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ON EINSTEIN HERMITIAN MANIFOLDS II

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Abstract. We show that on a Hermitian surface M, if M is weakly $*$-Einstein and has J-invariant Ricci tensor then M is Einstein, and vice versa. As a consequence, we obtain that a compact $*$-Einstein Hermitian surface with J-invariant Ricci tensor is Kähler. In contrast with the 4-dimensional case, we show that there exists a compact Einstein Hermitian $(4n+2)$-dimensional manifold which is not weakly $*$-Einstein.

1. Introduction

The Riemannian version of the Goldberg-Sachs Theorem [1] says that the self-dual Weyl tensor W^+ of an Einstein Hermitian surface M is degenerate, i.e., at least two of its three eigenvalues coincide. In fact, this implies that M is weakly $*$-Einstein. Another weak form of the Einstein condition is the J-invariant condition for the Ricci tensor. In this note, we show that, on a Hermitian surface, both weak versions of the Einstein condition together are equivalent to the Einstein condition. In contrast with the 4-dimensional case, we show that there exists a compact Einstein Hermitian $(4n+2)$-dimensional manifold which is not weakly $*$-Einstein. More precisely we obtain the followings:

Theorem 1. *Let $M=(M,J,g)$ be a Hermitian surface. If M is weakly $*$-Einstein and has J-invariant Ricci tensor, then M is Einstein, and vice versa.*

Corollary 1. *A compact $*$-Einstein Hermitian surface with J-invariant Ricci tensor is Kähler.*

Theorem 2. *If $M=(S^{2n+1}\times S^{2n+1}, J_C, g_{\text{prod}})$, then M is a compact Einstein Hermitian manifold which is not weakly $*$-Einstein, where J_C is the complex structure of Calabi-Eckmann and g_{prod} the standard product metric on the product of the spheres.*

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We recall that a Hermitian manifold (M, J, g) is called Einstein Hermitian if the compatible Riemannian metric g with complex structure J is Einstein by its Levi-Civita connection.

2. Preliminaries

Let $M = (M, J, g)$ be a Hermitian manifold with complex structure J and compatible Riemannian metric g, i.e., $g(JX, JY) = g(X, Y)$ for vector fields X, Y. Denote by $\Omega(X, Y)$ the Kähler form of M defined by $\Omega(X, Y) = g(JX, Y)$. We shall always consider M with the orientation determined by the complex structure J. The Riemannian curvature R, the Ricci tensor Ric and the scalar curvature s of M are defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

$$\mathrm{Ric}(X,Y) = \mathrm{Trace}\{Z \to R(Z,X)Y\}, s = \mathrm{Trace}_g \mathrm{Ric}$$

for vector fields X, Y, Z.

Furthermore, we define the $*$-Ricci tensor and $*$-scalar curvature of (J, g) by

$$\mathrm{Ric}^*(X,Y) = \mathrm{Trace}\{Z \to -JR(Z,X)JY\}, s^* = \mathrm{Trace}_g \mathrm{Ric}^* .$$

The $*$-Ricci tensor is in general neither symmetric nor skew-symmetric and satisfies the equation: $\mathrm{Ric}^*(JX, JY) = \mathrm{Ric}^*(Y, X)$. Note that on a Kähler manifold, the $*$-Ricci tensor and the Ricci tensor coincide; this is a consequence of the Kähler identity $R(X,Y)(JZ) = J(R(X,Y)Z)$, which itself follows from the fact that $\nabla J = 0$. We shall say that M is weakly $*$-Einstein if and only if the $*$-Ricci tensor is a functional multiple of the metric g, i.e., $\mathrm{Ric}^* = \lambda g$. Note that in contrast to Einstein manifolds the $*$-scalar curvature of a weakly $*$-Einstein manifold M need not be a constant and when this holds, we shall simply say that M is $*$-Einstein. We also consider the curvature tensor R as a (0,4)-tensor as follows: $R(X,Y,Z,W) = -g(R(X,Y)Z, W)$. Considering the Riemannian curvature tensor R as a (0,4)-tensor, we have the following well known $SO(2n)$-decomposition: $R = \frac{s}{4n(2n-1)} g \wedge g + \frac{1}{2n-2} \mathrm{Ric}_0 \wedge g + W$, where $\mathrm{Ric}_0 = \mathrm{Ric} - \frac{s}{2n} g$ is the traceless Ricci tensor and W is the Weyl tensor. Here the symbol $\wedge$ is the Nomizu-Kulkarni product of symmetric (0,2)-tensors generating a curvature type tensor. Note that $\mathrm{Ric}_0 = 0$ if and only if M is Einstein. Let $\{e_i\}_{i=1,\ldots,2n}$ be a local orthonormal frame and R_{ijkl}, r_{ij}, R_{ij}, W_{ijkl}, r^*_{ij}, R^*_{ij} be components of R, Ric, Ric_0, W, Ric^*, Ric^*_0 with respect to $\{e_i\}$ respectively, i.e.,

$$R_{ijkl} = R(e_i, e_j, e_k, e_l), r_{ij} = \mathrm{Ric}(e_i, e_j), R_{ij} = \mathrm{Ric}_0(e_i, e_j),$$

$$W_{ijkl} = W(e_i, e_j, e_k, e_l), r^*_{ij} = \mathrm{Ric}^*(e_i, e_j), R^*_{ij} = \mathrm{Ric}^*_0(e_i, e_j).$$

Here Ric^*_0 is the traceless $*$-Ricci tensor. Hence the Weyl tensor $W = (W_{ijkl})$ of M can be expressed as

$$W_{ijkl} = R_{ijkl} - \frac{1}{2n-2}(R_{ik}\delta_{jl} + R_{jl}\delta_{ik} - R_{il}\delta_{jk} - R_{jk}\delta_{il}) -$$

$$-\frac{s}{2n(2n-1)}(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}). \tag{1}$$

Here δ_{ij} is the Kronecker delta. We denote J_{ij} by

$$J_{ij}=g(Je_i,e_j).$$

From now on the components of tensors shall be considered under orthonormal frame. Indices with an overbar are the ones with respect to $\{Je_i\}$, for example,

$$R_{\bar{i}jkl}=R(Je_i,e_j,e_k,e_l).$$

Using this notation, we have

$$r_{ij}=\sum_a R_{iaja}, r^*_{ij}=\sum_a R_{ia\bar{j}\bar{a}}, s=\sum_{a,b} R_{abab}, s^*=\sum_{a,b} R_{ab\bar{a}\bar{b}}$$

and

$$r^*_{ij}=r^*_{\bar{j}\bar{i}}.$$

The Riemannian metric g induces a metric on the bundle $\bigwedge^2 M$ of 2-vectors on M by $\langle X_1\wedge X_2, X_3\wedge X_4\rangle=\det(g(X_i,X_j))$. Then we also consider the curvature tensor R as an endomorphism of the bundle $\bigwedge^2 M$ as follows: $\langle R(X\wedge Y), Z\wedge W\rangle=-g(R(X,Y)Z,W)$. In dimension 4, the Hodge star operator defines an endomorphism $*$ of $\bigwedge^2 M$ with $*^2=Id$. Hence $\bigwedge^2 M=\bigwedge^+ M\bigoplus\bigwedge^- M$, where $\bigwedge^+ M$(resp. $\bigwedge^- M$) is the subbundle of $\bigwedge^2 M$ corresponding to the eigenvalue $+1$ (resp. -1) of $*$. As an endomorphism of $\bigwedge^2 M$ the Weyl tensor W commutes with the Hodge star operator $*$ and so W preserves the decomposition $\bigwedge^2 M=\bigwedge^+ M\bigoplus\bigwedge^- M$. Note that as an endomorphism of $\bigwedge^2 M$ the curvature tensor R commutes with the Hodge star operator $*$, i.e., $*R=R*$ if and only if M is Einstein [5]. We denote the restriction of W to $\bigwedge^+ M$ (resp. $\bigwedge^- M$) by W^+ (resp. W^-) called the self-dual Weyl tensor (resp. anti-self-dual Weyl tensor). Hence, in dimension 4, the Riemannian curvature (0,4)-tensor R can be obtained as follows:

$$R=\frac{s}{24}g\wedge g+\frac{1}{2}\mathrm{Ric}_0\wedge g+W^++W^-.$$

Let M=(M,J,g) be a Hermitian surface (i.e., a Hermitian manifold of real dimension 4). From now on we identify 2-vectors with 2-forms. Then usual type decomposition

$$\bigwedge^{2} M\bigotimes C=\bigwedge^{2,0} M\bigoplus\bigwedge^{1,1} M\bigoplus\bigwedge^{0,2} M$$

of complexified 2-forms induces the decomposition

$$\bigwedge^{2} M=R\Omega\bigoplus(\bigwedge_0^{1,1} M)_R\bigoplus(\bigwedge^{2,0} M\bigoplus\bigwedge^{0,2} M)_R,$$

where $R\Omega$ is the line bundle generated by the Kähler form Ω and $\bigwedge_0^{1,1} M$ is the orthogonal complement of $R\Omega$ in $\bigwedge^{1,1} M$. Note that

$$R\Omega \bigoplus (\bigwedge^{2,0} M \bigoplus \bigwedge^{0,2} M)_R = \bigwedge^{+} M$$

and

$$(\bigwedge_0^{1,1} M)_R = \bigwedge^{-} M.$$

On the other hand we can extend J to act on 2-forms as follows: $J(A)(X,Y) = A(JX,JY)$ for a 2-form A.

3. Proof of Theorem 1 and Corollary 1

Let $M = (M, J, g)$ be a Hermitian surface. From now on we assume that all tensors are continued by complex linearity. For an orthonormal frame $\{e_1, Je_1, e_2, Je_2\}$ we set $Z_k = \frac{1}{\sqrt{2}}(e_k - iJe_k)$, $Z_{\bar{k}} = \frac{1}{\sqrt{2}}(e_k + iJe_k), k = 1, 2$. Let $\langle , \rangle$ be the Hermitian continuation of g on $\bigwedge^2 M \otimes C$ and $\alpha = (Z_1 \bigwedge Z_2), \beta = \frac{1}{\sqrt{2}}(Z_1 \bigwedge Z_{\bar{1}} + Z_2 \bigwedge Z_{\bar{2}}), \bar{\alpha} = (Z_{\bar{1}} \bigwedge Z_{\bar{2}}), \gamma = (Z_1 \bigwedge Z_{\bar{2}}), \delta = \frac{1}{\sqrt{2}}(Z_1 \bigwedge Z_{\bar{1}} - Z_2 \bigwedge Z_{\bar{2}}),\ \bar{\gamma} = (Z_{\bar{1}} \bigwedge Z_2)$. Then $\{\alpha, \beta, \bar{\alpha}\}$ and $\{\gamma, \delta, \bar{\gamma}\}$ are orthonormal frames of $\bigwedge^+ M \otimes C$ and $\bigwedge^- M \otimes C$ respectively. Consider W^+ as an endomorphism of $\bigwedge^+ M \otimes C$. Then the matrix of W^+ with respect to the frame $\{\alpha, \beta, \bar{\alpha}\}$ has the following form [2]:

$$W^+ = \left(\begin{array}{c|c|c} W_1^+ & W_2^+ & W_3^+ \\ \hline W_2^+ & -2W_1^+ & -W_2^+ \\ \hline W_3^+ & -W_2^+ & W_1^+ \end{array} \right).$$

Here $W_1^+ = \langle R\alpha, \alpha \rangle - \frac{s}{12}, W_2^+ = \langle R\alpha, \beta \rangle, W_3^+ = \langle R\alpha, \bar{\alpha} \rangle$. Since $\nabla_X Y \in T^{(1,0)}M$ for all $X, Y \in T^{(1,0)}M$, we have $W_3^+ = 0$. Hence, on a Hermitian surface M, W^+ is degenerate if and only if $W_2^+ = 0$. In order to show that Theorem 1 and Corollary 1 hold, we need the following;

Lemma 1. *Let $M = (M, J, g)$ be a Hermitian surface. Then we have*

$$(2)\ \ \mathrm{Ric}(X,Y) + \mathrm{Ric}(JX,JY) - \mathrm{Ric}^*(X,Y) - \mathrm{Ric}^*(JX,JY) = \frac{s - s^*}{2} g(X,Y).$$

Proof. Let $\{e_i\}_{i=1,\dots,4}$ be an orthonormal frame and $\{e^i\}_{i=1,\dots,4}$ its dual frame. Then we can write the Kähler form Ω as $\Omega = \frac{1}{2}\sum J_{ij} e^i \wedge e^j$ and we have

$$W(\Omega)_{kl} = \frac{1}{2}\sum_{i,j} W_{ijkl} J_{ij} = \frac{1}{2}\sum_{i} W_{i\bar{i}kl} = \sum_{i} W_{ik\bar{i}l}.$$

From (1), we have

$$\begin{aligned} W(\Omega)_{kl} &= -r^*_{k\bar{l}} - \frac{1}{2}(R_{\bar{k}l} - R_{k\bar{l}}) - \frac{s}{12}J_{kl} \\ &= -r^*_{(k\bar{l})} - r^*_{[k\bar{l}]} - \frac{1}{2}(R_{\bar{k}l} + R_{\bar{\bar{k}}\bar{l}}) - \frac{s}{12}J_{kl} \\ &= R^*_{(\bar{k}l)} + \frac{s^*}{4}J_{kl} - r^*_{[\bar{k}l]} - \frac{1}{2}(R_{\bar{k}l} + R_{\bar{\bar{k}}\bar{l}}) - \frac{s}{12}J_{kl}, \end{aligned}$$

where $A_{(ij)}$ and $A_{[ij]}$ are the symmetric and skew-symmetric part of a tensor A_{ij}, respectively. It is easy to see that only $B_{kl} = R^*_{(\bar{k}l)} - \frac{1}{2}(R_{\bar{k}l} + R_{\bar{\bar{k}}\bar{l}})$ is the component of a section of $(\bigwedge_0^{1,1} M)_R$. Since the Weyl tensor of M preserves the self-duality and $(\bigwedge_0^{1,1} M)_R$ is identified with $\bigwedge^- M$, we get $B_{kl} = 0$. Therefore the identity (2) holds. This completes the proof of Lemma 1. □

Suppose that a Hermitian surface M is weakly $*$-Einstein. Then the above identity (2) implies $\mathrm{Ric}(X,Y) + \mathrm{Ric}(JX,JY) = \frac{s}{2}g(X,Y)$. By the another assumption, i.e., M has J-invariant Ricci tensor, we get $\mathrm{Ric}(X,Y) = \frac{s}{4}g(X,Y)$ which means that M is Einstein. Conversely, by definition M is weakly $*$-Einstein if and only if $\mathrm{Ric}^*(X,Y) = \lambda g(X,Y)$. This is equivalent to $R(\Omega) = \lambda\Omega$. Using the frame $\{\alpha, \beta, \bar{\alpha}, \gamma, \delta, \bar{\gamma}\}$, we see that $R(\Omega) = \lambda\Omega$ if and only if $\langle R(\beta), \alpha\rangle = \langle R(\beta), \bar{\alpha}\rangle = \langle R(\beta), \gamma\rangle = \langle R(\beta), \delta\rangle = \langle R(\beta), \bar{\gamma}\rangle = 0$. Our assumption that M is an Einstein Hermitian surface implies $W_2^+ = 0$ since W^+ is degenerate. Hence we have $\langle R(\beta), \alpha\rangle = \langle R(\beta), \bar{\alpha}\rangle = 0$. Furthermore in dimension 4 the Einstein condition is equivalent to $*R = R*$ which implies that $\langle R(\beta), \gamma\rangle = \langle R(\beta), \delta\rangle = \langle R(\beta), \bar{\gamma}\rangle = 0$. Hence M is weakly $*$-Einstein and obviously its Ricci tensor is J-invariant. This completes the proof of Theorem 1. The given condition of Corollary 1 implies that M is also a compact Einstein Hermitian surface with constant s^* by Theorem 1. Hence by the well-known result in [4], we can conclude that M is Kähler.

4. A compact Einstein Hermitian and non-weakly $*$-Einstein manifold of dimension $(4n + 2)$

The product of odd dimensional spheres $S^{2n+1} \times S^{2m+1}$ can be provided with a complex structure [3], defined as follows: let N_1 and N_2 be the outward normals to the spheres S^{2n+1} and S^{2m+1} sitting inside $\mathbf{C}^{n+1}$ and $\mathbf{C}^{m+1}$, respectively, and let $J_1$and J_2 be the standard complex structures on these spaces. Since J_1N_1 and J_2N_2 are globally defined vector fields on the respective spheres, we can decompose any vector field X on $S^{2n+1} \times S^{2m+1}$ as

$$X = X_1 + X_2 + d_1(X)J_1N_1 + d_2(X)J_2N_2,$$

where X_1 is tangent to S^{2n+1} and perpendicular to J_1N_1, while X_2 is tangent to S^{2m+1} and perpendicular to J_2N_2. The notion of perpendicularity is defined using the standard metrics on $R^{2n+2} = \mathbf{C}^{n+1}$ and $R^{2m+2} = \mathbf{C}^{m+1}$, respectively.

Using this decomposition, we may now define the (1,1)-tensor J_C by

$$J_C X = J_1 X_1 + J_2 X_2 - d_2(X) J_1 N_1 + d_1(X) J_2 N_2.$$

This J_C on $S^{2n+1} \times S^{2m+1}$ is in fact a complex structure [3]. For each sphere factor in $S^{2n+1} \times S^{2m+1}$ we have the Hopf fibration onto a complex projective space and so their product produces a Riemannian submersion $S^{2n+1} \times S^{2m+1} \to CP^n \times CP^m$. We obtain a decomposition of the tangent space at each point into a horizontal and vertical component. For our purpose we consider the product of odd dimension sphere S^{2n+1} with itself. Obviously the product metric g_{prod} of the standard metric on each sphere factor in $S^{2n+1} \times S^{2n+1}$ is Einstein. Furthermore, the product metric g_{prod} on $S^{2n+1} \times S^{2n+1}$ is compatible with J_C. Under the complex structure J_C, the $*$-Ricci tensor of g_{prod} is not a functional multiple of the metric g_{prod}. In fact we have $\text{Ric}^*(V, V) = 0$ and $\text{Ric}^*(Y, Y) \neq 0$, where V is a vertical vector and Y is a non-trivial horizontal vector. Summing up the above argument, we can conclude that $M = (S^{2n+1} \times S^{2n+1}, J_C, g_{\text{prod}})$ is a compact Einstein Hermitian manifold which is not weakly $*$-Einstein. This completes the proof of Theorem 2.

References

[1] V. Apostolov and P. Gauduchon, *The Riemannian Goldberg-Sachs theorem*, Internat. J. Math. **8** (1997), no. 4, 421–439.

[2] Ch. Boyer, *Conformal duality and compact complex surfaces*, Math. Ann. **274** (1986), no. 3, 517–526.

[3] E. Calabi and B. Eckmann, *A class of compact, complex manifolds which are not algebraic*, Ann. of Math. (2) **58** (1953), 494–500.

[4] J. Kim, *On Einstein Hermitian manifolds*, Monatsh. Math. **152** (2007), no. 3, 251–254.

[5] ———, *Rigidity theorems for Einstein-Thorpe metrics*, Geom. Dedicata **80** (2000), no. 1-3, 281–287.

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