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LINEAR WEINGARTEN HYPERSURFACES IN A UNIT SPHERE

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Abstract. In this paper, we have considered linear Weingarten hypersurfaces in a sphere and obtained some rigidity theorems. The purpose of this paper is to give some extension of the results due to Cheng-Yau [3] and Li [7].

1. Introduction

Let M be a hypersurface in an $(n+1)$-dimensional unit sphere $S^{n+1}(1)$. As is well known to us, there are many rigidity results for hypersurfaces in a unit sphere with constant mean curvature or with constant scalar curvature. Now we want to introduce an well-known theorem due to Cheng-Yau [3] as follows:

Theorem 1.1. *Let M be an n-dimensional compact hypersurface in a unit sphere $S^{n+1}(1)$. If*
(1) *M has nonnegative sectional curvature,*
(2) *the normalized scalar curvature r of M is constant and $r \geq 1$,*
then M is either totally umbilical, or $M = S^{n-k}(a) \times S^k(\sqrt{1-a^2})$, $1 \leq k \leq n-1$.

On the other hand, Li [7] studied some hypersurfaces in a unit sphere with scalar curvature proportional to mean curvature and proved the following theorem.

Theorem 1.2. *Let M be an n-dimensional compact hypersurface in a unit sphere $S^{n+1}(1)$. If*
(1) *M has nonnegative sectional curvature,*

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(2) *the normalized scalar curvature* r *of* M *is proportional to mean curvature* H *of* M*, that is,*

$$r = aH, \qquad a^2 \geq \frac{4n}{n-1},$$

where a *is a constant, then* M *is either totally umbilical, or* $M = S^{n-k}(a) \times S^k(\sqrt{1-a^2})$, $1 \leq k \leq n-1$.

Now let us introduce a notion for linear Weingarten hypersurfaces in an $(n+1)$-dimensional unit sphere $S^{n+1}(1)$ as follows:

Definition 1.1. Let M be a hypersurface in an $(n+1)$-dimensional unit sphere $S^{n+1}(1)$. We call M a *linear Weingarten hypersurface* if $cR+dH+e=0$, where c, d and e are constants such that $c^2+d^2 \neq 0$, R and H respectively denote the scalar curvature and the mean curvature of M.

Remark 1.1. When the constant $d=0$ in Definition 1.1, a linear Weingarten hypersurface M reduces to a hypersurface with constant scalar curvature. When the constant $c=0$ in Definition 1.1, a linear Weingarten hypersurface M reduces to a hypersurface with constant mean curvature. In such a sense, the linear Weingarten hypersurfaces can be regarded as a natural generalization of hypersurfaces with constant scalar curvature or with constant mean curvature.

By investigating Cheng-Yau's operator $\square$ given in [3] and using some new estimations, we want to study linear Weingarten hypersurfaces in a unit sphere as follows:

Theorem 1.3. *Let* M *be an* n*-dimensional compact hypersurface in a unit sphere* $S^{n+1}(1)$*. If*

(1) M *has nonnegative sectional curvature,*

(2) *the normalized scalar curvature* r *and the mean curvature* H *of* M *satisfies the following conditions*: $r = aH + b$, $(n-1)a^2 - 4n + 4nb \geq 0$, *then* M *is either totally umbilical, or* $M = S^{n-k}(a) \times S^k(\sqrt{1-a^2})$, $1 \leq k \leq n-1$.

Remark 1.2. Since $R = n(n-1)r$, a hypersurface M in Theorem 1.3 satisfying $r = aH + b$ is just a linear Weingarten hypersurface in Definition 1.1.

Remark 1.3. When the constant a in above identically vanishes, our Theorem 1.3 reduces to Theorem 1.1. When the constant b vanishes, our Theorem 1.3 reduces to Theorem 1.2.

In all of theorems mentioned above, we have assumed that M has nonnegative sectional curvature. In the following theorem, we want to study linear Weingarten hypersurfaces in a unit sphere without the assumption of nonnegative sectional curvature. In fact, we prove the following:

Theorem 1.4. *Let* M *be a hypersurface in* $S^{n+1}(1)$*. If*

(1) $r = aH + b$, $(n-1)a^2 - 4n + 4nb \geq 0$,

(2) $|B|^2 \leq 2\sqrt{n-1}$,

then either $|B|^2 = 0$ *and* M *is a totally umbilical hypersurface or* $|B|^2 = 2\sqrt{n-1}$ *and* $M = S^1(c) \times S^{n-1}(\sqrt{1-c^2})$.

2. Preliminaries

There are many studies on compact hypersurfaces in an $(n+1)$-dimensional unit sphere $S^{n+1}(1)$ (see [1], [2], [3], [5]-[12]). In this paper, let us also denote by M a compact hypersurfaces in $S^{n+1}(1)$. We choose a local orthonormal frame $\{e_A\}_{1\leq A\leq n+1}$ in $S^{n+1}(1)$, with dual coframe $\{\omega_A\}_{1\leq A\leq n+1}$, such that, at each point of M, $e_1, \dots, e_n$ are tangent to M and e_{n+1} is the positively oriented unit normal vector. We shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, C, \dots, \leq n+1; \quad 1 \leq i, j, k, \dots, \leq n.$$

Then the structure equations of $S^{n+1}(1)$ are given by

$$d\omega_A = \sum_{B=1}^{n+1} \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = \sum_{C=1}^{n+1} \omega_{AC} \wedge \omega_{CB} - \omega_A \wedge \omega_B. \tag{2.2}$$

When restricted to M, we have $\omega_{n+1} = 0$ and

$$0 = d\omega_{n+1} = \sum_{i=1}^{n} \omega_{n+1i} \wedge \omega_i. \tag{2.3}$$

By Cartan's lemma, there exist functions h_{ij} such that

$$\omega_{n+1i} = \sum_{j=1}^{n} h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.4}$$

This gives the second fundamental form of M, $B = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1}$. The mean curvature H is defined by $H = \frac{1}{n}\sum_i h_{ii}$. From (2.1)-(2.4) we obtain the structure equations of M

$$d\omega_i = \sum_{j=1}^{n} \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.5}$$

$$d\omega_{ij} = \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{n} R_{ijkl}\omega_k \wedge \omega_l \tag{2.6}$$

and the Gauss equation

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + h_{ik}h_{jl} - h_{il}h_{jk}. \tag{2.7}$$

Then it follows that

$$n(n-1)(r-1) = n^2H^2 - |B|^2, \tag{2.8}$$

where R_{ijkl} denotes the components of the Riemannian curvature tensor of M, $n(n-1)r$ is the scalar curvature of M and $|B|^2 = \sum_{i,j=1}^n h_{ij}^2$ is the square norm of the second fundamental form of M.

Let h_{ijk} denote the covariant derivative of h_{ij}. We then have

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}. \tag{2.9}$$

Thus, by exterior differentiation of (2.4), we obtain the Codazzi equation

$$h_{ijk} = h_{ikj}. \tag{2.10}$$

The second covariant derivative of h_{ij} is defined by

$$\sum_i h_{ijkl}\omega_l = dh_{ijk} + \sum_m h_{mjk}\omega_{mi} + \sum_m h_{imk}\omega_{mj} + \sum_m h_{ijm}\omega_{mk}. \tag{2.11}$$

By exterior differentiation of (2.9), we can get that the following Ricci identities hold

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{2.12}$$

For a smooth function f defined on M, the gradient, the hessian (f_{ij}) and the Laplacian $\triangle f$ of f are defined by

$$df = \sum_i f_i\omega_i, \quad \sum_j f_{ij}\omega_j = df_i + \sum_j f_j\omega_{ji}, \quad \triangle f = \sum_i f_{ii}. \tag{2.13}$$

Let $\phi = \sum_{i,j} \phi_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor defined on M, where

$$\phi_{ij} = nH\delta_{ij} - h_{ij}. \tag{2.14}$$

Following Cheng-Yau [3], we introduce an operator $\Box$ associated to ϕ acting on any smooth function f by

$$\Box f = \sum_{i,j} \phi_{ij} f_{ij} = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}. \tag{2.15}$$

Since ϕ_{ij} is divergence-free, it follows that the operator $\Box$ is self-adjoint relative to the L^2 inner product of M, i.e.,

$$\int_M f\Box g dv = \int_M g\Box f dv. \tag{2.16}$$

We choose $e_1, \ldots, e_n$ such that

$$h_{ij} = \lambda_i\delta_{ij}, \tag{2.17}$$

then we have

$$\begin{aligned} \Box(nH) &= nH\triangle(nH) - \textstyle\sum_i \lambda_i(nH)_{,ii} \\ &= \tfrac{1}{2}\triangle(nH)^2 - \textstyle\sum_i (nH)_{,i}^2 - \sum_i \lambda_i(nH)_{,ii} \\ &= \tfrac{1}{2}n(n-1)\triangle r + \tfrac{1}{2}\triangle|B|^2 - n^2|\nabla H|^2 - \textstyle\sum_i \lambda_i(nH)_{,ii}. \end{aligned} \tag{2.18}$$

On the other hand, we can deduce from (2.10) and (2.12) that

$$\frac{1}{2}\triangle|B|^2=\sum_{i,j,k}h_{ijk}^2+\sum_i\lambda_i(nH)_{,ii}+\frac{1}{2}\sum_{i,j}R_{ijij}(\lambda_i-\lambda_j)^2. \tag{2.19}$$

Putting (2.19) into (2.18), we obtain

$$\Box(nH)=\frac{1}{2}n(n-1)\triangle r+|\nabla B|^2-n^2|\nabla H|^2+\frac{1}{2}\sum_{i,j}R_{ijij}(\lambda_i-\lambda_j)^2. \tag{2.20}$$

From the Gauss equation, we have $R_{ijij}=1+\lambda_i\lambda_j$, putting this into (2.20), we get

$$\begin{aligned}\Box(nH)\ &=\tfrac{1}{2}n(n-1)\triangle r+|\nabla B|^2-n^2|\nabla H|^2\\ &\quad+n|B|^2-n^2H^2-|B|^4+nH\textstyle\sum_i\lambda_i^3.\end{aligned} \tag{2.21}$$

Next we introduce the following lemma

Lemma 2.1. *Let M be a hypersurface in $S^{n+1}(1)$. If*

$$r=aH+b \quad \textit{and} \quad (n-1)a^2-4n+4nb\geq 0, \tag{2.22}$$

then we have

$$|\nabla B|^2\geq n^2|\nabla H|^2, \tag{2.23}$$

where $n(n-1)r$ is scalar curvature of M, H denotes the mean curvature of M.

Proof. By the formula (2.7), we have

$$|B|^2=n^2H^2+n(n-1)(1-r)=n^2H^2+n(n-1)(1-aH-b), \tag{2.24}$$

it follows that

$$2h_{ij}h_{ijk}=2n^2HH_{,k}-n(n-1)aH_{,k}, \tag{2.25}$$

then we obtain

$$4\sum_k\left(\sum_{i,j}h_{ij}h_{ijk}\right)^2=[2n^2H-n(n-1)a]^2|\nabla H|^2. \tag{2.26}$$

Hence we deduce that

$$\begin{aligned}4\left(\sum_{i,j}h_{ij}^2\right)\left(\sum_{i,j,k}h_{ijk}^2\right)&\geq 4\sum_k\left(\sum_{i,j}h_{ij}h_{ijk}\right)^2\\ &=[2n^2H-n(n-1)a]^2|\nabla H|^2,\end{aligned} \tag{2.27}$$

that is,

$$4|B|^2|\nabla B|^2\geq[2n^2H-n(n-1)a]^2|\nabla H|^2. \tag{2.28}$$

On the other hand,

$$\begin{aligned}
&[2n^2H - n(n-1)a]^2 - 4n^2|B|^2 \\
&= 4n^4H^2 + n^2(n-1)^2a^2 - 4n^3(n-1)Ha \\
&\quad - 4n^3[nH^2 + (n-1)(1 - aH - b)] \\
&= n^2(n-1)[(n-1)a^2 - 4n + 4nb] \\
&\geq 0.
\end{aligned} \tag{2.29}$$

From this, together with Lemma 2.1, we have

$$4|B|^2|\nabla B|^2 \geq [2n^2H - n(n-1)a]^2|\nabla H|^2 \geq 4n^2|B|^2|\nabla H|^2,$$

then we obtain either $|B|^2 = 0$ and $|\nabla B|^2 = n^2|\nabla H|^2 = 0$ or $|\nabla B|^2 \geq n^2|\nabla H|^2$. This completes the proof of Lemma 2.1. □

Remark 2.1. When the constant a vanishes, our Lemma 2.1 reduce to Lemma 3.2 in [6] due to Li.

In this paper, we also need the following lemma due to Okumura [9].

Lemma 2.2. *Let μ_i, $i = 1, \ldots, n$, be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta = \text{constant} \geq 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3 \tag{2.30}$$

and the equality holds if and only if at least $(n-1)$ of the μ_i are equal.

3. Proofs of Theorem 1.3 and Theorem 1.4

First in this section, we want to prove Theorem 1.3 as follows:

By (2.20), we can obtain

$$0 = \int_M (|\nabla B|^2 - n^2|\nabla H|^2)dv + \frac{1}{2}\int_M \sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2 dv. \tag{3.1}$$

By using of Lemma 2.1, we can deduce that

$$|\nabla B|^2 \geq n^2|\nabla H|^2. \tag{3.2}$$

Since M has nonnegative sectional curvature and (3.1), we know that

$$|\nabla B|^2 = n^2|\nabla H|^2, \quad \sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2 = 0, \tag{3.3}$$

that is,

$$|\nabla B|^2 = n^2|\nabla H|^2, \quad R_{ijij} = 0 \text{ if } \lambda_i \neq \lambda_j. \tag{3.4}$$

Then it follows from the Gauss equation $R_{ijij} = 1+\lambda_i\lambda_j$ and (3.4) that either M is totally umbilical or M has two distinct constant principal curvatures λ, μ and $1 + \lambda\mu = 0$. This completes the proof of Theorem 1.3. □

Next we want to give the proof of Theorem1.4 as follows:

• Case 1: $n = 2$.

In this case, we obtain from the Gauss equation that

$$2(r-1) = 4H^2 - |B|^2 = 2\lambda_1\lambda_2. \tag{3.5}$$

Combining (2.21) and the Gauss equation (3.5), we obtain

$$\begin{aligned}
\Box(2H) &= \triangle r + |\nabla B|^2 - 4|\nabla H|^2 + 2|B|^2 - 4H^2 - |B|^4 + 2H\sum_{i=1}^{2}\lambda_i^3 \\
&= \triangle r + |\nabla B|^2 - 4|\nabla H|^2 + 2|B|^2 - 4H^2 - |B|^4 \\
&\quad + 2H(\lambda_1+\lambda_2)(\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2) \\
&= \triangle r + |\nabla B|^2 - 4|\nabla H|^2 + 2|B|^2 - 4H^2 - |B|^4 \\
&\quad + 4H^2(|B|^2 - (r-1)) \\
&= \triangle r + |\nabla B|^2 - 4|\nabla H|^2 + |B|^2 - 2(r-1) - |B|^4 \\
&\quad + \{|B|^2 + 2(r-1)\}\{|B|^2 - (r-1)\} \\
&= \triangle r + |\nabla B|^2 - 4|\nabla H|^2 + r(|B|^2 + 2 - 2r) \\
&= \triangle r + |\nabla B|^2 - 4|\nabla H|^2 + (4H^2 + 2 - |B|^2)(|B|^2 - 2H^2),
\end{aligned} \tag{3.6}$$

then we get the following integral equality

$$\int_M (|\nabla B|^2 - 4|\nabla H|^2)dv + \int_M (4H^2 + 2 - |B|^2)(|B|^2 - 2H^2)dv = 0. \tag{3.7}$$

Since $|B|^2 \leq 2$, we know that

$$(4H^2 + 2 - |B|^2)(|B|^2 - 2H^2) \geq 0. \tag{3.8}$$

By Lemma 2.1, we have

$$|\nabla B|^2 - 4|\nabla H|^2 \geq 0. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), we obtain $|\nabla B|^2 - 4|\nabla H|^2 = 0$ and $(4H^2 + 2 - |B|^2)(|B|^2 - 2H^2) = 0$, that is, either $H = 0$ and $|B|^2 = 2$ or $|B|^2 - 2H^2 = 0$. If $H = 0$ and $|B|^2 = 2$, we know that $M = S^1(c) \times S^1(\sqrt{1-c^2})$ from a result of [5]. If $|B|^2 - 2H^2 = 0$, M is totally umbilical.

• Case 2: $n \geq 3$.

Let $\mu_i = \lambda_i - H$ and $|Z|^2 = \sum_i \mu_i^2$, we get

$$\sum_i \mu_i = 0, \quad |Z|^2 = |B|^2 - nH^2, \tag{3.10}$$

$$\sum_i \lambda_i^3 = \sum_i \mu_i^3 + 3H|Z|^2 + nH^3. \tag{3.11}$$

By (3.10) and (3.11), we obtain

$$\begin{aligned}\square(nH) = &\frac{1}{2}n(n-1)\triangle r + |\nabla B|^2 - n^2|\nabla H|^2 \\ &+ |Z|^2(n + nH^2 - |Z|^2) + nH\sum_i \mu_i^3.\end{aligned} \tag{3.12}$$

Combining (3.12) and Lemma 2.2, we get

$$\begin{aligned}\square(nH) \ \geq &\tfrac{1}{2}n(n-1)\triangle r + |\nabla B|^2 - n^2|\nabla H|^2 \\ &+|Z|^2\left(n + nH^2 - |Z|^2 - \tfrac{n(n-2)}{\sqrt{n(n-1)}}|H||Z|\right).\end{aligned} \tag{3.13}$$

Hence we have the following:

Lemma 3.1. *Let M be a compact hypersurface in $S^{n+1}(1)$. Then we have*

$$\begin{aligned}0 \geq &\int_M (|\nabla B|^2 - n^2|\nabla H|^2)dv \\ &+ \int_M |Z|^2\left(n + nH^2 - |Z|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||Z|\right)dv.\end{aligned} \tag{3.14}$$

Now, we assume $r = aH + b$ and $(n-1)a^2 - 4n + 4nb \geq 0$. For a real number $d = \frac{n+2\sqrt{n-1}}{n-2}\sqrt{n} > 0$, we have

$$2|H||Z| \leq dH^2 + \frac{1}{d}|Z|^2. \tag{3.15}$$

By Lemma 2.1, Lemma 3.1 and (3.15), we obtain

$$\begin{aligned}&\int_M |Z|^2\Bigg\{n + nH^2\left(2 - \frac{(n-2)d}{2\sqrt{n(n-1)}} + \frac{n(n-2)}{2\sqrt{n(n-1)}d}\right) \\ &- |B|^2\left(1 + \frac{n(n-2)}{2\sqrt{n(n-1)}d}\right)\Bigg\}dv \leq 0,\end{aligned}$$

that is,

$$0 \geq \int_M |Z|^2\left\{n - \frac{n}{2\sqrt{n-1}}|B|^2\right\}dv. \tag{3.16}$$

Noting that $|B|^2 \leq 2\sqrt{n-1}$, we know that the right hand side of (3.16) is nonnegative, it follows that $|\nabla B|^2 = n^2|\nabla H|^2$ and $\int_M |Z|^2\left\{n - \frac{n}{2\sqrt{n-1}}|B|^2\right\}dv = 0$. Then we have either $|Z|^2 = |B|^2 - nH^2 = 0$ and M is totally umbilical or $|B|^2 = 2\sqrt{n-1}$.

If M is not totally umbilical, we can see that

$$|B|^2 = 2\sqrt{n-1}, \tag{3.17}$$

it follows from Lemma 2.2, Lemma 3.1 and (3.17) that

$$\lambda_1 = \cdots = \lambda_{n-1} \neq \lambda_n, \tag{3.18}$$

and $M = S^1(c) \times S^{n-1}(\sqrt{1-c^2})$. This completes the proof of Theorem 1.4. $\square$

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