LIMIT RELATIVE CATEGORY THEORY APPLIED
TO THE CRITICAL POINT THEORY

Tacksun Jung and Q-Heung Choi

Abstract. Let H be a Hilbert space which is the direct sum of five closed
subspaces X_0, X_1, X_2, X_3 and X_4 with X_1, X_2, X_3 of finite dimension.
Let J be a $C^{1,1}$ functional defined on H with $J(0) = 0$. We show the
existence of at least four nontrivial critical points when the sublevels of J
(the torus with three holes and sphere) link and the functional J satisfies
sup-inf variational inequality on the linking subspaces, and the functional
J satisfies $(P.S.)_c^*$ condition and $f|_{X_0 \oplus X_4}$ has no critical point with level
c. For the proof of main theorem we use the nonsmooth version of the
classical deformation lemma and the limit relative category theory.

1. Introduction and statement of main result

Let H be a Hilbert space which is a direct sum of five closed subspaces
X_0, X_1, X_2, X_3 and X_4 with X_1, X_2, X_3 of finite dimension. Let J be a $C^{1,1}$
functional defined on H with $J(0) = 0$. In this paper we investigate the number
of nontrivial critical points of the $C^{1,1}$ functional J under some conditions on
the sublevels of J and the shape of J. We show the existence of at least four
nontrivial critical points when the sublevels of J (the torus with three holes
and sphere) link and the functional J satisfies the sup-inf variational inequality
on the linking subspaces, and the functional J satisfies $(P.S.)_c^*$ condition and
$J|_{X_0 \oplus X_4}$ has no critical point with level c. Micheletti and Saccon prove in [13]
that the functional J has at least two nontrivial critical points under the same
conditions on J except the condition that the sublevel sets are the torus with
one hole and the sphere. In this paper we improve this result to the case that
the sublevel sets are the torus with three holes and sphere.

Now, we state the main result:

Theorem 1.1. *Let H be a Hilbert space and $H = X_0 \oplus X_1 \oplus X_2 \oplus X_3 \oplus X_4$,*
where X_0, X_1, X_2, X_3 and X_4 are five closed subspaces of H with X_1, X_2, X_3

of finite dimension. Let $J : H \to R$ be a $C^{1,1}$ functional with $J(0) = 0$. Let $0 < \rho_1, \rho_2, \rho_3, r < R$, $R_1 > 0$; we define

$$S_i(\rho_i) = \{z \in X_i | \ \|z\| = \rho_i\}, \ i = 1, 2, 3,$$

$$S_i(\rho_i) - w_i = \{z - w_i | \ z \in S_i(\rho_i), w_i \in X_i\}, \ i = 1, 2, 3,$$

$$\Delta_R^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)$$
$$= \{z = (z_1 - w_1) + (z_2 - w_2) + (z_3 - w_3) + z_4 | \ z_i \in X_i, \ i = 1, 2, 3, 4,$$
$$\rho_1 \leq \|z_1 - w_1\| \leq R, \ \rho_2 \leq \|z_2 - w_2\| \leq R,$$
$$\rho_3 \leq \|z_3 - w_3\| \leq R, \ \|z_4\| \leq R_1, \ \|z\| \leq R\},$$

$$\Sigma_R^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)$$
$$= \{z = (z_1 - w_1) + (z_2 - w2) + (z_3 - w_3) + z_4 | \ z_i \in X_i, \ i = 1, 2, 3, 4,$$
$$\|z_4\| \leq R_1, \ \|z_1 - w_1\| = \rho_1, \ \|z_2 - w_2\| = \rho_2, \ \|z_3 - w_3\| = \rho_3, \ \|z\| = R\}$$
$$\cap \ \{z = z_1 + z_2 + z_3 + z_4 | \ z_i \in X_i, \ i = 1, 2, 3, 4, \ \|z_4\| = R_1,$$
$$\rho_1 \leq \|z_1 - w_1\| \leq R, \ \|z\| = R, w_1 \in X_1\}$$
$$\cap \ \{z = z_1 + z_2 + z_3 + z_4 | \ z_i \in X_i, \ , i = 1, 2, 3, 4,$$
$$\|z_4\| = R_1, \ \rho_2 \leq \|z_2 - w_2\| \leq R, \ \|z\| = R, w_2 \in X_2\}$$
$$\cap \ \{z = z_1 + z_2 + z_3 + z_4 | \ z_i \in X_i, \ i = 1, 2, 3, 4, \ \|z_4\| = R_1,$$
$$\rho_3 \leq \|z_3 - w_3\| \leq R, \ \|z\| = R, \ w_3 \in X_3\}.$$

Let

$$\alpha = \inf_{S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} J(z), \quad \beta = \sup_{\Delta_R^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} J(z).$$

Assume that

$$\sup_{z \in \Sigma_R^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} J(z) < 0 < \inf_{z \in S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} J(z).$$

Assume that the $(P.S.)_c^*$ condition holds for J $\forall c \in [\alpha, \beta]$. Assume that $J|_{X_0 \oplus X_4}$ has no critical points with $\alpha \leq J(z) \leq \beta$. Moreover we assume $\beta < +\infty$. Then there exist at least four nontrivial critical points z_1, z_2, z_3 and z_4 for J in $X_1 \oplus X_2 \oplus X_3$ such that

$$\inf_{z \in S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} J(z) \leq J(z_i) \leq \sup_{z \in \Delta_R^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} J(z),$$

$i = 1, 2, 3, 4$.

In Section 2, we introduce the notion of the limit relative category and the $(P.S.)_c^*$ condition and recall the suitable version of the deformation lemma and the multiplicity theorem in [13]. In Section 3, by the nonsmooth version of the classical suit deformation lemma, the limit relative category theory and the multiplicity Theorem 2.1 prove the main theorem.

2. Recall of the critical point theory on the manifold

Now, we consider the critical point theory on the manifold with boundary induced from the limit relative category. Let H be a Hilbert space and M be the closure of an open subset of H such that M can be endowed with the structure of C^2 manifold with boundary. Let $J : W \to R$ be a $C^{1,1}$ functional, where W is an open set containing M. For applying the usual topological methods of critical points theory we need a suitable notion of critical point for J on M. We recall the following notions: lower gradient of J on M, $(P.S.)_c^*$ condition and the limit relative category (see [7]).

Definition 2.1. If $u \in M$, the lower gradient of J on M at u is defined by

$$\text{grad}_M^- J(u) = \begin{cases} \nabla J(u) & \text{if } u \in \text{int}(M), \\ \nabla J(u) + [< \nabla J(u), \nu(u) >]^- \nu(u) & \text{if } u \in \partial M, \end{cases}$$

where we denote by $\nu(u)$ the unit normal vector to ∂M at the point u, pointing outwards. We say that u is a lower critical point for J on M, if $\text{grad}_M^- J(u) = 0$. Since the functional $J(u)$ is strongly indefinite, the notion of the $(P.S.)_c^*$ condition and the limit relative category is a very useful tool for the proof of the main theorems.

Let $(H_n)_n$ be a sequence of closed subspaces of H such that, for any n, $M_n = M \cap H_n$ is the closure of an open subset of H_n and has the structure of a C^2 manifold with boundary in H_n. We assume that for any n there exists a retraction $r_n : M \to M_n$. For given $B \subset H$, we will write $B_n = B \cap H_n$.

Definition 2.2. Let $c \in R$. We say that J satisfies the $(P.S.)_c^*$ condition with respect to $(M_n)_n$, on the manifold with boundary M, if for any sequence $(k_n)_n$ in $\mathcal{N}$ and any sequence $(u_n)_n$ in M such that $k_n \to \infty$, $u_n \in M_{k_n}$, $\forall n$, $J(u_n) \to c$, $\text{grad}_{M_{k_n}}^- J(u_n) \to 0$, there exists a subsequence of $(u_n)_n$ which converges to a point $u \in M$ such that $\text{grad}_M^- J(u) = 0$.

Let Y be a closed subspace of M.

Definition 2.3. Let B be a closed subset of M with $Y \subset B$. We define the relative category $\text{cat}_{M,Y}(B)$ of B in (M,Y), as the least integer h such that there exist $h + 1$ closed subsets $U_0, U_1, \ldots, U_h$ with the following properties:
$B \subset U_0 \cup U_1 \cup \cdots \cup U_h$;
$U_1, \ldots, U_h$ are contractible in M;
$Y \subset U_0$ and there exists a continuous map $F : U_0 \times [0,1] \to M$ such that

$$\begin{aligned} F(x,0) &= x & \forall x \in U_0, \\ F(x,t) &\in Y & \forall x \in Y, \ \forall t \in [0,1], \\ F(x,1) &\in Y & \forall x \in U_0. \end{aligned}$$

If such an h does not exist, we say that $\text{cat}_{M,Y}(B) = +\infty$.

Definition 2.4. Let (X, Y) be a topological pair and $(X_n)_n$ be a sequence of subsets of X. For any subset B of X we define the limit relative category of B in (X, Y), with respect to $(X_n)_n$, by

$$(2.1) \qquad \mathrm{cat}^*_{(X,Y)}(B) = \lim_{n \to \infty} \sup \, \mathrm{cat}_{(X_n, Y_n)}(B_n).$$

Let Y be a fixed subset of M. We set

$$(2.2) \qquad \mathcal{B}_i = \{ \mathrm{B} \subset \mathrm{M} | \ \mathrm{cat}^*_{(\mathrm{M},\mathrm{Y})}(\mathrm{B}) \geq i \},$$

$$(2.3) \qquad c_i = \inf_{B \in \mathcal{B}_i} \sup_{x \in B} J(x).$$

We recall the "nonsmooth" version of the classical Deformation Lemma in [5].

Lemma 2.1. (*Deformation Lemma*) *Let $h : H \to R \cup \{+\infty\}$ be a lower semicontinuous function and assume h to be φ-convex of order 2. Let $c \in R$, $\delta > 0$ and D be a closed set in H such that*

$$\inf\{\| \mathrm{grad}_M^- h(x)\| | \ c - \delta \leq h(x) \leq c + \delta, \qquad \mathrm{dist}(x, D) < \delta\} > 0.$$

Then there exist $\epsilon > 0$ and a continuous deformation $\eta : h^{c+\epsilon} \cap D \times [0, 1] \to h^{c+\epsilon} \cap D_\delta$ (D_δ is the δ-neighborhood of D and $h^c = \{x | h(x) \leq c\}$) such that

  (i) $\eta(x, 0) = x \qquad \forall x \in h^{c+\epsilon} \cap D,$
  (ii) $\eta(x, t) = x \qquad \forall x \in h^{c-\epsilon} \cap D, \forall t \in [0, 1],$
  (iii) $\eta(x, 1) \in h^{c-\epsilon} \quad \forall x \in h^{c+\epsilon} \cap D, \forall t \in [0, 1].$

We have the following multiplicity theorem, which was proved in [13].

Theorem 2.1. *Let $i \in \mathcal{N}$ and assume that*

  (1) $c_i < +\infty,$
  (2) $\sup_{x \in Y} J(x) < c_i,$
  (3) *the $(P.S.)^*_{c_i}$ condition with respect to $(M_n)_n$ holds.*

Then there exists a lower critical point x such that $J(x) = c_i$. If

$$(2.4) \qquad c_i = c_{i+1} = \cdots = c_{i+k-1} = c,$$

then

$$(2.5) \qquad \mathrm{cat}_M(\{x \in M | J(x) = c, \ \mathrm{grad}_M^- J(x) = 0\}) \geq k.$$

3. Proof of Theorem 1.1

Now we will show that J has at least four nontrivial critical points in the subspace $X_1 \oplus X_2 \oplus X_3$ of H.

Let $P_{X_1 \oplus X_2 \oplus X_3}$ be the orthogonal projection from H onto $X_1 \oplus X_2 \oplus X_3$ and

$$(3.1) \qquad C = \{z \in H | \ \|P_{X_1 \oplus X_2 \oplus X_3} z\| \geq 1\}.$$

Then C is the smooth manifold with boundary. Let $C_n = C \cap H_n$. Let us define a functional $\Psi : H \setminus \{X_0 \oplus X_4\} \to H$ by

(3.2)
$$\Psi(z) = z - \frac{P_{X_1 \oplus X_2 \oplus X_3} z}{\|P_{X_1 \oplus X_2 \oplus X_3} z\|} = P_{X_0 \oplus X_4} z + \left(1 - \frac{1}{\|P_{X_1 \oplus X_2 \oplus X_3} z\|}\right) P_{X_1 \oplus X_2 \oplus X_3} z.$$

We have

(3.3)
$$\nabla \Psi(z)(w)$$
$$= w - \frac{1}{\|P_{X_1 \oplus X_2 \oplus X_3} z\|} \left(P_{X_1 \oplus X_2 \oplus X_3} w - \left\langle \frac{P_{X_1 \oplus X_2 \oplus X_3} z}{\|P_{X_1 \oplus X_2 \oplus X_3} z\|}, w \right\rangle \frac{P_{X_1 \oplus X_2 \oplus X_3} z}{\|P_{X_1 \oplus X_2 \oplus X_3} z\|} \right).$$

Let us define the functional $\tilde{J} : C \to R$ by

(3.4)
$$\tilde{J} = J \circ \Psi.$$

Then $\tilde{J} \in C_{loc}^{1,1}$. We note that if $\tilde{z}$ is the critical point of $\tilde{J}$ and lies in the interior of C, then $z = \Psi(\tilde{z})$ is the critical point of J. So it suffices to find the nontrivial critical points of $\tilde{J}$. We note that

(3.5)
$$\| \operatorname{grad}_C^- \tilde{J}(\tilde{z})\| \geq \|P_{X_0 \oplus X_4} \nabla J(\Psi(\tilde{z}))\| \qquad \forall \tilde{z} \in \partial C.$$

Let us set

$$\tilde{S}_r = \Psi^{-1}(S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)),$$
$$\tilde{B}_r = \Psi^{-1}(B_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)),$$
$$\tilde{\Sigma}_R^3 = \Psi^{-1}(\Sigma_R^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)),$$
$$\tilde{\Delta}_R^3 = \Psi^{-1}(\Delta_R^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)).$$

We note that $\tilde{S}_r$, $\tilde{B}_r$, $\tilde{\Sigma}_R^3$ and $\tilde{\Delta}_R^3$ have the same topological structure as S_r, B_r, Σ_R^3 and Δ_R^3, respectively. By the condition of Theorem 1.1, there exist $0 < \rho_1, \rho_2, \rho_3, r < R$ and $R_1 > 0$ such that

$$\sup_{\tilde{z} \in \tilde{\Sigma}_R^3} \tilde{J}(\tilde{z}) = \sup_{z \in \Sigma_R^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} J(z)$$

$$< \inf_{z \in S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} J(z) = \inf_{\tilde{z} \in \tilde{S}_r} \tilde{J}(\tilde{z}),$$

$$\sup_{\tilde{z} \in \tilde{\Delta}_R^3} \tilde{J}(\tilde{z}) = \sup_{z \in \Delta_R^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} J(z) < \infty$$

and

$$\inf_{\tilde{z} \in \tilde{B}_r} \tilde{J}(\tilde{z}) = \inf_{z \in B_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} J(z) > -\infty.$$

By the condition of Theorem 1.1, $\tilde{J}$ satisfies the $(P.S.)_{\tilde{c}}^*$ condition with respect to $(C_n)_n$ for every real number $\tilde{c}$ such that

(3.6)
$$\inf_{\tilde{z} \in \tilde{S}_r} \tilde{J}(\tilde{z}) \leq \tilde{c} \leq \sup_{\tilde{z} \in \tilde{\Delta}_R^3} \tilde{J}(\tilde{z}).$$

Let

$$\Sigma_n^3 = \Sigma_R^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4) \cap H_n,$$
$$\Delta_n^3 = \Delta_R^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4) \cap H_n,$$
$$\tilde{\Sigma}_n^3 = \tilde{\Sigma}_R^3 \cap H_n, \qquad \tilde{\Delta}_n^3 = \tilde{\Delta}_R^3 \cap H_n.$$

We claim that

$$(3.7) \qquad\qquad \mathrm{cat}_{(C_n, \tilde{\Sigma}_n^3)}(\tilde{\Delta}_n^3) = 4.$$

In fact, we consider a continuous deformation $r : \tilde{S}_r \backslash X_0 \times [0,1] \to \tilde{S}_r \backslash X_0$ such that

- $r(x,0) = x, \ \forall x \in \tilde{S}_r \backslash X_0,$
- $r(x,t) = x, \ \forall x \in \tilde{S}_r \cap (X_1 \oplus X_2 \oplus X_3), \ \forall t \in [0,1],$
- $r(x,1) \in \tilde{S}_r \cap (X_1 \oplus X_2 \oplus X_3), \ \forall x \in \tilde{S}_r \backslash X_0.$

Now we can define, if $x = x_0 + x_{123} + x_4 \in X_0 \oplus (X_1 \oplus X_2 \oplus X_3) \oplus X_4, t \in [0,1],$

$$(3.8) \qquad\qquad r_1(x,t) = x_0 + \|x_{123} + x_4\| r\left(\frac{x_{123} + x_4}{\|x_{123} + x_4\|}, t\right).$$

Using r_1, it is easy to construct, for all n, a continuous deformation $\eta_n : C_n \times [0,1] \to C_n$ such that

- $\eta_n(x,0) = x, \quad \forall x \in C_n$
- $\eta_n(x,t) = x, \quad \forall x \in \tilde{\Delta}_n^3, \forall t \in [0,1],$
- $\eta_n(x,1) \in \tilde{\Delta}_n^3, \quad \forall x \in C_n,$
- $\eta_n(x,t) \in C_n \backslash \tilde{S}_r, \quad \forall x \in C_n \backslash \tilde{S}_r, \quad \forall t \in [0,1].$

The existence of η_n implies that

$$(3.9) \qquad\qquad \mathrm{cat}_{(C_n, \tilde{\Sigma}_n^3)}(\tilde{\Delta}_n^3) = \mathrm{cat}_{(\tilde{\Delta}_n^3, \tilde{\Sigma}_n^3)}(\tilde{\Delta}_n^3).$$

We note that the pair $(\tilde{\Delta}_n^3, \tilde{\Sigma}_n^3)$ is homeomorphic to the pair (Δ_n^3, Σ_n^3) and the pair (Δ_n^3, Σ_n^3) is homeomorphic to the pair $(\mathcal{B}^{p+1} \times \{(\mathcal{S}^{q_1-1} - w_1) \cup (\mathcal{S}^{q_2-1} - w_2) \cup (\mathcal{S}^{q_3-1} - w_3)\}, \mathcal{S}^p \times \{(\mathcal{S}^{q_1-1} - w_1) \cup (\mathcal{S}^{q_2-1} - w_2) \cup (\mathcal{S}^{q_3-1} - w_3)\})$, where $p = \dim X_0 \cap H_n$, $q_1 = \dim X_1 \cap H_n$, $q_2 = \dim X_2 \cap H_n$, $q_3 = \dim X_3 \cap H_n$ and $\mathcal{B}^r$, $\mathcal{S}^r$ denote the r-dimensional ball, the r-dimensional sphere, respectively. Thus the pair $(\tilde{\Delta}_n^3, \tilde{\Sigma}_n^3)$ is homeomorphic to the pair $(\mathcal{B}^{p+1} \times \{(\mathcal{S}^{q_1-1} - w_1) \cup (\mathcal{S}^{q_2-1} - w_2) \cup (\mathcal{S}^{q_3-1} - w_3)\}, \mathcal{S}^p \times \{(\mathcal{S}^{q_1-1} - w_1) \cup (\mathcal{S}^{q_2-1} - w_2) \cup (\mathcal{S}^{q_3-1} - w_3)\})$. This fact and the facts that $q_i = 1$, $i = 1, 2, 3$ and (b) of (3.7) in [7] imply that

$$\mathrm{cat}_{(C_n, \tilde{\Sigma}_n^3)}(\tilde{\Delta}_n^3) = 4.$$

Thus we have

$$(3.10) \qquad\qquad \mathrm{cat}^*_{(C, \tilde{\Sigma}_R^3)}(\tilde{\Delta}_R^3) = 4.$$

Let us set
(3.11)
$$\mathcal{A}_1 = \{A \subset C|\ \mathrm{cat}^*_{(C,\tilde{\Sigma}^3_R)}(A) \geq 1\}, \quad \mathcal{A}_2 = \{A \subset C|\ \mathrm{cat}^*_{(C,\tilde{\Sigma}^3_R)}(A) \geq 2\},$$
$$\mathcal{A}_3 = \{A \subset C|\ \mathrm{cat}^*_{(C,\tilde{\Sigma}^3_R)}(A) \geq 3\}, \quad \mathcal{A}_4 = \{A \subset C|\ \mathrm{cat}^*_{(C,\tilde{\Sigma}^3_R)}(A) \geq 4\}.$$

Since $\mathrm{cat}^*_{(C,\tilde{\Sigma}^3_R)}(\tilde{\Delta}^3_R) = 4$, $\tilde{\Delta}^3_R \in \mathcal{A}_i$, $i = 1, 2, 3$. Let us set

(3.12)
$$\tilde{c}_1 = \inf_{A \in \mathcal{A}_1} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}), \quad \tilde{c}_2 = \inf_{A \in \mathcal{A}_2} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}),$$
$$\tilde{c}_3 = \inf_{A \in \mathcal{A}_3} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}), \quad \tilde{c}_4 = \inf_{A \in \mathcal{A}_4} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}).$$

We first claim that $\tilde{c}_i < \infty$, $i = 1, 2, 3, 4$. In fact, from the facts that

$$\sup_{z \in \Delta^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} J(z) < \infty$$

and $\tilde{\Delta}^3_R \in \mathcal{A}_i$, $i = 1, 2, 3, 4$, we have that

$$\tilde{c}_i = \inf_{A \in \mathcal{A}_i} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}) \leq \sup_{\tilde{z} \in \tilde{\Delta}^3_R} \tilde{J}(\tilde{z})$$
$$= \sup_{z \in \Delta^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} J(z) < \infty.$$

We also claim that $\sup_{\tilde{z} \in \tilde{\Sigma}^3_R} \tilde{J}(\tilde{z}) \leq \tilde{c}_i$, $i = 1, 2, 3, 4$. In fact, for any $A \in \mathcal{A}_i$ with $\tilde{\Sigma}^3_R \subset A$, $i = 1, 2, 3, 4$,

(3.13)
$$\sup_{\tilde{z} \in \tilde{\Sigma}^3_R} \tilde{J}(\tilde{z}) \leq \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}),$$

and hence

(3.14)
$$\sup_{\tilde{z} \in \tilde{\Sigma}^3_R} \tilde{J}(\tilde{z}) \leq \inf_{A \in \mathcal{A}_i} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}) = \tilde{c}_i, \quad i = 1, 2, 3, 4.$$

By the condition of Theorem 1.1, $\tilde{J}$ satisfies the $(P.S.)^*_{\tilde{c}}$ condition with respect to $(C_n)_n$ for any real number $\tilde{c}$ with $\inf_{\tilde{z} \in \tilde{S}_r} \tilde{J}(\tilde{z}) \leq \tilde{c} \leq \sup_{\tilde{z} \in \tilde{\Delta}^3_R} \tilde{J}(\tilde{z})$. Thus, by Theorem 2.1, there exist four nontrivial critical points $\tilde{z}_1$, $\tilde{z}_2$, $\tilde{z}_3$, $\tilde{z}_4$ of the functional $\tilde{J}$ such that

(3.15)
$$\tilde{c}_1 = \tilde{J}(\tilde{z}_1), \quad \tilde{c}_2 = \tilde{J}(\tilde{z}_2), \quad \tilde{c}_3 = \tilde{J}(\tilde{z}_3), \quad \tilde{c}_4 = \tilde{J}(\tilde{z}_4).$$

We claim that

(3.16)
$$\inf_{\tilde{z} \in \tilde{S}_r} \tilde{J}(\tilde{z}) \leq \tilde{c}_1 \leq \tilde{c}_2 \leq \tilde{c}_3 \leq \tilde{c}_4 \leq \sup_{\tilde{z} \in \tilde{\Delta}^3_R} \tilde{J}(\tilde{z}).$$

Since $\mathrm{cat}^*_{(C,\tilde{\Sigma}^3_R)}(\tilde{\Delta}^3_R) = 4$, $\tilde{\Delta}^3_R \in \mathcal{A}_4$ and hence

(3.17)
$$\tilde{c}_4 = \inf_{A \in \mathcal{A}_4} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}) \leq \sup_{\tilde{z} \in \tilde{\Delta}^3_R} \tilde{J}(\tilde{z}), \quad \forall A \in \mathcal{A}_4.$$

For the proof of $\tilde{c}_1 \geq \inf_{\tilde{z} \in \tilde{S}_r} \tilde{J}(\tilde{z})$, we construct a deformation $\eta'_n : C_n \backslash \tilde{S}_r \times [0,1] \to C_n \backslash \tilde{S}_r$, for all n, such that

- $\eta'_n(x, 0) = x, \quad \forall x \in C_n \backslash \tilde{S}_r$,
- $\eta'_n(x, t) = x, \quad \forall x \in \tilde{\Sigma}_n^3, \forall t \in [0,1]$,
- $\eta'_n(x, 1) \in \tilde{\Sigma}_n^3, \quad \forall x \in C_n$.

Actually η'_n can be defined by taking the retraction of η_n on $C_n \backslash \tilde{S}_r$ followed by a retraction of $\tilde{\Delta}_n^3 \backslash \tilde{S}_r$ to $\tilde{\Sigma}_n^3$. The existence of η'_n, for all n, implies that any $A \in \mathcal{A}_1$ must intersect $\tilde{S}_r$. So $\sup \tilde{J}(A) \geq \inf_{\tilde{z} \in \tilde{S}_r} \tilde{J}(\tilde{z}) \ \forall A \in \mathcal{A}_1$. So we have $\tilde{c}_1 = \inf_{A \in \mathcal{A}_1} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}) \geq \inf_{\tilde{z} \in \tilde{S}_r} \tilde{J}(\tilde{z})$. Therefore there exist at least four nontrivial critical points $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4$ for the functional $\tilde{J}$ such that

$$\inf_{\tilde{z} \in \tilde{S}_r} \tilde{J}(\tilde{z}) \leq \tilde{J}(\tilde{z}_1) \leq \tilde{J}(\tilde{z}_2) \leq \tilde{J}(\tilde{z}_3) \leq \tilde{J}(\tilde{z}) \leq \sup_{\tilde{z} \in \tilde{\Delta}_R^3} \tilde{J}(\tilde{z}).$$

Setting $z_i = \Psi(\tilde{z}_i)$, $i = 1, 2, 3, 4$, we have

$$(3.18) \quad \begin{aligned} \inf_{z \in S_r} J(z) = \inf_{\tilde{z} \in \tilde{S}_r} \tilde{J}(\tilde{z}) &\leq J(z_1) \leq J(z_2) \\ &\leq J(z_3) \leq J(z_4) \leq \sup_{\tilde{z} \in \tilde{\Delta}_R^3} \tilde{J}(\tilde{z}) = \sup_{z \in \Delta_R^3} J(z). \end{aligned}$$

We claim that $\tilde{z}_i \notin \partial C$, that is $z_i \notin X_0 \oplus X_4$, which implies that z_i are the critical points for J in $X_1 \oplus X_2 \oplus X_3$. For this we assume by contradiction that $z_i \in X_0 \oplus X_4$. From (3.5), $P_{X_0 \oplus X_4} \nabla J(z_i) = 0$, namely, z_i, $i = 1, 2, 3, 4$, are the critical points for $J|_{X_0 \oplus X_4}$. By the condition of Theorem 1.1, the critical points z_i in $X_0 \oplus X_4$ has no critical values in

$$\left[\inf_{z \in S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} J(z), \sup_{z \in \Delta_R^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} J(z) \right],$$

which contradicts to (3.18). Thus $z_i \notin X_0 \oplus X_4$. This proves Theorem 1.1.

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Tacksun Jung
Department of Mathematics
Kunsan National University
Kunsan 573-701, Korea
E-mail address: tsjung@kunsan.ac.kr

Q-Heung Choi
Department of Mathematics Education
Inha University
Incheon 402-751, Korea
E-mail address: qheung@inha.ac.kr