting some of the details. Again, we apply LU-factorization. Here, we claim that

$$A^{(3,\pm)}_{\alpha,\alpha}(n) = L\cdot U,$$

where $L = A^{(3,\pm)}_{\beta,\alpha}(n)$ with $\beta = (1,0,0,\ldots)$ is a lower triangular matrix with 1's on the diagonal, and where

$$U = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix},$$

with

$$U_i = \begin{cases} (\underbrace{1,1,1,\ldots,1}_{n \text{ times}}) & i=1, \\ (\underbrace{0,0,\ldots,0}_{i-1 \text{ times}}, \underbrace{11,-7,1,1,-1,-1,\ldots,u_{i,n-1},u_{i,n}}_{n-i+1 \text{ times (6-periodic)}}) & i \overset{6}{\equiv} 0, \\ (\underbrace{0,0,\ldots,0}_{i-1 \text{ times}}, \underbrace{11,-5,3,-1,-1,-1,\ldots,u_{i,n-1},u_{i,n}}_{n-i+1 \text{ times (6-periodic)}}) & i \overset{6}{\equiv} 1, \\ (\underbrace{0,0,\ldots,0}_{i-1 \text{ times}}, \underbrace{1,1,1,-1,-1,-1,\ldots,u_{i,n-1},u_{i,n}}_{n-i+1 \text{ times (6-periodic)}}) & i \overset{6}{\equiv} 2, \\ (\underbrace{0,0,\ldots,0}_{i-1 \text{ times}}, \underbrace{1,1,3,1,1,-1,\ldots,u_{i,n-1},u_{i,n}}_{n-i+1 \text{ times (6-periodic)}}) & i \overset{6}{\equiv} 3, \\ (\underbrace{0,0,\ldots,0}_{i-1 \text{ times}}, \underbrace{1,-5,1,-1,1,-1,\ldots,u_{i,n-1},u_{i,n}}_{n-i+1 \text{ times (6-periodic)}}) & i \overset{6}{\equiv} 4, \\ (\underbrace{0,0,\ldots,0}_{i-1 \text{ times}}, \underbrace{11,-7,3,-1,1,-1,\ldots,u_{i,n-1},u_{i,n}}_{n-i+1 \text{ times (6-periodic)}}) & i \overset{6}{\equiv} 5, \end{cases}$$

and $(u_{i,n-1}, u_{i,n})$ is satisfied in Table 3.

Table 3.

$i\backslash n$	$n \overset{6}{\equiv} 0$	$n \overset{6}{\equiv} 1$	$n \overset{6}{\equiv} 2$	$n \overset{6}{\equiv} 3$	$n \overset{6}{\equiv} 4$	$n \overset{6}{\equiv} 5$
$i \overset{6}{\equiv} 0$	$(-1,11)$	$(11,-7)$	$(-7,1)$	$(1,1)$	$(1,-1)$	$(-1,-1)$
$i \overset{6}{\equiv} 1$	$(-1,-1)$	$(-1,11)$	$(11,-5)$	$(-5,3)$	$(3,-1)$	$(-1,-1)$
$i \overset{6}{\equiv} 2$	$(-1,-1)$	$(-1,-1)$	$(-1,1)$	$(1,1)$	$(1,1)$	$(1,-1)$
$i \overset{6}{\equiv} 3$	$(3,1)$	$(1,1)$	$(1,-1)$	$(-1,1)$	$(1,1)$	$(1,3)$
$i \overset{6}{\equiv} 4$	$(-5,1)$	$(1,-1)$	$(-1,1)$	$(1,-1)$	$(-1,1)$	$(1,5)$
$i \overset{6}{\equiv} 5$	$(11,-7)$	$(-7,3)$	$(3,-1)$	$(-1,1)$	$(1,-1)$	$(-1,11)$

The matrix U is an upper triangular one with diagonal entries

$$1, \underbrace{1,1,1,11,11,11,\ldots,u_{n-1,n-1},u_{n,n}}_{6-\text{periodic}},$$

where

$$(u_{n-1,n-1}, u_{n,n}) = \begin{cases} (11,11) & \text{if } n \overset{6}{\equiv} 0 \text{ or } 1, \\ (11,1) & \text{if } n \overset{6}{\equiv} 2, \\ (1,1) & \text{if } n \overset{6}{\equiv} 3 \text{ or } 4, \\ (1,11) & \text{if } n \overset{6}{\equiv} 5. \end{cases}$$

Since $\det L = 1$ and

$$\det U = \begin{cases} 11^{3l-1} & \text{if} \quad n = 6l, \qquad (l = 1, 2, \ldots) \\ 11^{3l} & \text{if} \quad n = 6l + r, \ (r = 1, 2, 3, 4, \ l = 0, 1, 2, \ldots) \\ 11^{3l+1} & \text{if} \quad n = 6l + 5, \ (l = 0, 1, 2, \ldots) \end{cases}$$

it is obvious that the claimed factorization of $A^{(3,\pm)}_{\alpha,\alpha}(n)$ implies the validity of the theorem.

Let us do the required calculation. Again, we assume that

$$L = (l_{i,j})_{1 \le i,j \le n} \quad \text{and} \quad U = (u_{i,j})_{1 \le i,j \le n}.$$

Then by definition, we have $l_{1,1} = 1$, $l_{1,j} = 0$, $l_{i,1} = 1$ for $2 \le i, j \le n$ and the entries $l_{i,j}$ for $2 \le i, j \le n$ satisfy

(27)
$$l_{i,j} = \begin{cases} l_{i-1,j-1} + l_{i-1,j} & \text{if } \ j \overset{4}{\equiv} 2, 3, 4 \\ l_{i-1,j-1} - l_{i-1,j} & \text{if } \ j \overset{4}{\equiv} 5, 0, 1. \end{cases}$$

Moreover, the jth column of U can be considered as follows.

(28)
$$(u_{1,j}, \ldots, u_{n,j})^T = \begin{cases} (1, 0, 0, \ldots, 0)^T & j = 1, \\ (1, \underbrace{-1, 1, 1, -7, 11, -1, \ldots, -7, 11}_{j-1 \text{ times } (6-\text{periodic})}, \underbrace{0, \ldots, 0}_{n-j})^T & j \overset{6}{\equiv} 0, \\ (1, \underbrace{-1, 1, -1, 3, -7, 11, \ldots, -7, 11}_{j-1 \text{ times } (6-\text{periodic})}, \underbrace{0, \ldots, 0}_{n-j})^T & j \overset{6}{\equiv} 1, \\ (1, \underbrace{1, -1, 1, -1, 1, -5, \ldots, -5, 1}_{j-1 \text{ times } (6-\text{periodic})}, \underbrace{0, \ldots, 0}_{n-j})^T & j \overset{6}{\equiv} 2, \\ (1, \underbrace{1, 1, -1, 1, 1, 3, \ldots, 3, 1, 1}_{j-1 \text{ times } (6-\text{periodic})}, \underbrace{0, \ldots, 0}_{n-j})^T & j \overset{6}{\equiv} 3, \\ (1, \underbrace{1, 1, 1, -1, -1, -1, \ldots, 1, 1, 1}_{j-1 \text{ times } (6-\text{periodic})}, \underbrace{0, \ldots, 0}_{n-j})^T & j \overset{6}{\equiv} 4, \\ (1, \underbrace{-1, 3, -5, 11, -1, -1, \ldots, -5, 11}_{j-1 \text{ times } (6-\text{periodic})}, \underbrace{0, \ldots, 0}_{n-j})^T & j \overset{6}{\equiv} 5. \end{cases}$$

In order to prove the claim we show that the (i, j)-entry of $L \cdot U$, that is

$$(L \cdot U)_{i,j} = \sum_{k=1}^{n} l_{i,k} u_{k,j},$$

satisfy $(L \cdot U)_{1,j} = 1$, $(L \cdot U)_{i,1} = 1$ for $1 \le i, j \le n$ and

(29)
$$(L \cdot U)_{i,j} = \begin{cases} (L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j} & j \overset{6}{\equiv} 2, 3, 4 \\ (L \cdot U)_{i-1,j-1} - (L \cdot U)_{i-1,j} & j \overset{6}{\equiv} 5, 0, 1 \end{cases}$$

for $2 \le i, j \le n$.

First assume that $i = 1$. Then, in accordance with the definition of $l_{1,j}$, we obtain

$$(L \cdot U)_{1,j} = \sum_{k=1}^{n} l_{1,k} u_{k,j} = l_{1,1} u_{1,j} = 1.$$

Next, suppose that $j = 1$. In this case by (28) we obtain

$$(L \cdot U)_{i,1} = \sum_{k=1}^{n} l_{i,k} u_{k,1} = l_{i,1} u_{1,1} = 1.$$

Finally, we assume that $2 \leq i, j \leq n$. In this case we must show that the entries $(L \cdot U)_{i,j}$ satisfy (29). Here, there are six cases to distinguish, according to $j \overset{6}{\equiv} 0, 1, 2, 3, 4$ or 5. Using similar arguments to those in the proof of Theorem 1.2, we see that the result is true in any cases. For instance, we assume that $j \overset{6}{\equiv} 4$. In this case, we must establish that

$$(L \cdot U)_{i,j} = (L \cdot U)_{i-1,j-1} + (L \cdot U)_{i-1,j}. \tag{30}$$

Since $j \overset{6}{\equiv} 4$, in according with (28) , the right hand of (30) is equal to

$$2 + 2 \sum_{k=1}^{\frac{j-4}{6}} l_{i-1,6k+1} + 2 \sum_{k=0}^{\frac{j-4}{6}} l_{i-1,6k+2} + 2 \sum_{k=0}^{\frac{j-4}{6}} l_{i-1,6k+3} + l_{i-1,j}. \tag{31}$$

Again, since $j \overset{4}{\equiv} 2$ by (24), we see that the left-hand of (30) is equal to

$$1 - \sum_{k=1}^{\frac{j-4}{6}} l_{i,6k} - \sum_{k=1}^{\frac{j-4}{6}} l_{i,6k+1} + \sum_{k=0}^{\frac{j-4}{6}} l_{i,6k+2} + \sum_{k=0}^{\frac{j-4}{6}} l_{i,6k+3} + \sum_{k=0}^{\frac{j-4}{6}} l_{i,6k+4} - \sum_{k=0}^{\frac{j-10}{6}} l_{i,6k+5}. \tag{32}$$

Now, if we substitute the corresponding value for $l_{i,6k+r}$ $(0 \leq r \leq 5)$ from (29), we can conclude

$$(L \cdot U)_{i,j} = 2 + 2 \sum_{k=1}^{\frac{j-4}{6}} l_{i-1,6k+1} + 2 \sum_{k=0}^{\frac{j-4}{6}} l_{i-1,6k+2} + 2 \sum_{k=0}^{\frac{j-4}{6}} l_{i-1,6k+3} + l_{i-1,j}$$

which results in (30). In this way the proof is completed. □

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